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Research Article

Combinatorial Aspects of the Generalized Euler's Totient

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A generalized Euler's totient is defined as a Dirichlet convolution of a power function and a product of the Souriau-Hsu-Möbius function with a completely multiplicative function. Two combinatorial aspects of the generalized Euler's totient, namely, its connections to other totients and its relations with counting formulae, are investigated.

1. Introduction

Let $\mathcal{A}$ be the unique factorization domain of arithmetic functions [1, 2] equipped with addition and (Dirichlet) convolution defined, respectively, by

$$(f+g)(n) = f(n) + g(n), \qquad (f * g)(n) = \sum_{ij=n} f(i)g(j). \tag{1.1}$$

The convolution identity $I \in \mathcal{A}$ is defined by

$$I(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{1.2}$$

For $f \in \mathcal{A}$, write f^{-1} for its convolution inverse whenever it exists. A nonzero arithmetic function f is said to be *multiplicative* if $f(mn) = f(m)f(n)$ whenever $\gcd(m,n) = 1$, and is called *completely multiplicative* if this equality holds for all $m, n \in \mathbb{N}$. For $\alpha \in \mathbb{C}$, *the Souriau-Hsu-Möbius (SHM) function* ([3, 4], [5, page 107]) is defined by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)}, \tag{1.3}$$

where $n = \prod p^{\nu_p(n)}$ denotes the unique prime factorization of $n \in \mathbb{N}$, $\nu_p(n)$ being the largest exponent of the prime p that divides n. This function generalizes the usual Möbius function, μ, because $\mu_1 = \mu$. Note that

$$\mu_0 = I, \quad \mu_{-1} = u \text{ the arithmetic unit function defined by } u(n) = 1 \quad (n \in \mathbb{N}), \tag{1.4}$$

and for $\alpha, \beta \in \mathbb{C}$, we have

$$\mu_{\alpha+\beta} = \mu_\alpha * \mu_\beta. \tag{1.5}$$

It is easily checked that μ_α is multiplicative; there are exactly two SHM functions that are completely multiplicative, namely, $\mu_0 = I$ and $\mu_{-1} = u$, and there is exactly one SHM function whose convolution inverse is completely multiplicative, namely, $\mu_1 = u^{-1}$. For a general reference on the Möbius function and its generalizations, see Chapter 2 of the encyclopedic work [5].

The classical *Euler's totient* $\phi(n)$ is defined as the number of positive integers $a \leq n$ such that $\gcd(a, n) = 1$. It is well known (page 7 of [1]) that

$$\phi(n) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{1.6}$$

For a general reference about Eulier's totient, its many facets and generalizations, see [5, Chapter 3]. Euler's totient has been given a good deal of generalizations. Of interest to us here is the one due to Wang and Hsu [6], defined for $k, r \in \mathbb{N}$ and completely multiplicative $f \in \mathcal{A}$ by

$$\phi_\tau^{(k)}(n) = \sum_{d|n} \left(\frac{n}{d}\right)^k f(d) \mu_r(d), \tag{1.7}$$

where $\tau = \mu_r f$. In [6] it is shown that $\phi_\tau^{(k)}$ possesses properties extending those of the classical Euler totient, such as the following.

(P1) $\phi_\tau^{(k)}(n) = n^k \prod_{p|n} (1 - f(p)/p^k)^r$ when n is r-powerful, that is, $\nu_p(n) \geq r$ for each prime factor p of n.

(P2) Let $\overrightarrow{a} := (a_1, \ldots, a_k) \in \mathbb{Z}^k$ (Theorem 2.3 of [6]). Then, for prime p, there uniquely exists an $r \times k$ matrix $B_p(\overrightarrow{a})$ over $\mathbb{Z}_p := \{0, 1, \ldots, p-1\}$ such that $\overrightarrow{a} \equiv (1, p, \ldots, p^{r-1}) B_p(\overrightarrow{a}) \pmod{p^r}$. Let A_p be a subset of $\mathbb{Z}_p^k$. Then, there uniquely exists a completely multiplicative $f \in \mathcal{A}$ with $f(p)$ being defined by the number of vectors in A_p. For $\overrightarrow{a} \in \mathbb{Z}_n^k$, we write $(\overrightarrow{a}, n)_A = 1$, if no row of $B_p(\overrightarrow{a})$ is in A_p for every prime divisor p of n. Then for n being r-powerful, $\phi_\tau^{(k)}(n)$ counts the number of k-vectors $\overrightarrow{a} := (a_1, \ldots, a_k) \in \mathbb{Z}_n^k$ such that $(\overrightarrow{a}, n)_A = 1$.

We take off from the work of Wang and Hsu, by defining our *generalized Euler totient* (or GET for short) as

$$\phi_{s,\alpha}^f(n) := (\zeta_s * \mu_\alpha f)(n) = \sum_{d|n} \left(\frac{n}{d}\right)^s f(d)\mu_\alpha(d), \tag{1.8}$$

where $\alpha \in \mathbb{C}$, $s \in \mathbb{R}$, $\zeta_s(n) = n^s$, $\zeta_0 = u$, and f is a completely multiplicative function. Comparing with the terminology of Wang-Hsu, we see that $\phi_{k,r}^f = \phi_\tau^{(k)}(\tau = \mu_r f)$. For brevity write

$$\phi_{s,\alpha} := \phi_{s,\alpha}^u, \qquad \phi_\alpha := \phi_{1,\alpha}. \tag{1.9}$$

There have appeared quite a number of results related to our GET, such as those in [4, 6–9], and the most complete collection to date can be found in [5, Chapter 3]. In the present paper, we consider two aspects of the GET. In the next section, its relations with other totients are investigated. Here we deal mostly with those results closely connected to our GET; for further and more complete collection up to 2004, we refer to the encyclopedic work in [5, Chapter 3]. In the last section, after proving a general inversion formula, various counting formulae related to the GET are derived.

Before listing a few properties of our GET generalizing the classical Euler's totient, we recall some auxiliary notions. The *log-derivation*, [10], is the operator $D : \mathcal{A} \to \mathcal{A}$ defined by

$$(Df)(n) = f(n)\log n \quad (n \in \mathbb{N}). \tag{1.10}$$

For $f \in \mathcal{A}$, $f(1) > 0$, the *Rearick logarithmic operator* of f (or logarithm of f; [11–13]), denoted by $\operatorname{Log} f \in \mathcal{A}$, is defined via

$$\begin{gathered}(\operatorname{Log} f)(1) = \log f(1), \\ (\operatorname{Log} f)(n) = \frac{1}{\log n}\sum_{d|n} f(d) f^{-1}\left(\frac{n}{d}\right)\log d = \frac{1}{\log n}\left(Df * f^{-1}\right)(n) \quad (n > 1),\end{gathered} \tag{1.11}$$

where D denotes the log-derivation. For $h \in \mathcal{A}$, the *Rearick exponential* $\operatorname{Exp} h$ is defined as the unique element $f \in \mathcal{A}$, $f(1) > 0$ such that $h = \operatorname{Log} f$. For $f \in \mathcal{A}$, $f(1) > 0$ and $\alpha \in \mathbb{R}$, the αth power function is defined as

$$f^\alpha = \operatorname{Exp}(\alpha \operatorname{Log} f). \tag{1.12}$$

It is not difficult to check that this agrees with the usual power function, should α be integral. From [11], we know that if f is multiplicative and $\alpha \in \mathbb{R} \setminus \{0\}$, then f^α is also multiplicative; the fact which automatically implies its converse.

Proposition 1.1. *Let $s \in \mathbb{R}$, $\alpha \in \mathbb{C}$, and f be a completely multiplicative function.*

(A) *We have the product representation*

$$\phi_{s,\alpha}^{f}(n) = n^s \prod_{p|n} \sum_{i=0}^{\nu_p(n)} (-1)^i \binom{\alpha}{i} \left(\frac{f(p)}{p^s}\right)^i \quad (n \in \mathbb{N}). \tag{1.13}$$

(B) *If $\alpha \in \mathbb{R}$, then $\zeta_s = \phi_{s,\alpha}^{f} * f^{\alpha}$.*

(C) *If $r \in \mathbb{N}$, then $\phi_{s,\alpha}^{f}(r) = \det[A_{ij}]_{r\times r}$, where*

$$A_{ij} = \begin{cases} 1, & \text{if } j \mid i,\ 1 \le j \le r-1, \\ 0, & \text{if } j \nmid i,\ 1 \le j \le r-1, \\ \sum_{d|i} \phi_{s,\alpha}^{f}(d), & \text{if } j = r. \end{cases} \tag{1.14}$$

Proof. Part (A) follows immediately from $\phi_{s,\alpha}^{f}$ being multiplicative. Part (B) follows from the fact that $\mu_\alpha f = f^{-\alpha}$ [14]. To prove Part (C), let $B(r) = [B_{ij}]_{r\times r}$, where

$$B_{ij} = \begin{cases} 1 & \text{if } j \mid i, \\ 0 & \text{if } j \nmid i. \end{cases} \tag{1.15}$$

Then

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & & & \\ B_{r1} & B_{22} & \cdots & B_{rr} \end{bmatrix} \begin{bmatrix} \phi_{s,\alpha}^{f}(1) \\ \phi_{s,\alpha}^{f}(2) \\ \vdots \\ \phi_{s,\alpha}^{f}(r) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{s,\alpha}^{f}(1) \\ \phi_{s,\alpha}^{f}(2) \\ \vdots \\ \phi_{s,\alpha}^{f}(r) \end{bmatrix} = \begin{bmatrix} \sum_{d|1} \phi_{s,\alpha}^{f}(d) \\ \sum_{d|2} \phi_{s,\alpha}^{f}(d) \\ \vdots \\ \sum_{d|r} \phi_{s,\alpha}^{f}(d) \end{bmatrix}. \tag{1.16}$$

Considering (1.16) as a system of simultaneous equations in the unknowns $\phi_{s,\alpha}^{f}(1)$, $\phi_{s,\alpha}^{f}(2), \ldots, \phi_{s,\alpha}^{f}(r)$ and appealing to Cramer's rule, the result follows. □

Part (C) generalizes a well-known identity on page 86 of [2], which is the case where $f = u$, $s = \alpha = 1$, stating that

$$\phi(r) = \det\left[a_{ij}\right]_{r\times r}, \tag{1.17}$$

where

$$a_{ij} = \begin{cases} 1, & \text{if } j \mid i,\ 1 \le j \le r-1, \\ 0, & \text{if } j \nmid i,\ 1 \le j \le r-1, \\ i, & \text{if } j = r. \end{cases} \tag{1.18}$$

2. Connections with Other Totients

Case I ($f = u$). When the parameters s and α take integer values, the GET does indeed represent a number of well-known arithmetic functions, namely,

$$\begin{gathered} \phi_{1,1} = \zeta * \mu = \phi \text{ (the classical Euler totient)}, \\ \phi_{0,-1} = u * u = \sigma_0 = \tau \text{ (the number of divisors function)}, \\ \phi_{s,-1} = \zeta_s * u = \sigma_s \text{ (the sum of the } s\text{th power of divisors function)}. \end{gathered} \tag{2.1}$$

When $s \in \mathbb{N}$ and $\alpha = 1$, this particular totient $\phi_{s,1} = \zeta_s * \mu$ is equivalent to quite a few classical totients.

(i.1) The Jordan totient $J_s(n)$ which counts the number of s-tuples $(x_1, \ldots, x_s)$ such that $1 \le x_1, \ldots, x_s \le n$ and $\gcd(x_1, \ldots, x_s, n) = 1$ ([1, page 13], [5, pages 186-187, page 275], [2, page 91]). Clearly, $J_1 = \phi$. From [5, pages 186-187], closely resembles the Jordan totient is the function

$$J_s'(n) = \#\{(x_1, \ldots, x_s) : 1 \le x_1 \le x_2 \le \cdots \le x_s \le n, \quad \gcd(x_1, \ldots, x_s, n) = 1\}. \tag{2.2}$$

While $J_s(n) = \sum_{d|n} \mu(d)(n/d)^s$, one has, on the other hand, $J_s'(n) = \sum_{d|n} \mu(n/d)\binom{d+s+1}{s}$, showing that J_s' is not of the form of our GET. Even more general is the Shonhiwa's totient, J_s^m [5, pages 187, 276], defined as the number of s-tuple $(x_1, \ldots, x_s)$ such that $1 \le x_1, \ldots, x_s \le n$ and $\gcd(x_1, \ldots, x_s, m) = 1$, whose representation is $J_s^m(n) = \sum_{d|m} \mu(d)[n/d]^s$.

(i.2) The von Sterneck function, [1, pages 14-15] and [5, pages 275-276],

$$H_s(n) := \sum_{\operatorname{lcm}(e_1,\ldots,e_s)=n} \phi(e_1) \cdots \phi(e_s), \tag{2.3}$$

where the sum is over all ordered s-tuples $(e_1, \ldots, e_s) \in \mathbb{Z}^s$ such that $1 \le e_i \le n$ ($i = 1, \ldots, s$) and $\operatorname{lcm}(e_1, \ldots, e_s) = n$.

(i.3) Eckford Cohen's totient $E_s(n)$ which counts the number of elements of a s-reduced residue system $(\operatorname{mod} n)$. For integers a, b not both 0, let

$$(a, b)_s \text{ denote the largest } s\text{th-power common divisor of } a \text{ and } b. \tag{2.4}$$

If $(a,b)_s = 1$, we say that a and b are relatively s-prime. We refer to the subset of a complete residue system M (mod n^s) consisting of all elements of M that are relatively s-prime to n^s as a s-reduced residue system (mod n), [2, pages 98-99] and [5, pages 275-276].

(i.4) $\phi_{s,1}(n) = \Phi_s(n^s)$, where $\Phi(n)$ is the Klee's totient, [15], which counts the number of integers $h \in \{1,2,\ldots,n\}$ for which $\gcd(h,n)$ is sth-power-free, that is, contains no sth-power divisors other than 1. The Klee's totient has a product representation of the form, [5, page 278], $\Phi_s(n) = n\prod_{p^s|n}(1-1/p^s)$.

(i.5) Haukkanen's totient, [5, page 276], $\phi_{km,1}(n) = \sum_{d|n} d^{km}\mu(d)$, which counts the number of m-tuples $(x_1,\ldots,x_m)$ mod n^k such that $(\gcd\ (x_1,x_2,\ldots,x_m),n^k)_k = 1$.

On the other hand, based on the combinatorial interpretation (P2) of $\phi_{s,\alpha}$ above, our GET includes several special totients [6] such as taking $\alpha = 1$ and

(i) $B_p(\overrightarrow{a}) = \mathbb{Z}_p$ when $p \le k$, or $B_p(\overrightarrow{a}) = \mathbb{Z}_k$ when $p > k$, we obtain Schemmel's totient, $S_k(n)$, which counts the number of sets of k consecutive integers each less than n and relatively prime to n. The function $S_k(n)$ has a product representation of the form $S_k(n) = n\prod_{p|n}(1-k/p)$, [5, page 276]. The case $k = 2$ was also called Schemmel totient function and was shown by Lehmer to have application in the enumeration of certain magic squares, [5, page 184]. There are many other totients closely connected to Schemmel's totient. As examples, we describe two more, taken from [5, Chapter 3], namely, Lucas's and Nageswara Rao's totients. For fixed integers $e_1,\ldots,e_k$, Lucas's totient counts the number of integers $h \in \{0,1,\ldots,n-1\}$ such that $h-e_1,\ldots,h-e_n$ are relatively prime to n and its product representation is $n\prod_{p|n}(1-\lambda/p)$, where λ is the number of distinct residues of $e_1,\ldots,e_k$ mod p. Nageswara Rao's totient counts the number of sets of k consecutive integers each less than n^s which are s-prime to n^s;

(ii) let $F = \{f_1(x),\ldots,f_k(x)\}$ be a set of polynomials with integer coefficients and $B_p(\overrightarrow{a}) = \{(a_1,\ldots,a_k);\ f_i(a_i) \equiv 0\,(\operatorname{mod} p),\ i = 1,\ldots,k\}$, we obtain Steven's totient which denotes the number of k-vectors $(a_1,\ldots,a_k) \pmod n$ such that $\gcd\,(f_1(a_1),\ldots,f_k(a_k),n) = 1)$. Following [5, pages 279-280], the product representation of Stevens's totient takes the form $n^k\prod_{j=1}^{r}(1-N_{1j}\cdots N_{kj}/p_j^k)$, where $n = \prod_{j=1}^{r} p_j^{a_j}$, N_{ij} is the number of incongruent solutions of $f_i(x) \equiv 0 \pmod{p_j}$. The Stevens's totient is multiplicative, and contains, as special cases

(1) the Jordan totient $J_k(n)$ (by taking $f_1(x) = \cdots = f_k(x) = x$);

(2) the Schemmel totient $S_t(n)$ (by taking $k = 1,\ f_1(x) = x(x+1)\cdots(x+t-1)$);

(3) Cashwell-Everett's totient (by taking $f_1(x) = \cdots = f_\ell(x) = nx,\ f_{\ell+1}(x) = \cdots = f_k(x) = x$), which counts the number of k-tuples $(a_1,\ldots,a_k)$ with $\ell \le a_k \le k$ such that $\gcd(a_{\ell+1},\ldots,a_k,n) = 1$. Its product representation is $n^k\prod_{p|n}(1-p^j/p^k)$.

In passing, let us mention that, our GET is closely connected to the generalized Ramanujan sum through

$$\phi_{1,\alpha}(r) = c^{(\alpha)}(n,r) \quad \text{whenever } r \mid n \ \left([9,\ \text{page } 4],\ [5,\ \text{pages } 277\text{-}278]\right). \tag{2.5}$$

Case II ($f \neq u$). The GET also includes a number of known totients in this case.

(ii.1) The Garcia-Ligh totient [16, 17], defined for fixed $s, d \in \mathbb{N}$, by

$$\phi(s, d, n) := \phi_{1,1}^{I_d}(n), \tag{2.6}$$

where $I_d(n) := I(\gcd(d, n))$ is easily shown to be completely multiplicative. This totient $\phi_{1,1}^{I_d}(n)$ counts the number of elements in the set $\{s, s+d, \ldots, s+(n-1)d\}$ that are relatively prime to n with $\phi(1, 1, n) \equiv \phi(n)$.

(ii.2) The Garcia-Ligh totient is a special case of the following totient taken from Exercise 1.21 on pages 34-35 of [1]. Let $g(x) \in \mathbb{Z}[x]$. The number of integers $x \in \{1, 2, \ldots, n\}$ and $\gcd(f(x), n) = 1$ is, using our terminology above,

$$\phi_{1,1}^{v_g}(n) = (\zeta_1 * v_g\mu)(n) = \sum_{d \mid n} d v_g\left(\frac{n}{d}\right)\mu\left(\frac{n}{d}\right), \tag{2.7}$$

where v_g is the completely multiplicative function defined over prime p by $v_g(p) = g_p$, the number of solutions of the congruence $g(x) \equiv 0 \pmod{p}$.

(ii.3) Martin G. Beumer's function (Section IV.2 on pages 72–74 of [2]) defined for $k \in \mathbb{N}$, by

$$\phi_{0,1-k}^{u} = \zeta_0 * (\mu_{1-k}u) = u * \left(u^{k-1}\right) = u^k, \tag{2.8}$$

where we have used a result of Haukkanen [14], that if f is a completely multiplicative function and $\alpha \in \mathbb{R}$, then $f^{\alpha} = \mu - \alpha f$.

(ii.4) The Dedekind ψ-function ([2, Problem 10, page 80], [5, page 284]) defined by $\psi(n) = n\prod_{p \mid n}(1 + 1/p)$ is clearly equivalent to

$$\phi_{1,1}^{f} = \zeta_1 * \mu_1 f, \tag{2.9}$$

where f is the completely multiplicative function defined for prime p by $f(p) = -1$.

(ii.5) H. L. Adler's totient ([2, Section V.6, page 102], [5, page 279]) is defined, for fixed $N \in \mathbb{N}$, as

$$\phi_{1,1}^{\epsilon_N}(r) = (\zeta_1 * \mu_1 \epsilon_N)(r) = r\prod_{p \mid r}\left(1 - \frac{\epsilon_N(p)}{p}\right), \tag{2.10}$$

where ϵ_N is the completely multiplicative function defined, for prime p, by

$$\epsilon_N(p) = \begin{cases} 1, & \text{if } p \mid N, \\ 2, & \text{if } p \nmid N. \end{cases} \tag{2.11}$$

The totient value $\phi_{1,1}^{\epsilon_N}(r)$ is the number of ordered pairs $(x, y) \in \mathbb{N}^2$ for which $x+y = N+r$, $\gcd(x, r) = \gcd(y, r) = 1$ and $1 \le x \le r$. Note that when $N = 0$, this is merely Euler's totient.

(ii.6) D. L. Goldsmith's totient (the main theorem on page 183 of [18]) is defined, for $m \in \mathbb{N}$, by

$$\psi(m) = \#S(m) = \#\{x; 0 \le x < m, \gcd(Q(x), m) = 1\}, \tag{2.12}$$

where $Q(x) \in \mathbb{Z}[x]$. From the meaning of $\psi(m)$, it is clear that $\psi(m)$ is a multiplicative function (Problem 5, page 31 in [19]). Thus,

$$\phi_{1,1}^{f}(m) = (\zeta_1 * \mu_1 f)(m) = \psi(m) = \prod_{p|m}\left(p^{\nu_p(m)} - \beta_p p^{\nu_p(m)-1}\right), \tag{2.13}$$

where β_p is the number of integers $x \in \{0, 1, \ldots, p-1\}$ such that $Q(x)$ is divisible by the prime p and f is the completely multiplicative function defined for prime p by $f(p) = \beta_p$. It is possible to enlarge the values of α in the Goldsmith's totient such as taking $\alpha = 2$ to get

$$\phi_{1,2}^{g}(m) = (\zeta_1 * \mu_2 g)(m) = \prod_{p|m}\left(p^{\nu_p(m)} - \beta_p p^{\nu_p(m)-1}\right), \tag{2.14}$$

where g is the completely multiplicative function defined for prime p by $g(p) = p + \sqrt{p^2 - p\beta_p}$.

Let $f, g \in \mathcal{A}$. For $k \in \mathbb{N}$, define the *k-convolution* of f and g by

$$(f *_k g)(n) := \sum_{d^k a = n} f(d) g(a). \tag{2.15}$$

It is easily checked that the k-convolution is neither commutative nor associative. Yet it preserves multiplicativity, that is, if f and g are multiplicative functions, then the $f *_k g$ is also multiplicative (Problem 1.26, page 37 of [1]). The kth *convolute* (page 53 of [1]) of $f \in \mathcal{A}$ is defined by

$$f^{[k]}(n) := \begin{cases} f(n^{1/k}), & \text{if } n \text{ is } a\ k\text{th power}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.16}$$

The k-convolution is connected to the usual (Dirichlet) convolution via

$$f *_k g = f^{[k]} * g. \tag{2.17}$$

We list here some examples of arithmetic functions which enjoy k-convolution relations.

(1) Klee's totient, $\Phi_k(n)$, which counts the number of integers $h \in \{1,2,\ldots,n\}$ for which $\gcd(h,n)$ is kth-power-free, satisfies (Problem 1.29 on pages 38-39 of [1]),

$$\Phi_k(n) = (\mu_1 *_k \zeta_1)(n) = \sum_{d^k|n} \mu_1(d)\frac{n}{d^k}. \tag{2.18}$$

(2) The number of divisors function $\phi_{0,-1} = \sigma_0 := \tau$ is related to the arithmetic function $\theta_k(n)$ which counts the number of k-free divisors of n by (Problem 1.27, page 37 of [1])

$$\tau(n) = \sum_{d^k|n} \theta_k\left(\frac{n}{d^k}\right) = (u *_k \theta)(n). \tag{2.19}$$

(3) The Liouville's function is defined by

$$\lambda(1) = 1; \quad \lambda(n) = (-1)^{r_1+\cdots+r_t} \quad \text{if } n = p_1^{r_1}\cdots p_t^{r_t}. \tag{2.20}$$

It is known (Problem 1.47 on page 45 of [1]) that λ is completely multiplicative and satisfies

$$\lambda(n) = (u *_2 \mu_1)(n) = \sum_{d^2|n} \mu_1\left(\frac{n}{d^2}\right). \tag{2.21}$$

(4) For $\kappa \in \mathbb{R}$, $t \in \mathbb{N}$, the Gegenbauer's function (page 55 of [1]) is defined as

$$\rho_{\kappa,t}(n) = \sum_{\substack{d|n \\ n|d \text{ is a } t\text{th power}}} d^\kappa = (u *_t \zeta_\kappa)(n) = \left(u^{[t]} * \zeta_\kappa\right)(n). \tag{2.22}$$

The notion of convolute enables us to give swift proofs of a number of identities such as the following ones which are generalizations of Problem 1.59 on page 48, Problem 1.78 on page 51, Problem 1.89 on page 55, and Problem 1.90 on page 56 of [1].

Proposition 2.1. *Let $s,h,\kappa \in \mathbb{R}$, and $k,t \in \mathbb{N}$. Then,*

(i) $\sum_{d|n} \lambda(d)\phi_{s,-1}(n/d) = \sum_{d^2|n}(n/d^2)^s$,

(ii) $\sum_{d^2|n} \phi_{s,1}(n/d^2) = \sum_{d|n} d^s\lambda(n/d)$,

(iii) $\sum_{d|n} d^h\rho_{\kappa,t}(n/d) = \sum_{d|n} d^\kappa\rho_{h,t}(n)$,

(iv) $\sum_{d|n} d^\kappa\phi_{h,1}(d)\rho_{\kappa,t}(n/d) = \rho_{h+\kappa,t}(n)$,

(v) $\sum_{d|n} \rho_{h,t}(d)\rho_{\kappa,t}(n/d) = n^\kappa \sum_{d^t|n} \tau(d)\phi_{h-\kappa,-1}(n/d^t)/d^{\kappa t}$.

Proof. We first note the identity $\lambda * u = u^{[2]}$ which follows from the facts that $\lambda * u$ is multiplicative and

$$(\lambda * u)\left(p^k\right) = \sum_{i=0}^{k} \lambda\left(p^i\right) = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.23}$$

Assertion (i) follows quickly from $u *_2 \zeta_s = u^{[2]} * \zeta_s = (\lambda * u) * \zeta_s = \lambda * \phi_{s,-1}$.
Assertion (ii) follows from $u *_2 \phi_{s,1} = u^{[2]} * \phi_{s,1} = (\lambda * u) * (\zeta_s * \mu) = \lambda * \zeta_s$.
Assertion (iii) follows from $\zeta_h * \rho_{\kappa,t} = \zeta_h * u^{[t]} * \zeta_\kappa = \rho_{h,t} * \zeta_\kappa$.
Assertion (iv) follows from

$$\begin{aligned} \zeta_\kappa \phi_{h,1} * \rho_{\kappa,t} &= \zeta_\kappa (\zeta_h * \mu) * \left(u^{[t]} * \zeta_\kappa\right) = \left(\zeta_{\kappa+h} * u^{[t]}\right) * (\zeta_\kappa \mu * \zeta_\kappa) \\ &= \rho_{\kappa+h,t} * \zeta_\kappa (\mu * u) = \rho_{\kappa+h,t} * I. \end{aligned} \tag{2.24}$$

To prove (v), we need the identity $u^{[t]} * u^{[t]} = \tau^{[t]}$. Now Assertion (v) follows from

$$\begin{aligned} \rho_{h,t} * \rho_{\kappa,t} &= \left(u^{[t]} * \zeta_h\right) * \left(u^{[t]} * \zeta_\kappa\right) = \left(u^{[t]} * u^{[t]}\right) * (\zeta_{h-\kappa} * u)\zeta_\kappa \\ &= \tau^{[t]} * (\phi_{h-\kappa,-1}\zeta_\kappa) = \tau *_t (\phi_{h-\kappa,-1}\zeta_\kappa). \end{aligned} \tag{2.25}$$

□

3. Inversion and Counting Formulae

In [20], see also Problem 1.25, pages 36-37 of [1], Suryanarayana proved the following inversion formula:

$$g(n) = \sum_{d^k a = n} f(a) \Longleftrightarrow f(n) = \sum_{d^k a = n} \mu(d) g(a). \tag{3.1}$$

Our objective now is to extend this inversion formula using our GET.

Theorem 3.1 (modified generalized Möbius inversion formula). *Let $k, n \in \mathbb{N}$ and $f, g \in \mathcal{A}$. For $\alpha \in \mathbb{C}$, one has*

$$g(n) = \sum_{d^k a = n} f(a) \Longleftrightarrow \left(f * \mu_{\alpha-1}^{[k]}\right)(n) = \sum_{d^k a = n} \mu_\alpha(d) g(a). \tag{3.2}$$

Proof. Recall that the summation over $d^k a = n$, with fixed $k \in \mathbb{N}$, means that n is written as $n = d^k a$ when a runs through all divisors of n for which n can be so written with $d \in \mathbb{N}$. The result follows at once from

$$\begin{aligned}\sum_{d^k a=n} \mu_\alpha(d) g(a) &= \sum_{d^k a=n} \mu_\alpha(d) \sum_{s^k t=a} f(t) = \sum_{d^k s^k t=n} \mu_\alpha(d) f(t) = \sum_{t|n} f(t) \sum_{d^k|n/t} \mu_\alpha(d) \\ &= \sum_{t|n} f(t) \sum_{d|\sqrt[k]{n/t}} \mu_\alpha(d) = \sum_{t|n} f(t) \mu_{\alpha-1}\left(\sqrt[k]{\frac{n}{t}}\right) = \sum_{t|n} f(t) \mu_{\alpha-1}^{[k]}\left(\frac{n}{t}\right) \\ &= \left(f * \mu_{\alpha-1}^{[k]}\right)(n).\end{aligned} \tag{3.3}$$

□

Theorem 1 of [20] is a special case of Theorem 3.1 when $\alpha = 1$. As is well known, the Möbius inversion formula has extensive applications which is also the case of our new inversion formula. Applying Suryanarayana's inversion formula to three examples in the last section, we obtain

$$\sum_{d^k|n} \Phi_k\left(\frac{n}{d^k}\right) = n, \quad \theta_k(n) = \sum_{d^k|n} \mu(d) \tau\left(\frac{n}{d^k}\right), \quad \mu(n) = \sum_{d^2|n} \mu(d) \lambda\left(\frac{n}{d^2}\right). \tag{3.4}$$

As an application to counting formulae, taking $\alpha = 1$, $g = \zeta_1$ in Theorem 3.1, we get the following special case of Theorem 2 in [21].

Corollary 3.2. *Let $r, n \in \mathbb{N}$ and Klee's function $T_r(n)$ be the number of integers k such that $1 \le k \le n$ and $\gcd(k, n)$ is not divisible by the rth power of any prime. Then,*

$$n = \sum_{d^r|n} T_r\left(\frac{n}{d^r}\right) \Longleftrightarrow T_r(n) = \sum_{d^r|n} \mu(d) \frac{n}{d^r}. \tag{3.5}$$

Following [5, page 136], a generalization of Klee's function T_r introduced by D. Suryanarayana is the function $T_r(x, n) = \sum_{d|n} \mu_r(d)[x/d]$; note that $T_r(n, n) = T_r(n)$.

To illustrate another application, consider a function of the form

$$\sum_{st=k} \mu(s) m^t, \tag{3.6}$$

which has been a subject of many investigations such as those in [22–25]. This function leads to a formula for the number of primitive elements over a finite field. Let $a, N \in \mathbb{N}$ and let all distinct prime factors of N be $p_1 \cdots p_k$. Define

$$F(a, N) = \sum_{d|N} \mu(d) a^{N/d} = a^N - \sum_i a^{N/p_i} + \sum_{i<j} a^{N/p_i p_j} + \cdots + (-1)^k a^{N/p_1 \cdots p_k}. \tag{3.7}$$

It is known ([24, pages 84–86], [5, pages 191–193]) that

$$F(a, N) \equiv 0 \pmod{N}, \tag{3.8}$$

a result which generalizes Fermat's little theorem when N is prime. This congruence has been much extended in [22, 25]. Dickson [24, pages 84–86] shows a connection with Euler's totient via the identity

$$F(a,N) = \sum_{d,a,N} \phi(d), \tag{3.9}$$

where the summation $\sum_{d,a,N}$ runs through proper divisors d of $a^N - 1$; by a *proper divisor* of $a^N - 1$ we mean a divisor of $a^N - 1$ that does not divide $a^m - 1$ if $0 < m < N$.

We aim now to extend these results even further to our GET. For $a, k, N \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, define

$$F_{k,\alpha}(a,N) := \sum_{d^k m = N} \mu(d) h^{(k)}_{\alpha,a}(m), \tag{3.10}$$

where

$$h^{(k)}_{\alpha,a}(m) = \sum_{d^k t = a^m - 1} \phi_\alpha(t). \tag{3.11}$$

Theorem 3.3. *Let $\alpha \in \mathbb{C}$ and $a, k, N \in \mathbb{N}$ with $a > 1$. Then,*

$$F_{k,\alpha}(a,N) = \sum_{d^k,a,N} \phi_\alpha(t). \tag{3.12}$$

Proof. Let

$$f_{k,\alpha}(m) = \sum_{d^k,a,m} \phi_\alpha(t), \quad H(N) = \sum_{n^k m = N} f_{k,\alpha}(m). \tag{3.13}$$

By Theorem 3.1, we have

$$f_{k,\alpha}(N) = \sum_{d^k,a,N} \phi_\alpha(t) = \sum_{n^k m = N} \mu(n) H(m). \tag{3.14}$$

On the other hand, from the definitions of H and f, we get

$$\begin{aligned}
H(N) &= \sum_{n^k m=N}\left(\sum_{d^k,a,m}\phi_\alpha(t)\right) \\
&= \sum_{\substack{d^k t=a^{j_1}-1\\ d^k\nmid a^{m_1}-1\\ m_1<j_1}}\phi_\alpha(t)+\sum_{\substack{d^k t=a^{j_2}-1\\ d^k\nmid a^{m_2}-1\\ m_2<j_2}}\phi_\alpha(t)+\cdots+\sum_{\substack{d^k t=a^{j_s}-1\\ d^k\nmid a^{m_s}-1\\ m_s<j_s}}\phi_\alpha(t)+\sum_{\substack{d^k t=a^{N}-1\\ d^k\nmid a^{m}-1\\ m<N}}\phi_\alpha(t) \\
&\qquad \text{when } N=n^k j_i,\ \text{where } 0<j_1<\cdots<j_s<N \text{ are all divisors of } N \text{ such that } N \\
&\qquad \text{can be so written with } n\in\mathbb{N} \text{ and } i\in\{1,2,\ldots,s\} \\
&= \sum_{d^k t=a^N-1}\phi_\alpha(t)=h^{(k)}_{\alpha,a}(N).
\end{aligned} \tag{3.15}$$

Consequently,

$$F_{k,\alpha}(a,N)=f_{k,\alpha}(N)=\sum_{d^k,a,N}\phi_\alpha(t). \tag{3.16}$$

□

Specializing $k=1,\alpha=1$ in Theorem 3.3, we recover the result of Dickson mentioned above, namely,

$$\sum_{d,a,N}\phi(d)=F_{1,1}(a,N)=\sum_{d|N}\mu(d)h^{(1)}_{1,a}\left(\frac{N}{d}\right)=\sum_{d|N}\mu(d)(\zeta*I)\left(a^{N/d}-1\right)=\sum_{d|N}\mu(d)a^{N/d}. \tag{3.17}$$

If we take $a=p^n$, a prime power, in (3.7), then it is well known [26, page 93] that the number of monic irreducible polynomials of order m over $\mathrm{GF}(p^n)$ is

$$\frac{1}{m}\sum_{d|m}\mu(d)p^{nm/d}=\frac{1}{m}\left(p^{mn}-\sum_{i}p^{nm/p_i}+\sum_{i<j}p^{nm/p_ip_j}+\cdots+(-1)^k p^{nm/p_1\cdots p_k}\right) \tag{3.18}$$

and $F(p^n,m)=\sum_{d|m}\mu(d)p^{nm/d}$ is the number of primitive elements of $\mathrm{GF}(p^{nm})/\mathrm{GP}(p^n)$. Taking $k=1,\ \alpha=0$ in Theorem 3.3 yields another beautiful formula

$$F_{1,0}(a,N)=\sum_{d|N}\mu(d)(\zeta*\mu_{-1})\left(a^{N/d}-1\right)=\sum_{d|N}\mu(d)\sigma\left(a^{N/d}-1\right). \tag{3.19}$$

Combining the results of Theorems 3.1 and 3.3 yields another relation between μ_α and ϕ_α.

Corollary 3.4. *Let $\alpha \in \mathbb{C}$ and $a, k, N \in \mathbb{N}$ with $a > 1$. Then,*

$$h_{\alpha,a}^{(k)}(N) = \sum_{d^k m = N} f_{k,\alpha}(m) \Longleftrightarrow \left(f_{k,\alpha} * \mu_{\alpha-1}^{[k]}\right)(N) = \sum_{d^k m = N} \mu_\alpha(d) h_{\alpha,a}^{(k)}(m), \tag{3.20}$$

where

$$h_{\alpha,a}^{(k)}(m) = \sum_{d^k t = a^m - 1} \phi_\alpha(t), \qquad f_{k,\alpha}(m) = \sum_{d^k, a, m} \phi_\alpha(t). \tag{3.21}$$

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