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Research Article

Benjamin-Ono Equation on a Half-Line

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We consider the initial-boundary value problem for Benjamin-Ono equation on a half-line. We study traditionally important problems of the theory of nonlinear partial differential equations, such as global in time existence of solutions to the initial-boundary value problem and the asymptotic behavior of solutions for large time.

1. Introduction

In this paper we study the large time asymptotic behavior of solutions to the initial-boundary value problem for the Benjamin-Ono equation on a half-line:

$$\begin{gathered} u_t + uu_x + \mathscr{H}u_{xx} = 0, \quad x > 0,\ t > 0, \\ u(x,0) = u_0(x), \quad x > 0, \\ u(0,t) = 0, \quad t > 0, \end{gathered} \tag{1.1}$$

where $\mathscr{H}u = \mathrm{PV}\int_0^{+\infty} u((y,t)/(y-x))dy$ is the Hilbert transformation, and PV means the principal value of the singular integral. We note that in the case of the whole line we have the relations $\mathscr{H}\partial_x^2 = \partial_x(-\partial_x^2)^{1/2}$ since the operator $\mathscr{H}$ can be written as follows: $\mathscr{H} = -\mathcal{F}^{-1}(i\xi/|\xi|)\mathcal{F} = (-\partial_x^2)^{-1/2}\partial_x$, where $(\mathcal{F}\varphi)(\xi) = (1/\sqrt{2\pi})\int \varphi(x)e^{-ix\xi}dx$ is the usual Fourier transform, and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. This equation is of great interest in many areas of Physics (see [1, 2]). The Cauchy problem (1.1) was studied by many authors.

The existence of solutions in the usual Sobolev spaces $\mathbf{H}^{s,0}$ was proved in [3–9] and the smoothing properties of solutions were studied in [10–14]. In paper [15] it was proved that for small initial data in $\mathbf{H}^{2,0} \cap \mathbf{H}^{1,1}$ solutions decay as $t \to \infty$ in $\mathbf{L}^{\infty}$ norm at the same rate $1/\sqrt{t}$ as for the case of the linear Benjamin-Ono equation, where

$$\mathbf{H}^{m,s} = \left\{ \phi \in \mathbf{L}^2 : \|\phi\|_{m,s} = \left\| \left(1 + x^2\right)^{s/2} \left(1 - \partial_x^2\right)^{m/2} \phi \right\|_{\mathbf{L}^2} < \infty \right\}. \tag{1.2}$$

The initial-boundary value problem (1.1) plays an important role in the contemporary mathematical physics. For the general theory of nonlinear equations on a half-line we refer to the book [16], where it was developed systematically a general theory of the initial-boundary value problems for nonlinear evolution equations with pseudodifferential operators on a half-line, where pseudodifferential operator $\mathbb{K}$ on a half-line was introduced by virtue of the inverse Laplace transformation of the product of the symbol $K(p) = O(p^{\beta})$ which is analytic in the right complex half-plane, and the Laplace transform of the derivative $\partial_x^{[\beta]} u$. Thus, for example, in the case of $K(p) = p^{3/2}$ we get the following definition of the fractional derivative $\partial_x^{3/2}$:

$$\partial_x^{3/2} \phi = \mathcal{L}^{-1} \left\{ p^{3/2} \left(\mathcal{L}\phi - \frac{\phi(0)}{p} \right) \right\}. \tag{1.3}$$

Here and below p^{β} is the main branch of the complex analytic function in the complex half-plane $\operatorname{Re} p \geq 0$, so that $1^{\beta} = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). Note that due to the analyticity of p^{β} for all $\operatorname{Re} p > 0$ the inverse Laplace transform gives us the function which is equal to 0 for all $x < 0$. In spite of the importance and actuality there are few results about the initial-boundary value problem for pseudodifferential equations with nonanalytic symbols. For example, in paper [17] there was considered the case of rational symbol $K(p)$ which have some poles in the right complex half-plane. There was proposed a new method for constructing the Green operator based on the introduction of some necessary condition at the singularity points of the symbol $K(p)$. In the paper [18] one of the authors considered the initial-boundary value problem for a pseudodifferential equation with symbol $K(p) = |p|^{1/2}$ and nonlinearity $|u|^{\sigma} u$.

As far as we know the case of nonanalytic conservative symbols $K(p)$ was not studied previously. In the present paper we fill this gap, considering as example the Benjamin-Ono equation (1.1) with a symbol $K(p) = -p|p|$. There are many natural open questions which we need to study. First we consider the following question: how many boundary data we should pose on problem (1.1) for its correct solvability? Also we study traditionally important problems of a theory of nonlinear partial differential equations, such as global in time existence of solutions to the initial-boundary value problem and the asymptotic behavior of solutions for large time. We adopt here the approach of book [16] based on the estimates of the Green function. The main difficulty for nonlocal equation (1.1) on a half-line is that the symbol $K(p) = -p|p|$ is non analytic in the complex plane. Therefore we cannot apply the Laplace theory directly. To construct Green operator we proposed a new method based on the integral representation for sectionally analytic function and theory of singular integrodifferential equations with Hilbert kernel and the discontinues coefficients (see [18, 19]).

To state precisely the results of the present paper we give some notations. We denote $\langle t\rangle = \sqrt{1+t^2}$, $\{t\} = t/\langle t\rangle$. Direct Laplace transformation $\mathcal{L}_{x\to\xi}$ is

$$\widehat{u}(\xi) \equiv \mathcal{L}_{x\to\xi}u = \int_0^{+\infty} e^{-\xi x}u(x)dx, \tag{1.4}$$

and the inverse Laplace transformation $\mathcal{L}^{-1}_{\xi\to x}$ is defined by

$$u(x) \equiv \mathcal{L}^{-1}_{\xi\to x}\widehat{u} = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} e^{\xi x}\widehat{u}(\xi)d\xi. \tag{1.5}$$

Weighted Lebesgue space is $\mathbf{L}^{q,a}(\mathbf{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{q,a}} < \infty\}$, where

$$\|\varphi\|_{\mathbf{L}^{q,a}} = \left(\int_0^{+\infty} x^{aq}|\varphi(x)|^q dx\right)^{1/q} \tag{1.6}$$

for $a > 0$, $1 \le q < \infty$ and

$$\|\varphi\|_{\mathbf{L}^\infty} = \operatorname*{ess\,sup}_{x\in\mathbf{R}^+}|\varphi(x)|. \tag{1.7}$$

Sobolev space is

$$\mathbf{H}^1(\mathbf{R}^+) = \left\{\varphi \in \mathcal{S}'; \|\langle\partial_x\rangle\varphi\|_{\mathbf{L}^2} < \infty\right\}. \tag{1.8}$$

We define a linear functional f:

$$f(\phi) = \int_0^{+\infty} y\phi(y)dy. \tag{1.9}$$

Now we state the main results.

Theorem 1.1. *Suppose that the initial data $u_0 \in \mathbf{Z} \equiv \mathbf{H}^1(\mathbf{R}^+) \cap \mathbf{L}^{1,a+1}(\mathbf{R}^+)$ with $a \in (0,1)$ are such that the norm*

$$\|u_0\|_{\mathbf{Z}} \le \varepsilon \tag{1.10}$$

is sufficiently small. Then there exists a unique global solution

$$u \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}^+)\right) \tag{1.11}$$

to the initial-boundary value problem (1.1). *Moreover the following asymptotic is valid in $\mathbf{L}^\infty(\mathbf{R}^+)$:*

$$u = \frac{1}{t}A\Lambda\left(xt^{-1/2}\right) + \min\left(1, \frac{x}{\sqrt{t}}\right)O\left(t^{-1-(a/2)}\right) \tag{1.12}$$

for $t \to \infty$, *where* $\Lambda(xt^{-1/2}) \in \mathbf{L}^{\infty}(\mathbf{R}^{+}), \Lambda(0) = 0$ *is defined below by the formula* (2.191), *and the constant*

$$A = f(u_0) - \int_0^{+\infty} f(\mathcal{N}(u))d\tau, \qquad \mathcal{N}(u) = u_x u. \tag{1.13}$$

Remark 1.2. Note that the time decay rate of the solution is faster comparing with the case of the corresponding Cauchy problem. So the nonlinearity uu_x in (1.1) is not the super critical case for our problem.

Remark 1.3. In the case of the negative half line $x < 0$ we expect that the solutions have an oscillation character, and the time decay rate of the solution is the same as the case of the corresponding Cauchy problem. so the nonlinearity uu_x in (1.1) will be the super critical case.

2. Preliminaries

In subsequent consideration we will have frequently to use certain theorems of the theory of functions of complex variable, the statements of which we now quote. The proofs can be found in [19].

Theorem 2.1. *Let* $\phi(q)$ *be a complex function, which obeys the Hölder condition for all finite* q *and tends to a definite limit* ϕ_∞ *as* $|q| \to \infty$, *such that for large* q *the following inequality holds:*

$$\left|\phi(q) - \phi_\infty\right| \le C\left|q\right|^{-\mu}, \quad \mu > 0. \tag{2.1}$$

Then Cauchy type integral

$$F(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - z} dq \tag{2.2}$$

constitutes a function analytic in the left and right semiplanes. Here and below these functions will be denoted $F^{+}(z)$ *and* $F^{-}(z)$, *respectively. These functions have the limiting values* $F^{+}(p)$ *and* $F^{-}(p)$ *at all points of imaginary axis* $\operatorname{Re} p = 0$, *on approaching the contour from the left and from the right, respectively. These limiting values are expressed by Sokhotzki-Plemelj formula:*

$$\begin{aligned} F^{+}(p) &= \lim_{z \to p,\, \operatorname{Re} z<0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - z} dq = \frac{1}{2\pi i} PV\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - p} dq + \frac{1}{2}\phi(p), \\ F^{-}(p) &= \lim_{z \to p,\, \operatorname{Re} z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - z} dq = \frac{1}{2\pi i} PV\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q - p} dq - \frac{1}{2}\phi(p). \end{aligned} \tag{2.3}$$

Subtracting and adding the formula (2.3) we obtain the following two equivalent formulas:

$$F^{+}(p) - F^{-}(p) = \phi(p),$$
$$F^{+}(p) + F^{-}(p) = \frac{1}{\pi i} PV \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p} dq, \tag{2.4}$$

which will be frequently employed hereafter.

Theorem 2.2. *An arbitrary function $\phi(p)$ given on the contour* Re $p = 0$*, satisfying the Hölder condition, can be uniquely represented in the form*

$$\phi(p) = U^{+}(p) - U^{-}(p), \tag{2.5}$$

where $U^{\pm}(p)$ are the boundary values of the analytic functions $U^{\pm}(z)$ and the condition $U^{\pm}_{\infty} = 0$ holds. These functions are determined by formula

$$U(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z} dq. \tag{2.6}$$

Theorem 2.3. *An arbitrary function $\varphi(p)$ given on the contour* Re $p = 0$*, satisfying the Hölder condition, and having zero index,*

$$\operatorname{ind} \varphi(t) := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d \ln \varphi(p) = 0, \tag{2.7}$$

is uniquely representable as the ratio of the functions $X^{+}(p)$ and $X^{-}(p)$, constituting the boundary values of functions, $X^{+}(z)$ and $X^{-}(z)$, analytic in the left and right complex semiplane and having in these domains no zero. These functions are determined to within an arbitrary constant factor and given by formula

$$X^{\pm}(z) = e^{\Gamma^{\pm}(z)}, \qquad \Gamma(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \varphi(q) dq. \tag{2.8}$$

We consider the following linear initial-boundary value problem on half-line

$$u_t - PV \int_{0}^{+\infty} \frac{u_{yy}(y,t)}{x-y} dy = 0, \quad t > 0,\ x > 0,$$
$$u(x,0) = u_0(x), \quad x > 0,$$
$$u(0,t) = 0, \quad t > 0. \tag{2.9}$$

Setting

$$K(q) = -|q|q, \qquad K_1(q) = -q^2, \qquad k(\xi) = |\xi|^{1/2} e^{(1/2) i \arg \xi}, \tag{2.10}$$

where $\operatorname{Re} k(\xi) > 0$ for $\operatorname{Re} \xi > 0$, we define

$$\mathcal{G}(t)\phi = \int_0^{+\infty} G(x,y,t)\phi(y)dy, \tag{2.11}$$

where the function $G(x,y,t)$ is given by formula

$$\begin{aligned} G(x,y,t) = {} & \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} dp e^{px}\frac{1}{K_1(p)+\xi}\left(e^{-py}+\Psi(\xi,y)\right) \\ & - \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} dp e^{px}\frac{1}{K_1(p)+\xi}Y^-(p,\xi) \\ & \times \lim_{z\to p, \operatorname{Re} z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K_1(q)-K(q)}{K_1(q)+\xi}\left(e^{-qy}+\Psi(\xi,y)\right)dq \end{aligned} \tag{2.12}$$

for $\varepsilon > 0, x > 0, y > 0, t > 0$. Here and below

$$Y^{\pm} = e^{\Gamma^{\pm}} w^{\pm}. \tag{2.13}$$

$\Gamma^+(p,\xi)$ and $\Gamma^-(p,\xi)$ are a left and right limiting values of sectionally analytic function $\Gamma(z,\xi)$ given by

$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\left\{\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right)\frac{w^-(q)}{w^+(q)}\right\}dq, \tag{2.14}$$

where

$$\begin{aligned} w^-(z) &= \left(\frac{z}{z+k(\xi)}\right)^{1/2}, \qquad w^+(z) = \left(\frac{z}{z-k(\xi)}\right)^{1/2}, \\ \Psi(\xi,y) &= -\frac{e^{-k(\xi)y}}{Y^-(k(\xi),\xi)} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-k(\xi)}\frac{1}{Y^+(q,\xi)}\frac{K_1(q)-K(q)}{K_1(q)+\xi}e^{-qy}dq. \end{aligned} \tag{2.15}$$

All the integrals are understood in the sense of the principal values.

Proposition 2.4. *Let the initial data be* $u_0 \in \mathbf{L}^1(\mathbf{R}^+)$. *Then there exists a unique solution* $u(x,t)$ *of the initial-boundary value problem* (2.9), *which has integral representation*

$$u(x,t) = \mathcal{G}(t)u_0. \tag{2.16}$$

Proof. To derive an integral representation for the solutions of the problem (2.9) we suppose that there exists a solution $u(x,t)$ of problem (2.9), which is continued by zero outside of $x > 0$:

$$u(x,t) = 0, \quad \forall x < 0. \tag{2.17}$$

Let $\phi(p)$ be a function of the complex variable p, which obeys the Hölder condition for all finite p and tends to 0 as $p \to \pm i\infty$. We define the operator

$$\mathbb{P}\phi(z) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\phi(q)\,dq. \tag{2.18}$$

Since the operator $\mathbb{P}$ is defined by a Cauchy type integral, it is readily observed that $\mathbb{P}\phi(z)$ constitutes a function analytic in the entire complex plane, except for points of the contour of integration $\operatorname{Re} z = 0$. Also by Sokhotzki-Plemelj formula we have for $\operatorname{Re} p = 0$

$$\begin{aligned}\mathbb{P}^{+}\phi &= \frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty} \frac{1}{q-p}\phi(q)\,dq + \frac{1}{2}\phi(p),\\ \mathbb{P}^{-}\phi &= \frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty} \frac{1}{q-p}\phi(q)\,dq - \frac{1}{2}\phi(p).\end{aligned} \tag{2.19}$$

Here $\mathbb{P}^{+}\phi$ and $\mathbb{P}^{-}\phi$ are limits of $\mathbb{P}\phi$ as z tends to p from the left and right semi-plane, respectively.

We have for the Laplace transform

$$\mathcal{L}\{\mathscr{H}u_{xx}\} = \mathbb{P}\left\{-|p|p\left(\mathcal{L}\{u\} - \frac{u(0,t)}{p} - \frac{u_x(0,t)}{p^2}\right)\right\}. \tag{2.20}$$

Since $\mathcal{L}\{u\}$ is analytic for all $\operatorname{Re} q > 0$, we have

$$\widehat{u}(q,t) = \mathcal{L}\{u\} = \mathbb{P}\widehat{u}(p,t). \tag{2.21}$$

Therefore applying the Laplace transform with respect to x to problem (2.9) we obtain for $t > 0$

$$\begin{gathered}\mathbb{P}\left\{\widehat{u}_t + K(p)\widehat{u}(p,t) - \frac{K(p)}{p}u(0,t) - \frac{K(p)}{p^2}u_x(0,t)\right\} = 0,\\ \widehat{u}(p,0) = \widehat{u}_0(p),\end{gathered} \tag{2.22}$$

where

$$K(p) = -|p|p. \tag{2.23}$$

We rewrite (2.22) in the form

$$\begin{gathered}\widehat{u}_t + K(p)\widehat{u}(p,t) - \frac{K(p)}{p}u(0,t) - \frac{K(p)}{p^2}u_x(0,t) = \Phi(p,t),\\ \widehat{u}(p,0) = \widehat{u}_0(p),\end{gathered} \tag{2.24}$$

with some function $\Phi(p,t)$ such that for all Re $p > 0$

$$\mathbb{P}\{\Phi(p,t)\} = 0 \tag{2.25}$$

and for $|p| > 1$

$$|\Phi(p,t)| \le C\frac{1}{|p|}. \tag{2.26}$$

Applying the Laplace transformation with respect to time variable to problem (2.24) we find for Re $p > 0$

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\left(\widehat{u}_0(p) + \frac{K(p)}{p}\widehat{u}(0,\xi) + \frac{K(p)}{p^2}\widehat{u}_x(0,\xi) + \widehat{\Phi}(p,\xi)\right). \tag{2.27}$$

Here the functions $\widehat{\widehat{u}}(p,\xi), \widehat{\Phi}(p,\xi), \widehat{u}(0,\xi)$, and $\widehat{u}_x(0,\xi)$ are the Laplace transforms for $\widehat{u}(p,t), \Phi(p,t), u(0,t)$, and $u_x(0,t)$ with respect to time, respectively. We will find the function $\widehat{\Phi}(p,\xi)$ using the analytic properties of function $\widehat{\widehat{u}}$ in the right-half complex planes Re$\,p > 0$ and Re $\xi > 0$. We have for Re $p = 0$

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{\pi i}PV\int_{-i\infty}^{i\infty}\frac{1}{q-p}\widehat{\widehat{u}}(q,\xi)\,dq. \tag{2.28}$$

In view of Sokhotzki-Plemelj formula via (2.27) the condition (2.28) can be written as

$$\Theta^{+}(p,\xi) = -\Lambda^{+}(p,\xi), \tag{2.29}$$

where the sectionally analytic functions $\Theta(z,\xi)$ and $\Lambda(z,\xi)$ are given by Cauchy type integrals:

$$\Theta(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\widehat{\Phi}(q,\xi)\,dq, \tag{2.30}$$

$$\Lambda(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\left(\widehat{u}_0(q) + \frac{K(q)}{q}\widehat{u}(0,\xi) + \frac{K(q)}{q^2}\widehat{u}_x(0,\xi)\right)dq. \tag{2.31}$$

To perform the condition (2.29) in the form of nonhomogeneous Riemann problem we introduce the sectionally analytic function:

$$\Omega(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\Psi(q,\xi)\,dq, \tag{2.32}$$

where

$$\Psi(p,\xi) = \frac{K(p)}{K(p)+\xi}\widehat{\Phi}(p,\xi). \tag{2.33}$$

Taking into account the assumed condition (2.25) and making use of Sokhotzki-Plemelj formula (2.3) we get for limiting values of the functions $\Omega(z,\xi)$ and $\Theta(z,\xi)$

$$\Omega^-(p,\xi) = -\xi\Theta^-(p,\xi). \tag{2.34}$$

Also observe that from (2.30) and (2.32) by formula (2.4)

$$K(p)\left(\Theta^+(p,\xi) - \Theta^-(p,\xi)\right) = \Psi(p,\xi) = \Omega^+(p,\xi) - \Omega^-(p,\xi). \tag{2.35}$$

Substituting (2.29) and (2.34) into this equation we obtain nonhomogeneous Riemann problem

$$\Omega^+(p,\xi) = \frac{K(p)+\xi}{\xi}\Omega^-(p,\xi) - K(p)\Lambda^+(p,\xi). \tag{2.36}$$

It is required to find two functions for some fixed point ξ, Re $\xi > 0$: $\Omega^+(z,\xi)$, analytic in Re $z < 0$ and $\Omega^-(z,\xi)$, analytic in Re $z > 0$, which satisfy on the contour Re $p = 0$ the relation (2.36). Here, for some fixed point ξ, Re $\xi > 0$, the functions

$$W(p,\xi) = \frac{K(p)+\xi}{\xi}, \qquad g(p,\xi) = -K(p)\Lambda^+(p,\xi) \tag{2.37}$$

are called the coefficient and the free term of the Riemann problem, respectively.

Note that bearing in mind formula (2.33) we can find unknown function $\widehat{\Phi}(p,\xi)$ which involved in the formula (2.27) by the relation

$$\widehat{\Phi}(p,\xi) = \frac{K(p)+\xi}{K(p)}\left(\Omega^+(p,\xi) - \Omega^-(p,\xi)\right). \tag{2.38}$$

The method for solving the Riemann problem $A^+(p) = \varphi(p)A^-(p) + \phi(p)$ is based on the Theorems 2.2 and 2.3.

In the formulations of Theorems 2.2 and 2.3 the coefficient $\varphi(p)$ and the free term $\phi(p)$ of the Riemann problem are required to satisfy the Hölder condition on the contour Re $p = 0$. This restriction is essential. On the other hand, it is easy to observe that both functions $W(p,\xi)$ and $g(p,\xi)$ do not have limiting value as $p \to \pm i\infty$. The principal task now is to get an expression equivalent to the boundary value problem (2.36), such that the conditions of theorems are satisfied. First, let us introduce some notation and let us establish certain auxiliary relationships. Setting

$$K_1(p) = -p^2, \tag{2.39}$$

we introduce the function

$$\widetilde{W}(p,\xi) = \left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}, \tag{2.40}$$

where for some fixed point $k(\xi)$ (Re $k(\xi) > 0$)

$$w^-(z) = \left(\frac{z}{z+k(\xi)}\right)^{1/2}, \qquad w^+(z) = \left(\frac{z}{z-k(\xi)}\right)^{1/2}. \tag{2.41}$$

We make a cut in the plane z from point $k(\xi)$ to point $-\infty$ through 0. Owing to the manner of performing the cut the functions $w^-(z)$, $K_1(z)$ are analytic for Re $z > 0$ and the function $w^+(z)$ is analytic for Re $z < 0$.

We observe that the function $\widetilde{W}(p,\xi)$ given on the contour Re $p = 0$ satisfies the Hölder condition and under the assumption Re $K_1(p) > 0$ does not vanish for any Re $\xi > 0$. Also we have

$$\text{Ind.}\widetilde{W}(p,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\ln\widetilde{W}(p,\xi) = 0. \tag{2.42}$$

Therefore in accordance with Theorem 2.3 the function $\widetilde{W}(p,\xi)$ can be represented in the form of the ratio

$$\widetilde{W}(p,\xi) = \frac{X^+(p,\xi)}{X^-(p,\xi)}, \tag{2.43}$$

where

$$X^{\pm}(p,\xi) = e^{\Gamma^{\pm}(p,\xi)}, \qquad \Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\widetilde{W}(q,\xi)dq. \tag{2.44}$$

Now we return to the nonhomogeneous Riemann problem (2.36). Multiplying and dividing the expression $(K(p)+\xi)/\xi$ by $(1/(K_1(p)+\xi))(w^-(p)/w^+(p))$ and making use of the formula (2.43) we get

$$W(p,\xi) = \frac{K(p)+\xi}{\xi} = \frac{Y^+(p,\xi)}{Y^-(p,\xi)}\left(\frac{K_1(p)+\xi}{\xi}\right), \tag{2.45}$$

where

$$Y^{\pm}(p,\xi) = X^{\pm}(p,\xi)w^{\pm}(p). \tag{2.46}$$

Replacing in (2.36) the coefficient of the Riemann problem $W(p,\xi)$ by (2.45) we reduce the nonhomogeneous Riemann problem (2.36) to the form

$$\frac{\Omega^+(p,\xi)}{Y^+(p,\xi)} = \left(\frac{K_1(p)+\xi}{\xi}\right)\frac{\Omega^-(p,\xi)}{Y^-(p,\xi)} - \frac{1}{Y^+(p,\xi)}K(p)\Lambda^+(p,\xi). \tag{2.47}$$

Now we perform the function $\Lambda(z,\xi)$ given by formula (2.31) as

$$\Lambda(z,\xi) = \Lambda_1(z,\xi) + \Lambda_2(z,\xi), \tag{2.48}$$

where

$$\Lambda_1(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{K_1(q)+\xi}\left(\hat{u}_0(q) + \frac{K_1(q)}{q}\hat{u}(0,\xi) + \frac{K_1(q)}{q^2}\hat{u}_x(0,\xi)\right)dq, \tag{2.49}$$

$$\Lambda_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{K_1(q)-K(q)}{(K(q)+\xi)(K_1(q)+\xi)}\left(\hat{u}_0(q) - \frac{\xi}{q}\hat{u}(0,\xi) - \hat{u}_x(0,\xi)\right)dq. \tag{2.50}$$

Firstly we calculate the left limiting value $\Lambda_1^+(p,\xi)$. Since there exists only one root $k(\xi)$ of equation $K_1(z) = -\xi$ such that $\operatorname{Re} k(\xi) > 0$ for all $\operatorname{Re}\xi > 0$, therefore, taking limit $z \to p$ from the left-hand side of complex plane, by Cauchy theorem we get

$$\Lambda_1^+(p,\xi) = -\frac{k'(\xi)}{p-k(\xi)}\left(\hat{u}_0(k(\xi)) - \frac{\xi}{p}\hat{u}(0,\xi) - \hat{u}_x(0,\xi)\right). \tag{2.51}$$

The last relation implies that $(K_1(p)+\xi)\Lambda_1^+(p,\xi)$ can be expressed by the function $\Lambda_3(z,\xi)$ which is analytic in $\operatorname{Re} z > 0$:

$$(K_1(p)+\xi)\Lambda_1^+(p,\xi) = \Lambda_3^-(p,\xi), \tag{2.52}$$

where

$$\Lambda_3(z,\xi) = -k'(\xi)\left(\frac{K_1(z)+\xi}{z-k(\xi)}\right)\left(\hat{u}_0(k(\xi)) - \frac{\xi}{p}\hat{u}(0,\xi) - \hat{u}_x(0,\xi)\right). \tag{2.53}$$

By Sokhotzki-Plemelj formula (2.4) we express the left limiting value $\Lambda_2^+(p,\xi)$ in the term of the right limiting value $\Lambda_2^-(p,\xi)$ as

$$\Lambda_2^+(p,\xi) = \Lambda_2^-(p,\xi) + \tilde{g}_1(p,\xi), \tag{2.54}$$

where

$$\tilde{g}_1(p,\xi) = \frac{K_1(p)-K(p)}{(K(p)+\xi)(K_1(p)+\xi)}\left(\hat{u}_0(p) - \frac{\xi}{p}\hat{u}(0,\xi) - \hat{u}_x(0,\xi)\right). \tag{2.55}$$

Bearing in mind the representation (2.48) and making use of (2.52), (2.55), and (2.45) after simple transformations we get

$$-K(p)\Lambda^{+}=-\frac{\Upsilon^{+}}{\Upsilon^{-}}\left[\Lambda_3^{-}+\left(K_1(p)+\xi\right)\Lambda_2^{-}\right]+\xi\Lambda^{+}-g_1(p,\xi), \tag{2.56}$$

where

$$g_1(p,\xi)=\left(K(p)+\xi\right)\tilde{g}_1(p,\xi)=\frac{K_1(p)-K(p)}{K_1(p)+\xi}\left(\hat{u}_0(p)-\frac{\xi}{p}\hat{u}(0,\xi)-\hat{u}_x(0,\xi)\right). \tag{2.57}$$

Replacing in (2.47) $-K(p)\Lambda^{+}(p,\xi)$ by (2.56), we reduce the nonhomogeneous Riemann problem (2.47) in the form

$$\frac{\Omega_1^{+}(p,\xi)}{\Upsilon^{+}(p,\xi)}=\frac{\Omega_1^{-}(p,\xi)}{\Upsilon^{-}(p,\xi)}-\frac{1}{\Upsilon^{+}(p,\xi)}g_1(p,\xi), \tag{2.58}$$

where

$$\begin{aligned}\Omega_1^{+}(p,\xi)&=\Omega^{+}(p,\xi)-\xi\Lambda^{+}(p,\xi),\\ \Omega_1^{-}(p,\xi)&=\left(K_1(p)+\xi\right)\left(\xi^{-1}\Omega^{-}(p,\xi)-\Lambda_2^{-}(p,\xi)\right)-\Lambda_3^{-}(p,\xi).\end{aligned} \tag{2.59}$$

In subsequent consideration we will have to use the following property of the limiting values of a Cauchy type integral, the statement of which we now quote. The proofs may be found in [19].

Lemma 2.5. *If L is a smooth closed contour and $\phi(q)$ a function that satisfies the Hölder condition on L, then the limiting values of the Cauchy type integral*

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{1}{q-z}\phi(q)\,dq \tag{2.60}$$

also satisfy this condition.

Since $g_1(p,\xi)$ satisfies on $\operatorname{Re} p=0$ the Hölder condition, on basis of this Lemma the function $(1/\Upsilon^{+}(p,\xi))g_1(p,\xi)$ also satisfies this condition. Therefore in accordance with Theorem 2.2 it can be uniquely represented in the form of the difference of the functions $U^{+}(p,\xi)$ and $U^{-}(p,\xi)$, constituting the boundary values of the analytic function $U(z,\xi)$, given by formula

$$U(z,\xi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{\Upsilon^{+}(q,\xi)}g_1(q,\xi)\,dq. \tag{2.61}$$

Therefore the problem (2.58) takes the form

$$\frac{\Omega_1^+(p,\xi)}{\Upsilon^+(p,\xi)}+U^+(p,\xi)=\frac{\Omega_1^-(p,\xi)}{\Upsilon^-(p,\xi)}+U^-(p,\xi). \tag{2.62}$$

The last relation indicates that the function $(\Omega_1^+/\Upsilon^+)+U^+$, analytic in Re $z<0$, and the function $(\Omega_1^-/\Upsilon^-)+U^-$, analytic in Re $z>0$, constitute the analytic continuation of each other through the contour Re $z=0$. Consequently, they are branches of unique analytic function in the entire plane. According to generalize Liouville theorem this function is some arbitrary constant A. Thus, bearing in mind the representations (2.59) and (2.52) we get

$$\begin{aligned}\Omega^+(p,\xi)&=\Upsilon^+(A-U^+)+\xi\Lambda^+,\\ \Omega^-(p,\xi)&=\frac{\xi}{K_1(p)+\xi}\Upsilon^-(A-U^-)+\xi(\Lambda_1^++\Lambda_2^-).\end{aligned} \tag{2.63}$$

Since there exists only one root $k(\xi)$ of equation $K_1(z)=-\xi$ such that Re $k(\xi)>0$ for all Re $\xi>0$, therefore, in the expression for the function $\Omega^-(z,\xi)$ the factor $\xi/(K_1(z)+\xi)$ has a pole in the point $z=k(\xi)$. Also the function $\xi\Lambda_1^+$ has a pole in the point $z=k(\xi)$. Thus in general case the problem (2.36) is insolvable. It is soluble only when the functions $U^-(z,\xi)$ and $\xi\Lambda_1^+$ satisfy additional conditions. For analyticity of $\Omega^-(z,\xi)$ in points $z=k(\xi)$ it is necessary that

$$\operatorname{Res}_{p=k(\xi)}\left\{\frac{1}{K_1(p)+\xi}\Upsilon^-(A-U^-)+\Lambda_1^+\right\}=0. \tag{2.64}$$

We reduce (2.64) to the form

$$A\Upsilon^-(k(\xi),\xi)-\Upsilon^-(k(\xi),\xi)U^-(k(\xi),\xi)+\left(-\frac{\xi}{k(\xi)}\widehat{u}(0,\xi)+\widehat{u}_0(k(\xi))-\widehat{u}_x(0,\xi)\right)=0. \tag{2.65}$$

Multiplying the last relation by $1/\Upsilon^-(k(\xi),\xi)$ and taking limit $\xi\to\infty$ we get that $A=0$. This implies that for solubility of the nonhomogeneous problem (2.36) it is necessary and sufficient that the following condition is satisfied:

$$\Upsilon^-(k(\xi),\xi)U^-(k(\xi),\xi)+\frac{\xi}{k(\xi)}\widehat{u}(0,\xi)-\widehat{u}_0(k(\xi))+\widehat{u}_x(0,\xi)=0. \tag{2.66}$$

Therefore, we need to put in the problem (2.9) one boundary data and the rest of boundary data can be found from (2.66). Thus, for example, if we put $u(0,t)=0$ from (2.66) we obtain for the Laplace transform of $u_x(0,t)$,

$$\begin{aligned}&-\left[\Upsilon^-(k(\xi),\xi)I(k(\xi),\xi)-1\right]\widehat{u}_x(0,\xi)\\ &\quad=\widehat{u}_0(k(\xi))-\Upsilon^-(k(\xi),\xi)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-k(\xi)}\frac{\widehat{u}_0(q)}{\Upsilon^+(q,\xi)}\frac{K_1(q)-K(q)}{K_1(q)+\xi}dq,\end{aligned} \tag{2.67}$$

where

$$I(k(\xi),\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \frac{1}{Y^+(q,\xi)} \frac{K_1(q)-K(q)}{K_1(q)+\xi} dq. \tag{2.68}$$

Now we prove that the coefficient of $\widehat{u}_x(0,\xi)$ does not vanish for all Re $\xi > 0$. We represent the function $I(k(\xi),\xi)$ in the form

$$I = I_1 + I_2, \tag{2.69}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \frac{1}{Y^+(q,\xi)} dq, \\ I_2 &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \frac{1}{Y^+(q,\xi)} \frac{K(q)+\xi}{K_1(q)+\xi} dq. \end{aligned} \tag{2.70}$$

Since, for Re $z \neq 0$,

$$\int_{-i\infty}^{i\infty} \frac{1}{q-z} dq = -\pi i \operatorname{sgn}(\operatorname{Re} z), \tag{2.71}$$

making use of analytic properties of the function $((1/Y^+(q,\xi))-1)$ by Cauchy Theorem we have

$$I_1 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} dq + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \left(\frac{1}{Y^+(q,\xi)} - 1\right) dq = -\frac{1}{2}, \tag{2.72}$$

where ξ is some fixed point, Re $\xi > 0$. To calculate the function I_2 we will use the identity (2.43). Observe that the function $1/Y^-(q,\xi)$ is analytic for all Re $q > 0$. Therefore, setting the relation (2.43) into definition of I_2 and making use of Cauchy Theorem we find

$$I_2 = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \frac{1}{Y^-(q,\xi)} dq = \left(\frac{1}{Y^-(k(\xi),\xi)} - 1\right) + \frac{1}{2}. \tag{2.73}$$

Thus, from (2.72) and (2.73) we obtain the following relation for the function I :

$$I(k(\xi),\xi) = \frac{1}{Y^-(k(\xi),\xi)} - 1. \tag{2.74}$$

Substituting this formula into (2.67) we get

$$\widehat{u}_x(0,\xi) = \frac{\widehat{u}_0(k(\xi))}{Y^-(k(\xi),\xi)} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-k(\xi)} \frac{\widehat{u}_0(q)}{Y^+(q,\xi)} \frac{K_1(q)-K(q)}{K_1(q)+\xi} dq. \tag{2.75}$$

Now we return to problem (2.36). From (2.63) under the conditions $u(0,t)=0$ and (2.75) the limiting values of solution of (2.36) are given by

$$\Omega^+(p,\xi) = -\Upsilon^+U^+ + \xi\Lambda_2^+,$$
$$\Omega^-(p,\xi) = -\frac{\xi}{K_1(p)+\xi}\Upsilon^-U^- + \xi\Lambda_2^-. \tag{2.76}$$

From (2.76) with the help of the integral representations (2.61) and (2.50), for sectionally analytic functions $U(z,\xi)$ and $\Lambda_2(z,\xi)$, making use of Sokhotzki-Plemelj formula (2.3) and relation (2.45) we can express the difference limiting values of the function $\Omega(z,\xi)$ in the form

$$\begin{aligned}
&\Omega^+(p,\xi) - \Omega^-(p,\xi) \\
&\quad = -\Upsilon^+U^+ + \frac{\xi}{K_1(p)+\xi}\Upsilon^-U^- + \xi\left(\Lambda_2^+ - \Lambda_2^-\right) \\
&\quad = -\Upsilon^+\left(U^+ - \frac{\xi}{K(p)+\xi}U^-\right) + \xi\left(\Lambda_2^+ - \Lambda_2^-\right) \\
&\quad = -\frac{1}{2}\frac{K(p)}{K(p)+\xi}g_1(p,\xi) \\
&\qquad - \frac{K(p)}{K(p)+\xi}\Upsilon^+(p,\xi)\frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{\Upsilon^+(q,\xi)}g_1(q,\xi)\,dq.
\end{aligned} \tag{2.77}$$

We now proceed to find the unknown function $\widehat{\Phi}(p,\xi)$ involved in the formula (2.27) for the solution $\widehat{\widehat{u}}(p,\xi)$ of the problem (2.9). Replacing the difference $\Omega^+(p,\xi)-\Omega^-(p,\xi)$ in the relation (2.38) by formula (2.77) we get

$$\begin{aligned}
\widehat{\Phi}(p,\xi) &= \frac{K(p)+\xi}{K(p)}\left(\Omega^+(p,\xi) - \Omega^-(p,\xi)\right) \\
&= -\frac{1}{2}g_1(p,\xi) - \Upsilon^+(p,\xi)\frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{\Upsilon^+(q,\xi)}g_1(q,\xi)\,dq.
\end{aligned} \tag{2.78}$$

It is easy to observe that $\widehat{\Phi}(p,\xi)$ is boundary value of the function analytic in the left complex semi-plane and therefore satisfies our basic assumption for all Re $z>0$

$$\mathbb{P}\{\Phi\} = 0. \tag{2.79}$$

Having determined the function $\widehat{\Phi}(p,\xi)$ bearing in mind formula (2.27) and conditions $u(0,t)=0$ we determine required function $\widehat{\widehat{u}}$:

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\left(\widehat{u}_0(p)-\widehat{u}_x(0,\xi)\right) - \frac{1}{2}\frac{1}{K(p)+\xi}g_1(p,\xi) - \frac{1}{K(p)+\xi}Y^+(p,\xi)\frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{Y^+(q,\xi)}g_1(q,\xi)\,dq, \tag{2.80}$$

where the function $g_1(p,\xi)$ is given by formula (2.55):

$$g_1(p,\xi) = \frac{K_1(q)-K(q)}{K_1(q)+\xi}\left(\widehat{u}_0(q)-\widehat{u}_x(0,\xi)\right). \tag{2.81}$$

Now we prove that, in accordance with last relation, the function $\widehat{\widehat{u}}(p,\xi)$ constitutes the limiting value of an analytic function in $\operatorname{Re} z>0$.

With the help of the integral representations (2.61), (2.31), and (2.50) for sectionally analytic functions $U(z,\xi), \Lambda(z,\xi)$, and $\Lambda_2(z,\xi)$, and making use of Sokhotzki-Plemelj formula (2.3) we have

$$\frac{1}{K(p)+\xi}\left(\widehat{u}_0(p)-\widehat{u}_x(0,\xi)\right) = \Lambda^+ - \Lambda^-, \qquad \frac{1}{K(p)+\xi}g_1(p,\xi) = \Lambda_2^+ - \Lambda_2^-,$$
$$\frac{1}{2\pi i}\mathrm{PV}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\frac{1}{Y^+(q,\xi)}g_1(q,\xi)\,dq = \frac{1}{2}\left(U^+ + U^-\right). \tag{2.82}$$

Substituting these relations into (2.80) we express the function $\widehat{\widehat{u}}$ in the following form:

$$\widehat{\widehat{u}} = \left(\Lambda^+ - \Lambda^-\right) - \frac{1}{2}\left(\Lambda_2^+ - \Lambda_2^-\right) - \frac{1}{2}\frac{1}{K(p)+\xi}Y^+\left(U^+ + U^-\right). \tag{2.83}$$

If it is taken into account that $\Lambda(z,\xi) = \Lambda_1(z,\xi)+\Lambda_2(z,\xi)$ by virtue of the relation (2.45), the last expression agrees with formula

$$\widehat{\widehat{u}} = \Lambda_1^+ - \Lambda_1^- + \frac{1}{2}\left(\Lambda_2^+ - \Lambda_2^-\right) - \frac{1}{2}\frac{1}{K(p)+\xi}Y^+U^+ - \frac{1}{2}\frac{1}{K_1(p)+\xi}Y^-U^-. \tag{2.84}$$

Expressing the function U^+ in the last equation in terms of U^-

$$U^+ = U^- + \frac{1}{Y^+}\left(K(p)+\xi\right)\left(\Lambda_2^+ - \Lambda_2^-\right), \tag{2.85}$$

we arrive at the following relation:

$$\widehat{\widehat{u}} = \Lambda_1^+ - \Lambda_1^- - \frac{1}{K_1(p)+\xi}Y^-U^-, \tag{2.86}$$

where by virtue of (2.49) and (2.66),

$$\Lambda_1^+ - \Lambda_1^- = \frac{1}{K_1(p) + \xi}\left(\widehat{u}_0(p) - \widehat{u}_x(0,\xi)\right). \tag{2.87}$$

Thus the function $\widehat{\widehat{u}}$ is the limiting value of an analytic function in Re $z > 0$. Note the fundamental importance of the proven fact that the solution $\widehat{\widehat{u}}$ constitutes an analytic function in Re $z > 0$ and, as a consequence, its inverse Laplace transform vanishes for all $x < 0$. We now return to solution $u(x,t)$ of the problem (2.9).

Under assumption $u(0,t) = 0$ the integral representation (2.61) takes form

$$U(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K_1(q) - K(q)}{K_1(q) + \xi}\left(\widehat{u}_0(q) - \widehat{u}_x(0,\xi)\right)dq, \tag{2.88}$$

where $\widehat{u}_x(0,\xi)$ is defined by (2.75). Substituting this relation into (2.86) and taking inverse Laplace transform with respect to time and inverse Fourier transform with respect to space variables we obtain

$$u(x,t) = \mathcal{G}(t)u_0 = \int_0^{\infty} G(x,y,t)u_0(y)dy, \tag{2.89}$$

where the function $G(x,y,t)$ was defined by formula (2.12). Proposition 2.4 is proved. □

Now we collect some preliminary estimates of the Green operator $\mathcal{G}(t)$. Let the contours $\mathcal{C}_i$ be defined as

$$\mathcal{C}_1 = \left\{p \in \left(\infty e^{-i(\pi/2+\varepsilon)}, 0\right) \bigcup \left(0, \infty e^{i(\pi/2+\varepsilon)}\right)\right\}, \tag{2.90}$$

$$\mathcal{C}_2 = \left\{q \in \left(\infty e^{-i((\pi/2)+2\varepsilon)}, 0\right) \bigcup \left(0, \infty e^{i((\pi/2)+2\varepsilon)}\right)\right\}, \tag{2.91}$$

$$\mathcal{C}_3 = \left\{q \in \left(\infty e^{-i((\pi+\varepsilon)/2)}, 0\right) \bigcup \left(0, \infty e^{i((\pi+\varepsilon)/2)}\right)\right\}, \tag{2.92}$$

where $\varepsilon > 0$ can be chosen such that all functions under integration are analytic and Re $k(\xi) > 0$ for $\xi \in \mathcal{C}_1$.

Lemma 2.6. *The function $G(x,y,t)$ given by formula* (2.12) *has the following representation:*

$$\begin{aligned} G(x,y,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp \\ &- \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\mathcal{C}_1} d\xi e^{\xi t}\int_{\mathcal{C}_2} e^{px}\frac{e^{\Gamma(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}I(p,\xi,y)dp, \end{aligned} \tag{2.93}$$

where

$$I(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{(q-p)(q-k(\xi))}\frac{1}{e^{\Gamma^{+}(q,\xi)}w^{+}(q,\xi)}e^{-qy}dq,$$
$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\ln\left\{\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right)\frac{w^{-}(q)}{w^{+}(q)}\right\}dq. \tag{2.94}$$

The functions $w^{\pm}(q,\xi)$, $k(\xi)$ *were defined in formulas* (2.13) *and* (2.10).

Proof. We rewrite formula (2.12) in the form

$$G(x,y,t) = J_1(x-y,t) + J_2(x,y,t) + J_3(x,y,t) + J_4(x,y,t), \tag{2.95}$$

where

$$J_1(x-y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K_1(p)t}dp,$$
$$J_2(x,y,t) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\xi e^{\xi t}\Psi(\xi,y)\int_{-i\infty}^{i\infty} e^{px}\frac{1}{K_1(p)+\xi}dp, \tag{2.96}$$
$$J_3(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\Upsilon_1^{-}(p,\xi,y)dp,$$

$$J_4(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\xi e^{\xi t}\Psi(\xi,y)\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\Upsilon_1^{-}(p,\xi,0)dp. \tag{2.97}$$

Here

$$\Upsilon_1(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{\Upsilon^{+}(q,\xi)}\frac{K_1(q)-K(q)}{K_1(q)+\xi}e^{-qy}dq,$$
$$\Psi(\xi,y) = -\frac{e^{-k(\xi)y}}{\Upsilon^{-}(k(\xi),\xi)} + \Upsilon_1(k(\xi),\xi,y), \tag{2.98}$$
$$\Upsilon^{\pm} = e^{\Gamma^{\pm}}w^{\pm},$$
$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\ln\left\{\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right)\frac{w^{-}(q)}{w^{+}(q)}\right\}dq. \tag{2.99}$$

Firstly we consider the sectionally analytic function $\Upsilon_1(z,\xi,y)$ given by Cauchy type integral (2.98).

On basis of the definition (2.98) its limiting value can be represent in the form

$$\Upsilon_1^-(p,\xi,y) = I_1(p,\xi,y) + I_2(p,\xi,y), \tag{2.100}$$

where

$$\begin{aligned} I_1(p,\xi,y) &= \lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{Y^+(q,\xi)}e^{-qy}dq, \\ I_2(p,\xi,y) &= -\lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{Y^+(q,\xi)}\frac{K(q)+\xi}{K_1(q)+\xi}e^{-qy}dq. \end{aligned} \tag{2.101}$$

Making use of analytic properties of the functions $(1/Y^+(q,\xi)-1)$, for Re $q<0$, and e^{-qy}, for Re $q>0$, by Cauchy theorem we have

$$\begin{aligned} I_1(p,\xi,y) \quad &= \lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}e^{-qy}dq \\ &\quad + \lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\left(\frac{1}{Y^+(q,\xi)}-1\right)\left(e^{-qy}-1\right)dq \\ &\quad + \lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\left(\frac{1}{Y^+(q,\xi)}-1\right)dq \\ &= -e^{-py} + \lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\left(\frac{1}{Y^+(q,\xi)}-1\right)\left(e^{-qy}-1\right)dq, \end{aligned} \tag{2.102}$$

where ξ is some fixed point, Re $\xi>0$.

To calculate the function $I_2(p,\xi,y)$ we will use the following identity:

$$\frac{1}{Y^+(q,\xi)}\frac{K(\mathrm{q})+\xi}{K_1(q)+\xi} = \frac{1}{Y^-(q,\xi)}. \tag{2.103}$$

Observe that the function $1/Y^-(q,\xi)$ is analytic for all Re $q>0$. Therefore, setting the relation (2.103) into definition of $I_2(p,\xi,y)$ and making use of Cauchy theorem we find

$$I_2(p,\xi,y) = -\lim_{z\to p,\mathrm{Re}\ z>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{1}{Y^-(q,\xi)}e^{-qy}dq = \frac{1}{Y^-(p,\xi)}e^{-py}. \tag{2.104}$$

Thus from (2.102) and (2.104) we obtain the following relation:

$$\Upsilon_1^-(p,\xi,y) = -e^{-py} + \Upsilon^-(p,\xi,y) + \frac{1}{Y^-(p,\xi)}e^{-py}, \tag{2.105}$$

where

$$\Upsilon(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\left(\frac{1}{\Upsilon^{+}(q,\xi)} - 1\right)e^{-qy}dq. \tag{2.106}$$

In the same way (see also proof of relation (2.74)) we can prove that

$$\Upsilon_1^{-}(p,\xi,0) = -1 + \frac{1}{\Upsilon^{-}(p,\xi)}. \tag{2.107}$$

Also we observe that

$$\Psi(\xi,y) = -e^{-k(\xi)y} + \Upsilon(k(\xi),\xi,y). \tag{2.108}$$

Inserting into definition (2.96) the expression (2.105) for $\Upsilon_1^{-}(p,\xi,y)$ we obtain the function $J_3(x,y,t)$ in the form

$$J_3(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\left(-e^{-py} + \Upsilon^{-}(p,\xi,y)\right)dp - J_1(x-y,t). \tag{2.109}$$

Replacing in formula (2.97) the functions $\Psi(\xi,y)$ and $\Upsilon_1^{-}(p,\xi,0)$ by (2.108) and (2.107), respectively, we reduce the function $J_4(x,y,t)$ in the form

$$\begin{aligned} J_4(x,y,t) &= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\Psi(\xi,y)\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\left(-1 + \frac{1}{\Upsilon^{-}(p,\xi)}\right)dp \\ &= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\left(e^{-k(\xi)y} - \Upsilon(k(\xi),\xi,y)\right)dp - J_2(x,y). \end{aligned} \tag{2.110}$$

Therefore inserting into definition (2.95) expressions (2.109) and (2.110), for $J_3(x,y,t)$ and $J_4(x,y,t)$, respectively, we obtain the function $G(x,y,t)$ in the form

$$G(x,y,t) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{\Upsilon^{-}(p,\xi)}{K_1(p)+\xi}\Xi_1(p,\xi,y)dp, \tag{2.111}$$

where

$$\Xi_1(p,\xi,y) = \left(e^{-py} - e^{-k(\xi)y}\right) - \left(\Upsilon^{-}(p,\xi,y) - \Upsilon(k(\xi),\xi,y)\right). \tag{2.112}$$

Also, note that since

$$\begin{aligned}&\Upsilon^{-}(p,\xi,y)-\Upsilon(k(\xi),\xi,y)\\&\quad=\lim_{z\to p,\,\mathrm{Re}\,z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{1}{q-z}-\frac{1}{q-k(\xi)}\right)\left(\frac{1}{Y^{+}(q,\xi)}-1\right)e^{-qy}dq\\&\quad=\lim_{z\to p,\,\mathrm{Re}\,z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{z-k(\xi)}{(q-z)(q-k(\xi))}\right)\left(\frac{1}{Y^{+}(q,\xi)}-1\right)e^{-qy}dq,\end{aligned}\tag{2.113}$$

we obtain

$$\begin{aligned}&\left(e^{-py}-e^{-k(\xi)y}\right)-\left(\Upsilon^{-}(p,\xi,y)-\Upsilon(k(\xi),\xi,y)\right)\\&\quad=\lim_{z\to p,\,\mathrm{Re}\,z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(-\frac{1}{q-z}+\frac{1}{q-k(\xi)}\right)\left[e^{-qy}+\left(\frac{1}{Y^{+}(q,\xi)}-1\right)e^{-qy}\right]dq\\&\quad=\lim_{z\to p,\,\mathrm{Re}\,z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{k(\xi)-z}{(q-z)(q-k(\xi))}\right)\frac{1}{Y^{+}(q,\xi)}e^{-qy}dq.\end{aligned}\tag{2.114}$$

So,

$$G(x,y,t)=-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{Y^{-}(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^{-}(p,\xi,y)dp,\tag{2.115}$$

where

$$I(z,\xi,y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-z)(q-k(\xi))}\frac{1}{Y^{+}(q,\xi)}e^{-qy}dq.\tag{2.116}$$

Using relation

$$\frac{K(p)+\xi}{K_1(p)+\xi}=\frac{Y^{+}}{Y^{-}},\tag{2.117}$$

we rewrite last formula in the following form:

$$G(x,y,t)=-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{Y^{+}(p,\xi)(p-k(\xi))}{K(p)+\xi}I^{-}(p,\xi,y)dp.\tag{2.118}$$

On the basis of definitions (2.116) and in accordance with the Sohkotzki-Plemelj formula (2.3) we have

$$I^{-}(p,\xi,y)=I^{+}(p,\xi,y)-\frac{1}{(p-k(\xi))}\frac{1}{Y^{+}(p,\xi)}e^{-py}.\tag{2.119}$$

So we get

$$G(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp - \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{Y^{+}(p,\xi)(p-k(\xi))}{K(p)+\xi}I^{+}(p,\xi,y)dp. \tag{2.120}$$

Now we consider for Re $\xi > 0$

$$\begin{aligned}\Gamma^{+}(p,\xi) &= \frac{1}{2\pi i}\lim_{z\to p\,\mathrm{Re}\,z<0}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\left\{\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right)\frac{w^{-}(q)}{w^{+}(q)}\right\}dq \\ &= \frac{1}{2\pi i}\int_{0}^{\infty e^{i((\pi/2)+3\varepsilon)}}\frac{1}{q-p}\ln\left\{\left(\frac{iq^2+\xi}{-q^2+\xi}\right)\frac{w^{-}(q)}{w^{+}(q)}\right\}dq \\ &\quad + \frac{1}{2\pi i}\int_{\infty e^{i((-\pi/2)-3\varepsilon)}}^{0}\frac{1}{q-p}\ln\left\{\left(\frac{-iq^2+\xi}{-q^2+\xi}\right)\frac{w^{-}(q)}{w^{+}(q)}\right\}dq \\ &\quad + \ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^{-}(p)}{w^{+}(p)}\right\} \\ &= \Gamma_1(p,\xi) + \ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^{-}(p)}{w^{+}(p)}\right\}.\end{aligned} \tag{2.121}$$

Note that $\Gamma_1(p,\xi)$ is analytic in domain $0 \le \arg\xi < (\pi/2)+3\varepsilon$, $(-\pi/2)-3\varepsilon < \arg\xi \le 0$, and $\pi/2 \le \arg p < (\pi/2)+3\varepsilon$ and $(-\pi/2)-3\varepsilon < \arg p \le -\pi/2$. So we get

$$G(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp - \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}\frac{e^{\Gamma_1(p,\xi)}w^{-}(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^{+}(p,\xi,y)dp, \tag{2.122}$$

where

$$I^{+}(p,\xi,y) = \frac{1}{2\pi i}\lim_{z\to p\,\mathrm{Re}\,z<0}\int_{-i\infty}^{i\infty}\frac{1}{(q-z)(q-k(\xi))}\frac{1}{w^{-}e^{\Gamma_1(q,\xi)}}\frac{K_1(q)+\xi}{K(q)+\xi}e^{-qy}dq. \tag{2.123}$$

Changing the contour of integration with respect to ξ by Cauchy theorem we get

$$G(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp - \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2} e^{px}\frac{e^{\Gamma_1(p,\xi)}w^-(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^+(p,\xi,y)dp - \operatorname{res}_{\xi=-K(q)}\left\{\int_{C_2} e^{px}\frac{e^{\Gamma_1(p,\xi)}w^-(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^+(p,\xi,y)dp\right\}, \tag{2.124}$$

where

$$\operatorname{res}_{\xi=-K(q)}\left\{\int_{C_2} e^{px}\frac{e^{\Gamma_1(p,\xi)}w^-(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^+(p,\xi,y)dp\right\} = \frac{1}{2\pi i}\int_{C_2} e^{px}dp\int_{C_3} dq\frac{e^{-qy-K(q)t}}{q-p}\phi(p,q),$$

$$\phi(p,q) = \frac{e^{\Gamma_1(p,-K(q))}w^-(p)(p-k(-K(q)))}{K_1(p)-K(q)}\left(\frac{K_1(q)-K(q)}{e^{\Gamma_1(q,-K(q))}w^-(q)(q-k(-K(q)))}\right). \tag{2.125}$$

Since for $q\in C_3, \operatorname{Re}(-K(q))<0$, from (2.121)

$$e^{-\Gamma_1(q,-K(q))} = e^{-\Gamma^+(q,-K(q))}\left(\frac{K(q)-K(q)}{K_1(q)-K(q)}\right)\frac{w^-(q)}{w^+(q)} = 0, \tag{2.126}$$

and therefore

$$\phi(p,q) = 0. \tag{2.127}$$

Thus using relation (2.121) we get relation (2.93). Lemma is proved. □

Lemma 2.7. *The estimates are true, provided that the right-hand sides are finite:*

$$\left\|\mathcal{G}(t)\phi\right\|_{\mathbf{H}^1} \le \left\|\phi\right\|_{\mathbf{H}^1}, \tag{2.128}$$

$$\left\|\partial_x^n\mathcal{G}\phi\right\|_{\mathbf{L}^{s,\mu}} \le Ct^{(-1/2)(n+1+\delta-(1/s)-\mu)}\left\|\phi\right\|_{\mathbf{L}^{1,\delta}}, \tag{2.129}$$

$$\mathcal{G}(t)\phi - t^{-1}\Lambda\left(xt^{-1/2}\right)f(\phi) = \min\left(xt^{-1/2},1\right)t^{-1-a}\left\|\phi\right\|_{\mathbf{L}^{1,1+a}}, \quad a>0, \tag{2.130}$$

where $\mu > 0$, $(1/s)+\mu < n+1, n = 0,1, \delta \in [0,1]$, *and* $f(\phi)$ *is given by* (1.9). *The function* $\Lambda(xt^{-1/2}) \in \mathbf{L}^\infty(\mathbf{R}^+), \Lambda(0) = 0$ *is defined below (see* (2.191)*).*

Proof. From Lemma 2.6 we have

$$G(x,y,t) = J_1(x,y,t) + J_2(x,y,t), \tag{2.131}$$

where

$$J_1(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}\,dp, \tag{2.132}$$

$$J_2(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2} e^{px}\frac{w^+(p,\xi)e^{\Gamma(p,\xi)}(p-k(\xi))}{K(p)+\xi} I(p,\xi,y)\,dp, \tag{2.133}$$

$$I(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-p)(q-k(\xi))}\frac{1}{w^+e^{\Gamma^+}}e^{-qy}\,dq, \tag{2.134}$$

$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\left\{\left(\frac{K(q)+\xi}{K_1(q)+\xi}\right)\frac{w^-(q)}{w^+(q)}\right\}dq. \tag{2.135}$$

The functions $w^{\pm}(q,\xi)$, $k(\xi)$ were defined in formulas (2.13) and (2.10). The contours $\mathcal{C}_1, \mathcal{C}_2$ was defined by (2.90), and (2.91).

For subsequent considerations it is required to investigate the behavior of the function $\Gamma(z,\xi)$. Set

$$\phi(p,\xi) = \ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}\right\} \neq 0, \quad \operatorname{Re} p = 0,\ \operatorname{Re}\xi < 0. \tag{2.136}$$

Observe that the function $\phi(p,\xi)$ obeys the Hölder condition for all finite p and tends to a definite limit $\phi_\infty(\xi)$ as $p \to \pm i\infty$:

$$\begin{aligned}
\phi(\infty,\xi) &= \lim_{p\to\pm i\infty}\ln\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}\right\} \\
&= \lim_{p\to\pm i\infty}\ln\left|\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p)}{w^+(p)}\right| + i\lim_{p\to\pm i\infty}\arg\left\{\left(\frac{K(p)+\xi}{K_1(p)+\xi}\right)\frac{w^-(p,\xi)}{w^+(p,\xi)}\right\} = -i\frac{\pi}{2}.
\end{aligned} \tag{2.137}$$

Also there can be easily obtained that for large p and some fixed ξ the following inequality holds:

$$\left|\phi(p,\xi) - \phi_\infty(\xi)\right| \le C\left(\frac{|\xi|^\mu}{|p|^{2\mu}}\right), \quad \mu > 0. \tag{2.138}$$

Therefore

$$\left|e^{\Gamma^\pm(z,\xi)}\right| \le C \tag{2.139}$$

for all $\xi \in \mathcal{C}_1$ and

$$e^{\Gamma^+(p,\xi)} - e^{1/2} = O\left(\frac{|\xi|^{\mu}}{|p|^{2\mu}}\right), \quad \mu \in [0,1]. \tag{2.140}$$

Denote

$$\mathcal{J}_j(t)\phi = \theta(x)\int_0^{+\infty} J_j(x,y,t)\phi(y)dy, \tag{2.141}$$

By Plansherel Theorem it is easily to see that

$$\|\mathcal{J}_1(t)\phi\|_{\mathbf{H}^1} \le C\|\phi\|_{\mathbf{H}^1}. \tag{2.142}$$

Now we estimate $\mathcal{J}_2(t)\phi$.

From the integral representation (2.134), making use the relation

$$e^{\Gamma^+(p,\xi)-\Gamma^-(p,\xi)}\frac{w^+}{w^-} = \frac{K(p)+\xi}{K_1(p)+\xi} \tag{2.143}$$

and the estimate (2.138) we have for $y > 0, \xi \in \mathcal{C}_1$, and $p \in \mathcal{C}_2$

$$\begin{aligned} I(p,\xi,y) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-p)}\frac{1}{(q-k(\xi))}\frac{1}{w^+e^{\Gamma^+}}e^{-qy}dq \\ &= \lim_{z\to p,\,\mathrm{Re}\, z<0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-z)}\frac{1}{(q-k(\xi))}\frac{1}{w^-e^{\Gamma^-}}\frac{(K_1(q)+\xi)}{(K(q)+\xi)}e^{-qy}dq \\ &= O\left(y^{-1+\gamma}\int_{\mathcal{C}_3}\frac{|q|^{-1+\gamma}}{|q-p|}\frac{1}{|q-k(\xi)|}dq\right). \end{aligned} \tag{2.144}$$

Here $\gamma \in [0,1), K(q) = -q|q| = -q^2\exp(-i\theta), \theta = \arg q$ and

$$\mathcal{C}_3 = \left\{q \in \left(\infty e^{-i((\pi-\varepsilon)/2)},0\right)\bigcup\left(0,\infty e^{i((\pi-\varepsilon)/2)}\right)\right\}, \quad \varepsilon > 0. \tag{2.145}$$

After this observation in accordance with the integral representation (2.133) by the Hölder inequality we have arrived at the following estimate for $n = 0,1$

$$\begin{aligned} &\left\|\partial_x^n\int_0^{+\infty}J_2(\cdot,y,t)\phi(y)dy\right\|_{\mathbf{L}^2} \\ &\quad \le C\|\phi\|_{\mathbf{H}^1}\int_{\mathcal{C}_1}d\xi e^{-C|\xi|t}\int_{\mathcal{C}_2}dp\frac{|p|^n|p-k(\xi)|}{|K(p)+\xi|}\int_{\mathcal{C}_3}\frac{|q|^{-1+\gamma}}{|q-p|}\frac{1}{|q-k(\xi)|}dq \le C\|\phi\|_{\mathbf{H}^1}. \end{aligned} \tag{2.146}$$

Therefore according to (2.142) and (2.146) we obtain the estimate (2.129) of the lemma. Now we prove the second estimate of the lemma.

We rewrite the function $I(p,\xi,y)$ (see (2.134)) for $p \in C_2, \xi \in C_1$ in the form

$$I(p,\xi,y) = I_1(p,\xi,y) + \frac{1}{(p-k(\xi))}\frac{1}{w^+(p,\xi)e^{\Gamma(p,\xi)}} + e^{-1/2}\frac{1}{(p-k(\xi))}\left(e^{-k(\xi)y}-1\right), \tag{2.147}$$

where

$$I_1(p,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-p)(q-k(\xi))}\left(\frac{1}{w^+(q,\xi)e^{\Gamma^+(q,\xi)}} - e^{-1/2}\right)\left(e^{-qy}-1\right)dq. \tag{2.148}$$

Here we used that since $w^+(q,\xi)e^{\Gamma^+(q,\xi)}$ is analytic for $p \in C_2, \xi \in C_1$, and $\operatorname{Re} k(\xi) > 0$ by Cauchy theorem,

$$\begin{gathered}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-p)(q-k(\xi))}\frac{1}{w^+(q,\xi)e^{\Gamma^+(q,\xi)}}dq = \frac{1}{(p-k(\xi))}\frac{1}{w^+(p,\xi)e^{\Gamma^+(p,\xi)}}\\ \frac{e^{-1/2}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-p)(q-k(\xi))}\left(e^{-qy}-1\right)dq = e^{-1/2}\frac{1}{(p-k(\xi))}\left(e^{-k(\xi)y}-1\right).\end{gathered} \tag{2.149}$$

Substituting (2.147) into definition of the function $J_2(x,y,t)$ (see (2.133)) we get

$$\begin{aligned}J_2(x,y,t) = &-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1}d\xi e^{\xi t}\int_{C_2}e^{px}\frac{\Upsilon^+(p,\xi)(p-k(\xi))}{K(p)+\xi}I_1(p,\xi,y)dp\\ &-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1}d\xi e^{\xi t}\int_{C_2}e^{px}\frac{1}{K(p)+\xi}dp\\ &-\frac{e^{-1/2}}{2\pi i}\int_{C_1}d\xi e^{\xi t}\left(e^{-k(\xi)y}-1\right)\int_{C_2}e^{px}\frac{\Upsilon^+(p,\xi)}{K(p)+\xi}dp.\end{aligned} \tag{2.150}$$

Since by Cauchy theorem

$$-\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1}d\xi e^{\xi t}\int_{C_2}e^{px}\frac{1}{K(p)+\xi}dp = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px-K(p)t}dp, \tag{2.151}$$

we obtain the following form for Green function $G(x,y,t)$ (see (2.131)):

$$G(x,y,t) = \sum_{j=1}^{3}F_j(x,y,t), \tag{2.152}$$

where

$$F_1(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}\left(e^{-py}-1\right)dp, \tag{2.153}$$

$$F_2(x,y,t) = \frac{1}{2\pi i}\frac{e^{-1/2}}{2\pi i}\int_{C_1} d\xi e^{\xi t}\left(e^{-k(\xi)y}-1\right)\int_{C_2} e^{px}\frac{\Upsilon^+(p,\xi)}{K(p)+\xi}dp, \tag{2.154}$$

$$F_3(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2} e^{px}\frac{\Upsilon^+(p,\xi)(p-k(\xi))}{K(p)+\xi}I_1(p,\xi,y)dp, \tag{2.155}$$

$$I_1(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{(q-z)(q-k(\xi))}\left(\frac{1}{\Upsilon^+(q,\xi)}-e^{-1/2}\right)\left(e^{-qy}-1\right)dq. \tag{2.156}$$

Denote

$$\mathcal{F}_j\phi = \int_0^{+\infty} F_j(x,y,t)\phi(y)dy. \tag{2.157}$$

Now we prove estimate

$$\left\|\partial_x^n\mathcal{F}_1\phi\right\|_{\mathbf{L}^{s,\mu}} \le Ct^{(-1/2)(n+1+\delta-(1/s)-\mu)}\|\phi\|_{\mathbf{L}^{1,\delta}}. \tag{2.158}$$

We have

$$\begin{aligned} F_1(x,y,t) &= \frac{1}{2\pi i\sqrt{t}}\int_{-i\infty}^{i\infty} e^{|p|p}e^{pxt^{-1/2}}\left(e^{-pyt^{-1/2}}-1\right)dp \\ &= \frac{1}{\pi\sqrt{t}}\operatorname{Re}\int_0^{\infty} e^{ip^2-ipyt^{-1/2}}e^{ipxt^{-1/2}}\left(1-e^{ipyt^{-1/2}}\right)dp \\ &= \frac{1}{\pi\sqrt{t}}\operatorname{Re}\int_0^{y/\sqrt{t}} e^{ip^2}e^{ipxt^{-1/2}}\left(e^{-ipyt^{-1/2}}-1\right)dp \\ &\quad+\frac{1}{\pi\sqrt{t}}\operatorname{Re}\int_{y/\sqrt{t}}^{\infty e^{i\phi}} e^{ip^2-ipyt^{-1/2}}e^{ipxt^{-1/2}}\left(1-e^{ipyt^{-1/2}}\right)dp = I_{11}+I_{12}, \end{aligned} \tag{2.159}$$

where $\phi > 0$ is such that $\operatorname{Re}\, ip^2 > 0$.

By Bonnet Theorem we get

$$\begin{aligned}
|\partial_x^n I_{11}| &\le Ct^{(-1/2)(1+n)}\left|1-e^{iy^2t^{-1}}\right|\left|\int_k^{y/\sqrt{t}} e^{ip^2}e^{ipxt^{-1/2}}p^n\,dp\right| \\
&\le Ct^{(-1/2)(1+n)}\left|1-e^{iy^2t^{-1}}\right|\left|\int_k^{\infty} e^{ip^2}e^{ipxt^{-1/2}}p^n\,dp-\int_{y/\sqrt{t}}^{\infty} e^{ip^2}e^{ipxt^{-1/2}}p^n\,dp\right| \\
&\le Cy^{\delta}t^{(-1/2)(1+\delta+n)}\int_0^{\infty e^{i\phi}} e^{-C|p|^2}e^{-C|p|xt^{-1/2}}|p|^n|dp|,
\end{aligned} \tag{2.160}$$

where $k\in[0,y/\sqrt{t}]$ is intermediate point. Therefore

$$\|\partial_x^n I_{11}\|_{\mathbf{L}^{s,\mu}} \le Cy^{\delta}t^{(-1/2)(1+\delta+n-(1/s)-\mu)}\int_0^{\infty e^{i\phi}} e^{-C|p|^2}|p|^{n-(1/s)-\mu}dp, \tag{2.161}$$

where

$$\frac{1}{s}+\mu<n+1. \tag{2.162}$$

Now we get estimate for I_{12} from (2.159).

Using for $p=|p|e^{i\phi}, |p|>yt^{-1/2}, C_1=\sin\phi<1$,

$$\left|e^{ip^2-ipyt^{-1/2}}\right| \le C\frac{1}{\left||p|^2-C_1|p|yt^{-1/2}\right|^{\mu_1}},\quad \mu_1\ge 0, \tag{2.163}$$

we get

$$\left|\partial_x^n I_{12}(x,y,t)\right| \le Ct^{(-1/2)(1+n)}\int_{yt^{-1/2}}^{\infty}\frac{1}{|p^2-C_1pyt^{-1/2}|^{\mu_1}}e^{-C|p|xt^{-1/2}}|p|^n\left|e^{-C|p|yt^{-1/2}}-1\right|dp. \tag{2.164}$$

Therefore from (2.170) we obtain

$$\begin{aligned}
\left\|\partial_x^n \mathrm{I}_{12}(\cdot,y,t)\right\|_{\mathbf{L}^{s,\mu}} &\le Ct^{(-1/2)(n+1-(1/s)-\mu+\delta)}y^{\delta}\int_{yt^{-1/2}}^{\infty}\frac{p^{n-(1/s)-\mu+\delta}}{|p^2-Cpyt^{-1/2}|^{\mu_1}}dp \\
&\le Ct^{(-1/2)(n+2-(1/s)-\mu)}y^{\delta}\left(\int_0^1 p^{n+\delta-(1/s)-\mu}dp+\int_1^{\infty}\frac{p^{n+\delta-(1/s)-\mu}}{p^{2\mu_1}}dp\right) \\
&\le Ct^{(-1/2)(n+1-(1/s)-\mu+\delta)}y^{\delta},
\end{aligned} \tag{2.165}$$

where

$$\frac{1}{s} + \mu < n + 1 + \delta. \tag{2.166}$$

Therefore

$$\left\| \partial_x^n \mathcal{I}_{12} \phi \right\|_{\mathbf{L}^{s,\mu}} \le C t^{(-1/2)(n+1+\delta-(1/s)-\mu)} \|\phi\|_{\mathbf{L}^{1,\delta}}. \tag{2.167}$$

From (2.161) and (2.167) it follows estimate (2.158).

Now we estimate $\mathcal{F}_2\phi$.

Making the change of variables $\xi = qt$ and $p = \sqrt{t}z$ in (2.154) we get

$$\left| \partial_x^n F_2(x,y,t) \right| \le C t^{(-1/2)(1+\delta+n)} \int_{C_1} dq e^{-C|q|} O\left((k(q)y)^{\delta} \right) \int_{C_2} e^{-C|z|xt^{-1/2}} \frac{|z|^n \{|z|\}^{1/2}}{|K(z)+q|} |dz|. \tag{2.168}$$

Here we used the following estimations:

$$\begin{gathered} e^{-k(\xi)y} - 1 = O\left((k(\xi)y)^{\delta} \right), \quad \delta \in [0,1], \\ \left| w^+ e^{\Gamma(p,\xi)} \right| \le C\{|p|\}^{1/2}. \end{gathered} \tag{2.169}$$

Since

$$\begin{aligned} \left\| e^{-C|z|xt^{-1/2}} \right\|_{\mathbf{L}^{s,\mu}} &\le C \left(\int_0^{+\infty} x^{\mu s} e^{-Cst^{-1/2}|z|} dx \right)^{1/s} \\ &\le C \left(|z| t^{-1/2} \right)^{(-1/s)-\mu} \left(\int_0^{+\infty} x_1^{\mu s} e^{-Csx_1} dx_1 \right)^{1/s}, \end{aligned} \tag{2.170}$$

we get

$$\begin{aligned} \|F_2\|_{\mathbf{L}^{s,\mu}} &\le C t^{(-1/2)(1+\delta+n-(1/s)-\mu)} y^{\delta} \int_{C_1} e^{-C|q|} k^{\delta}(|q|) dq \int_{C_2} \frac{|z|^{n-(1/s)-\mu} \{|z|\}^{1/2}}{|K(z)+q|} |dz| \\ &\le C t^{(-1/2)(1+\delta+n-(1/s)-\mu)} y^{\delta}, \end{aligned} \tag{2.171}$$

where

$$\frac{1}{s} + \mu < n + \frac{3}{2}, \quad (\delta, \mu) \in [0,1]. \tag{2.172}$$

From the estimate (2.171) we have under condition (2.172)

$$\left\| \partial_x^n \mathcal{F}_2 \phi \right\|_{\mathbf{L}^{s,\mu}} \le C t^{(-1/2)(1+\delta+n-(1/s)-\mu)} \|\phi\|_{\mathbf{L}^{1,\delta}}, \tag{2.173}$$

Now we estimate $\mathcal{F}_3\phi$. From estimate (2.138) for $\gamma \in (0,1)$ we have

$$\frac{1}{e^{\Gamma^+}w^+} - e^{(-1/2)} = O\left(\frac{|\xi|^{\gamma}}{|p|^{2\gamma}}\right), \tag{2.174}$$

and therefore for $p \in \mathcal{C}_2$

$$\left|I_1^+(p,\xi,y)\right| \le Cy^{\delta}\int_{-i\infty}^{i\infty}\frac{|q|^{\delta}}{|q-p||q-k(\xi)|}\frac{|\xi|^{\gamma}}{|q|^{2\gamma}}dq. \tag{2.175}$$

Thus after the change of variables $\xi t = q_1, p = z\sqrt{t}, q = q_2\sqrt{t}$ we get

$$\begin{aligned}\partial_x^n F_3(x,y,t) \le\, & Cy^{\delta}t^{(-1/2)(1+\delta+n)}\int_{\mathcal{C}_1}dq_1 e^{-C|q_1|}|q_1|^{\gamma}\\ &\times\int_{\mathcal{C}_2}dz e^{-C|z|xt^{-1/2}}\frac{|z-k(q_1)||z|^n\{|z|\}^{1/2}}{|K(z)+q_1|}\\ &\times\int_{-i\infty}^{i\infty}\frac{|q_2|^{\delta-2\gamma}}{|q_2-z||q_2-k(q_1)|}dq_2.\end{aligned} \tag{2.176}$$

From (2.170) we obtain for $\gamma \in [0,(1+\delta)/2), \delta \in [0,1]$

$$\begin{aligned}\left\|\partial_x^n\mathcal{F}_3\phi\right\|_{\mathbf{L}^{s,\mu}} \le\, & Ct^{(-1/2)(1+\delta+n-(1/s)-\mu)}\left\|y^{\delta}\phi\right\|_{\mathbf{L}^1}\\ &\times\int_{\mathcal{C}_1}e^{-C|q_1|}|q_1|^{\gamma}dq_1\int_{\mathcal{C}_2}dz\frac{|z-k(q_1)|}{|K(z)+q_1|}|z|^{n-(1/s)-\mu}\{|z|\}^{1/2}\\ &\times\int_{-i\infty}^{i\infty}\frac{|q_2|^{\delta-2\gamma}}{|q_2-z||q_2-k(q_1)|}dq_2 \le Ct^{(-1/2)(1+\delta+n-(1/s)-\mu)}\|\phi\|_{\mathbf{L}^{1,\delta}},\end{aligned} \tag{2.177}$$

where

$$\frac{1}{s}+\mu < n+\frac{3}{2}, \quad (\delta,\mu) \in [0,1]. \tag{2.178}$$

Thus from (2.171), (2.177), and (2.158) it follows the second estimate (2.129) of the lemma.

Now we prove the asymptotic formula (2.130).

We will use the formula (2.115) from Lemma 2.6:

$$G(x,y,t) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}d\xi e^{\xi t}\int_{-i\infty}^{i\infty}e^{px}\frac{\Upsilon^-(p,\xi)(p-k(\xi))}{K_1(p)+\xi}I^-(p,\xi,y)dp, \tag{2.179}$$

where

$$I(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{(q-z)(q-k(\xi))} \frac{1}{\Upsilon^{+}(q,\xi)} \left(e^{-qy}-1\right) dq. \tag{2.180}$$

Since for Re $\xi > 0$

$$\int_{-i\infty}^{i\infty} \frac{\Upsilon^{-}(p,\xi)(p-k(\xi))}{K_1(p)+\xi} I^{-}(p,\xi,y)\, dp = 0, \tag{2.181}$$

we have $G(0,y,t) = 0$ and therefore

$$G(x,y,t) = \sum_{j=1}^{3} \widetilde{F}_j(x,y,t), \tag{2.182}$$

where

$$\begin{aligned}
\widetilde{F}_1(x,y,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(p)t}\left(e^{px}-1\right)\left(e^{-py}-1\right) dp, \\
\widetilde{F}_2(x,y,t) &= \frac{1}{2\pi i}\frac{e^{-1/2}}{2\pi i}\int_{C_1} d\xi e^{\xi t}\left(e^{-k(\xi)y}-1\right)\int_{C_2}\left(e^{px}-1\right)\frac{e^{\Gamma^{+}(p,\xi)}w^{+}(p,\xi)}{K(p)+\xi} dp, \\
\widetilde{F}_3(x,y,t) &= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2}\left(e^{px}-1\right)\frac{e^{\Gamma^{+}(p,\xi)}w^{+}(p,\xi)(p-k(\xi))}{K(p)+\xi} I_1(p,\xi,y)\, dp.
\end{aligned} \tag{2.183}$$

Here the function $I_1(p,\xi,y)$ was defined by (2.148). Since

$$I_1(p,\xi,y) = \widetilde{I}_1^{+}(p,\xi,y) - y\frac{p}{(p-k(\xi))}\left(\frac{1}{e^{\Gamma^{+}(p,\xi)}w^{+}(p,\xi)} - e^{-1/2}\right), \tag{2.184}$$

where

$$\begin{aligned}
\widetilde{I}_1(z,\xi,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} &\frac{1}{(q-z)(q-k(\xi))}\left(\frac{1}{e^{\Gamma^{+}(q,\xi)}w^{+}(q,\xi)} - e^{-1/2}\right) \\
&\times \left(e^{-qy}-1+qy\right) dq,
\end{aligned} \tag{2.185}$$

we have

$$\begin{aligned}
&\widetilde{F}_3(x,y,t)\\
&= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2}(e^{px}-1)\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}\widetilde{I}_1^+(p,\xi,y)dp \\
&\quad + \frac{1}{2\pi i}\frac{y}{2\pi i}\int_{C_1} d\xi e^{\xi t}\int_{C_2}(e^{px}-1)\frac{\Upsilon^+(p,\xi)p}{K(p)+\xi}\left(\frac{1}{e^{\Gamma^+(p,\xi)}w^+(p,\xi)}-e^{-1/2}\right)dp.
\end{aligned} \tag{2.186}$$

So

$$G(x,y,t) = \frac{y}{t}\Lambda\left(xt^{-1/2}\right) + \sum_{k=1}^{3} R_k(x,y,t), \tag{2.187}$$

where

$$\begin{aligned}
\Lambda\left(xt^{-1/2}\right) &= \frac{e^{-1/2}}{2\pi^2 i}\int_{C_1} d\xi e^{\xi}\int_{C_2}\left(e^{pxt^{-1/2}}-1\right)\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}dp,\\
R_1(x,y,t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(p)t}(e^{px}-1)(e^{-py}-1+py)dp,\\
R_2(x,y,t) &= \frac{1}{2\pi i}\frac{e^{-1/2}}{2\pi i}\int_{C_1} d\xi e^{\xi t}\left(e^{-k(\xi)y}-1+k(\xi)y\right)\\
&\quad\times\int_{C_2}(e^{px}-1)\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)}{K(p)+\xi}dp,\\
R_3(x,y,t) &= -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{C_1} d\xi e^{\xi t}\\
&\quad\times\int_{C_2}(e^{px}-1)\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}\widetilde{I}_1^+(p,\xi,y)dp.
\end{aligned} \tag{2.188}$$

Using $e^{-py}-1+py = O(p^{1+\mu}y^{1+\mu})$, $e^{-k(\xi)y}-1+k(\xi)y = O(k(\xi)^{1+\mu}y^{1+\mu})$, $\mu\in(0,1)$ by the same method as in the proof of estimate (2.129) we obtain that

$$\sum_{j=1}^{3}\left|\int_0^\infty dy R_j(x,y,t)\phi(y)\right| \le Ct^{-1-(\mu/2)}\min\left(xt^{-1/2},1\right)\|\phi\|_{\mathbf{L}^{1,1+\mu}}. \tag{2.189}$$

Since

$$\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}-\frac{e^{1/2}}{\sqrt{p^2+1}}=O\left(\frac{\xi}{p^2}\right),$$
$$\int_{C_1}d\xi e^{\xi}\int_{C_2}\left(e^{pxt^{-1/2}}-1\right)\frac{1}{\sqrt{p^2+1}}dp=0, \tag{2.190}$$

we rewrite

$$\Lambda\left(xt^{-1/2}\right)=\frac{e^{-1/2}}{2\pi^2 i}\int_{C_1}d\xi e^{\xi}\int_{C_2}\left(e^{pxt^{-1/2}}-1\right)\left(\frac{e^{\Gamma^+(p,\xi)}w^+(p,\xi)(p-k(\xi))}{K(p)+\xi}-\frac{e^{1/2}}{\sqrt{p^2+1}}\right)dp. \tag{2.191}$$

So $\Lambda(xt^{-1/2})\in\mathbf{L}^\infty(\mathbf{R}^+)$ and $\Lambda(0)=0$. From (2.189),(2.187) it follows (2.130) and consequently,

$$\left\|\mathcal{G}\phi-t^{-1}\Lambda(xt^{-1/2})f(\phi)\right\|_{\mathbf{L}^\infty}\le Ct^{-1-(\mu/2)}\left\|x^{1+\mu}\phi\right\|_{\mathbf{L}^1}. \tag{2.192}$$

The lemma is proved. □

3. Proof of Theorem 1.1

By Proposition 2.4 we rewrite the initial-boundary value problem (1.1) as the following integral equation:

$$u(t)=\mathcal{G}(t)u_0-\int_0^t\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau,$$
$$\mathcal{N}(u)=u_x u, \tag{3.1}$$

where $\mathcal{G}$ is the Green operator of the linear problem (2.9). We choose the space

$$\mathbf{Z}=\left\{\phi\in\mathbf{H}^1(\mathbf{R}^+)\cap\mathbf{L}^1(\mathbf{R}^+)\cap\mathbf{L}^{1,1+a}(\mathbf{R}^+)\right\} \tag{3.2}$$

with $0<a<1$ being small and the space

$$\mathbf{X}=\left\{\partial_x^n\phi\in\mathbf{C}\left([0,\infty);\mathbf{L}^2(\mathbf{R}^+)\bigcap(0,\infty);\mathbf{L}^{2,n+(a/2)}(\mathbf{R}^+)\right):\|\phi\|_{\mathbf{X}}<\infty\right\},\quad n=0,1, \tag{3.3}$$

where now the norm $\mu\in[0,a]$

$$\|\phi\|_{\mathbf{X}}=\sup_{t\ge0}\left(\sum_{n=0}^{1}\left(t^{(1-\mu)/4}\langle t\rangle^{1/2}\|\partial_x^n\phi(t)\|_{\mathbf{L}^{2,n+(\mu/2)}}+\langle t\rangle^{(1/2)(n+(3/2))}\|\partial_x^n\phi(t)\|_{\mathbf{L}^2}\right)\right) \tag{3.4}$$

reflects the optimal time decay properties of the solution. We apply the contraction mapping principle in a ball $\mathbf{X}_\rho = \{\phi \in \mathbf{X} : \|\phi\|_{\mathbf{X}} \leq \rho\}$ in the space $\mathbf{X}$ of a radius

$$\rho = \frac{1}{2C}\|u_0\|_{\mathbf{Z}} > 0. \tag{3.5}$$

For $v \in \mathbf{X}_\rho$ we define the mapping $\mathcal{M}(u)$ by formula

$$\mathcal{M}(u) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau. \tag{3.6}$$

We first prove that

$$\|\mathcal{M}(u)\|_{\mathbf{X}} \leq \rho, \tag{3.7}$$

where $\rho > 0$ is sufficiently small. By virtue of Lemma 2.7 we have

$$\begin{gathered} \|\mathcal{G}(t)\phi\|_{\mathbf{H}^1} \leq \|\phi\|_{\mathbf{H}^1}, \\ \|\partial_x^n \mathcal{G}\phi\|_{\mathbf{L}^{2,\mu}} \leq Ct^{(-1/2)(n+\delta+(1/2)-\mu)}\|\phi\|_{\mathbf{L}^{1,\delta}} \end{gathered} \tag{3.8}$$

for all $t \geq 0, \mu < n + (1/2)$. Therefore for $t < 1$

$$\|\mathcal{G}u_0\|_{\mathbf{H}^1} \leq C\|u_0\|_{\mathbf{Z}}, \tag{3.9}$$

$$\|\partial_x^n \mathcal{G}u_0\|_{\mathbf{L}^{2,n+(a/2)}} \leq Ct^{(-1/2)(n+(1/2)-n-(a/2))}\|u_0\|_{\mathbf{L}^1} \leq Ct^{-(1-a)/4}\|u_0\|_{\mathbf{Z}}. \tag{3.10}$$

For $t > 1$

$$\begin{aligned} \|\partial_x^n \mathcal{G}u_0\|_{\mathbf{L}^2} &\leq Ct^{(-1/2)(n+(3/2))}\|u_0\|_{\mathbf{L}^{1,1}} \leq Ct^{(-1/2)(n+(3/2))}\|u_0\|_{\mathbf{Z}}, \\ \|\partial_x^n \mathcal{G}u_0\|_{\mathbf{L}^{2,n+(a/2)}} &\leq Ct^{(-1/2)(n+(3/2)-n-(a/2))}\|\phi\|_{\mathbf{L}^{1,1}} \leq Ct^{-(3-a)/4}\|u_0\|_{\mathbf{Z}}. \end{aligned} \tag{3.11}$$

Thus

$$\|\mathcal{G}u_0\|_{\mathbf{X}} \leq C\|u_0\|_{\mathbf{Z}}. \tag{3.12}$$

Also since $v \in \mathbf{X}_\rho$ we get for all $\tau > 0$

$$\begin{gathered} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,1}} \leq C\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^{2,1}} \leq C\{\tau\}^{-1/4}\langle\tau\rangle^{-3/2}\|u\|_{\mathbf{X}}^2, \\ \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} \leq C\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^2} \leq C\langle\tau\rangle^{-2}\|u\|_{\mathbf{X}}^2. \end{gathered} \tag{3.13}$$

Therefore by Lemma 2.7 we get

$$\begin{aligned}
&\left\| \partial_x^n \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \right\|_{\mathbf{L}^2} \\
&\quad \leq \int_0^{t/2} \langle t-\tau\rangle^{(-1/2)(n+(3/2))} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,1}} d\tau + \int_{t/2}^{t} (t-\tau)^{(-1/2)(n+1-(1/2))} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} d\tau \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \left[\int_0^{t/2} \langle t-\tau\rangle^{(-1/2)(n+2-(1/2))} \{\tau\}^{-1/2} \langle\tau\rangle^{-(2-(1/2))} d\tau \right. \\
&\qquad\qquad \left. + \int_{t/2}^{t} (t-\tau)^{(-1/2)(n+1-(1/2))} \langle\tau\rangle^{-2} d\tau \right] \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \left[\langle t\rangle^{(-1/2)(n+2-(1/2))} + \langle t\rangle^{-2} t^{(-1/2)(n+1-(1/2))+1} \right] \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \langle t\rangle^{(-1/2)(n+(3/2))}
\end{aligned} \tag{3.14}$$

for all $t \geq 0$. In the same manner by virtue of Lemma 2.7 we have

$$\begin{aligned}
&\left\| \partial_x^n \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \right\|_{\mathbf{L}^{2,n+(a/2)}} \\
&\quad \leq \int_0^{t/2} (t-\tau)^{-(3-a)/4} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,1}} d\tau + \int_{t/2}^{t} (t-\tau)^{-1/4} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} d\tau \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \left[\int_0^{t/2} (t-\tau)^{-(3-a)/4} \{\tau\}^{-1/4} \langle\tau\rangle^{-3/2}\, d\tau + \int_{t/2}^{t} (t-\tau)^{-1/4} \langle\tau\rangle^{-2}\, d\tau \right] \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \left[\langle t\rangle^{(-1/2)(n+2-(1/2)-((1+a)/2))} + \langle t\rangle^{-2} t^{(-1/2)(n+1-(1/2))+1} \right] \\
&\quad \leq C\|u\|_{\mathbf{X}}^2 \langle t\rangle^{(-(3-a))/4} \{t\}^{-1/4}
\end{aligned} \tag{3.15}$$

for all $t \geq 0$. Thus we get

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \right\|_{\mathbf{X}} \leq C\|u\|_{\mathbf{X}}^2, \tag{3.16}$$

and hence in view of (3.6) and (3.9)

$$\begin{aligned}
\|\mathcal{M}(u)\|_{\mathbf{X}} &\leq \|\mathcal{G}u_0\|_{\mathbf{X}} + \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau \right\|_{\mathbf{X}} \\
&\leq C\|u_0\|_{\mathbf{Z}} + C\|u\|_{\mathbf{X}}^2 \leq \frac{\rho}{2} + C\rho^2 < \rho
\end{aligned} \tag{3.17}$$

since $\rho > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball $\mathbf{X}_\rho$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbf{X}} \leq \frac{1}{2}\|w - v\|_{\mathbf{X}}, \tag{3.18}$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore we see that there exists a unique solution $u \in \mathbf{C}([0,\infty); \mathbf{L}^1(\mathbf{R}^+) \cap \mathbf{L}^{1,a}(\mathbf{R}^+)) \cap \mathbf{C}((0,\infty); \mathbf{H}^1_\infty)$ to the initial-boundary value problem (1.1). Now we can prove asymptotic formula

$$u(x,t) = A_1 \Lambda\left(xt^{-1/2}\right)t^{-1} + \min\left(xt^{-1/2}, 1\right)O\left(t^{-1-\gamma}\right), \tag{3.19}$$

where

$$A_1 = f(u_0) - \int_0^{+\infty} d\tau \int_0^{+\infty} y u_y u \, dy. \tag{3.20}$$

Denote

$$G_0(t) = t^{-1}\Lambda\left(xt^{-1/2}\right). \tag{3.21}$$

From Lemma 2.7 we have

$$t\left\|\mathcal{G}(t)\phi - G_0(t) f(\phi)\right\|_{\mathbf{L}^\infty} \leq C\left\|\phi\right\|_{\mathbf{Z}} \tag{3.22}$$

for all $t > 1$. Also in view of the definition of the norm $\mathbf{X}$ we have

$$\left| f(\mathcal{N}(u(\tau)))\right| \leq \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1,1}} \leq C\{\tau\}^{-1/2}\langle\tau\rangle^{-3/2}\|u\|^2_{\mathbf{X}}. \tag{3.23}$$

By a direct calculation we have for some small $\gamma_1 > 0, \gamma > 0$

$$\begin{aligned} &\left\| \int_0^{t/2} e^{-\tau} |G_0(t-\tau) - G_0(t)| f(\mathcal{N}(u(\tau))) d\tau \right\|_{\mathbf{L}^\infty} \\ &\leq \langle t\rangle^{-1} C\|u\|^2_{\mathbf{X}} \int_0^{t/2} \|(G_0(t-\tau) + G_0(t))\|_{\mathbf{L}^\infty} \{\tau\}^{-\gamma_1}\langle\tau\rangle^{-\gamma_2} d\tau \\ &\leq C\langle t\rangle^{-2} \int_0^{t/2} \{\tau\}^{-\gamma_1}\langle\tau\rangle^{-\gamma_2} d\tau \leq C\langle t\rangle^{-\gamma-1} \end{aligned} \tag{3.24}$$

and in the same way

$$\left\| \langle t\rangle^{\gamma} G_0(t) \int_{t/2}^{\infty} f(\mathcal{N}(u(\tau))) d\tau \right\|_{\mathbf{L}^\infty} \leq C\|u\|^2_{\mathbf{X}}. \tag{3.25}$$

Also we have

$$\begin{aligned}
&\left\|\int_0^{t/2}\left(\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))-G_0(t-\tau)f(\mathcal{N}(u(\tau)))\right)d\tau\right\|_{\mathbf{L}^\infty}+\left\|\int_{t/2}^{t}\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau\right\|_{\mathbf{L}^\infty}\\
&\quad\le C\int_0^{t/2}(t-\tau)^{-1}\|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1}d\tau+C\int_{t/2}^{t}\|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1}d\tau\le Ct^{-1-\gamma}\|u\|_{\mathbf{X}}^2
\end{aligned}\tag{3.26}$$

for all $t>1$. By virtue of the integral equation (3.1) we get

$$\begin{aligned}
\langle t\rangle^{\gamma+1}\|(u(t)-AG_0(t))\|_{\mathbf{X}}&\le\left\|(\mathcal{G}(t)u_0-G_0(t)f(u_0)\right\|_{\mathbf{L}^\infty}\\
&\quad+\langle t\rangle^{\gamma+1}\left\|\int_0^{t/2}\left(\mathcal{G}(t-\tau)\mathcal{N}(a(\tau))-G_0(t-\tau)f(\mathcal{N}(u(\tau)))\right)d\tau\right\|_{\mathbf{L}^\infty}\\
&\quad+\langle t\rangle^{\gamma+1}\left\|\int_{t/2}^{t}\mathcal{G}(t-\tau)\mathcal{N}(a(\tau))d\tau\right\|_{\mathbf{L}^\infty}+\langle t\rangle^{\gamma+1}\left\|G_0(t)\int_{t/2}^{\infty}f(\mathcal{N}(u(\tau)))d\tau\right\|_{\mathbf{L}^\infty}\\
&\quad+\langle t\rangle^{\gamma+1}\left\|\int_0^{t/2}(G_0(t-\tau)-G_0(t))f(\mathcal{N}(u(\tau)))d\tau\right\|_{\mathbf{L}^\infty}.
\end{aligned}\tag{3.27}$$

The all summands in the right-hand side of (3.27) are estimated by $C\|u_0\|_{\mathbf{Z}}+C\|u\|_{\mathbf{X}}^2$ via estimates (3.24)–(3.26). Thus by (3.27) the asymptotic (3.19) is valid. Theorem is proved.

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