Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 459085, 9 pages
doi:10.1155/2011/459085

Research Article

Common Fixed-Point Problem for a Family Multivalued Mapping in Banach Space

Zhanfei Zuo

Department of Mathematics and Computer Science, Chongqing Three Gorges University, Wanzhou 404000, China

Correspondence should be addressed to Zhanfei Zuo, zuozhanfei@163.com

Received 11 December 2010; Revised 15 March 2011; Accepted 17 March 2011

Academic Editor: Vittorio Colao

It is our purpose in this paper to prove two convergents of viscosity approximation scheme to a common fixed point of a family of multivalued nonexpansive mappings in Banach spaces. Moreover, it is the unique solution in F to a certain variational inequality, where $F := \cap_{n=0}^{\infty} F(T_n)$ stands for the common fixed-point set of the family of multivalued nonexpansive mapping $\{T_n\}$.

1. Introduction

Let X be a Banach space with dual X^*, and let K be a nonempty subset of X. A gauge function is a continuous strictly increasing function $\varphi : R^+ \to R^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$. The duality mapping $J_\varphi : X \to X^*$ associated with a gauge function φ is defined by $J_\varphi(x) := \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}$, $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the particular case $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the normalized duality map. We note that $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$. It is known that if X is smooth, then J_φ is single valued and norm to weak* continuous (see [1]). When $\{x_n\}$ is a sequence in X, then $x_n \to x (x_n \rightharpoonup x, x_n \rightharpoonup^* x)$ will denote strong (weak, weak*) convergence of the sequence $\{x_n\}$ to x. s

Following Browder [2], we say that a Banach space X has the weakly continuous duality mapping if there exists a gauge function φ for which the duality map J_φ is single valued and weak to weak* sequentially continuous, that is, if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak* to $J_\varphi(x)$. It is known that $l_p (1 < p < 1)$ spaces have a weakly continuous duality mapping J_φ with a gauge $\varphi(t) = t^{p-1}$. Setting

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad t \ge 0, \tag{1.1}$$

it is easy to see that $\Phi(t)$ is a convex function and $J_\varphi(x) = \partial\Phi(\|x\|)$, for $x \in X$, where ∂ denotes the subdifferential in the sense of convex analysis. We will denote by 2^X the family of all subsets of X, by $CB(X)$ the family of all nonempty closed bounded subsets of X, and by $C(X)$ the family of all nonempty compact subsets of X. A multivalued mapping $T : K \to 2^X$ is said to be nonexpansive (resp., contractive) if

$$H(Tx,Ty) \leq \|x-y\|, \quad x,y \in K, \\ (\text{resp., } H(Tx,Ty) \leq k\|x-y\|, \quad \text{for some } k \in (0,1)), \tag{1.2}$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) := \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad A,B \in CB(X). \tag{1.3}$$

Since Banach's contraction mapping principle was extended nicely to multivalued mappings by Nadler in 1969 (see [3]), many authors have studied the fixed-point theory for multivalued mappings.

In this paper, we construct two viscosity approximation sequences for a family of multivalued nonexpansive mappings in Banach spaces. Let K be a nonempty closed convex subset of Banach space X and let $T_n : K \to C(K)$, $n = 1,2\ldots$ be a family of multivalued nonexpansive mapping, $f : K \to K$ is a contraction mapping with constant $\alpha \in (0,1)$. Let $\alpha_n \in (0,1)$, $\beta_n \in (0,1)$. For any given $x_0 \in K$, let $y_0 \in T_0x_0$ such that

$$x_1 = \alpha_0 f(x_0) + (1-\alpha_0)y_0. \tag{1.4}$$

From Nadler Theorem (see [3]), we can choose $y_1 \in T_1x_1$ such that

$$\|y_0 - y_1\| \leq H(T_0x_0, T_1x_1). \tag{1.5}$$

Inductively, we can get the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad \forall n \in N, \tag{1.6}$$

where, for each $n \in N$, $y_n \in T_nx_n$ such that

$$\|y_{n+1} - y_n\| \leq H(T_{n+1}x_{n+1}, T_nx_n). \tag{1.7}$$

Similarly, we also have the following multivalued version of the modified Mann iteration:

$$x_{n+1} = \beta_n f(x_n) + \alpha_n x_n + (1-\alpha_n-\beta_n)y_n, \tag{1.8}$$

and $y_n \in T_nx_n$ such that $\|y_{n+1} - y_n\| \leq H(T_{n+1}x_{n+1}, T_nx_n)$. Then, $\{x_n\}$ is said to satisfy Condition (A$'$) if for any subsequence $x_{n_k} \rightharpoonup x$ and $d(x_{n+1}, T_n(x_n)) \to 0$ implies that $x \in F$, where $F := \cap_{i=0}^{\infty} F(T_n) \neq \emptyset$ is the common fixed-point set of the family of multivalued mapping $\{T_n\}$. We give an example of a family of multivalued nonexpansive mappings with Condition (A$'$) as follows.

Example 1.1. Take $X = R$ and $T_n = T$ (for all $n \geq 0$), where T is defined by

$$T(x) = \begin{cases} \{0\}, & x \leq 1, \\ \left\{x - \dfrac{1}{2}, \dfrac{1}{2} - x\right\}, & \text{otherwise}. \end{cases} \tag{1.9}$$

Let $f : R \longrightarrow \{0\}$ and $\alpha_n = 1/n$, $n \geq 2$, then $F = \{0\}$ and the iteration (1.6), reduced to

$$x_{n+1} = \left(1 - \frac{1}{n+2}\right) y_n, \quad \forall n \geq 0, \tag{1.10}$$

where $y_n \in Tx_n$, and it satisfies Condition (A$'$). In fact, if $x_0 \leq 1$, then (for all $n \in \mathbb{N}, n > 0$)$x_n = 0$ and Condition (A$'$) is automatically satisfied. If $x_0 > 1$, then there exists an integer $p \geq 2$, such that

$$x_0 \in \left(\frac{p(p+1)}{4} - \frac{1}{2}, \frac{(p+1)(p+2)}{4} - \frac{1}{2}\right], \qquad x_{p-1} = \frac{1}{p}\left(x_0 - \frac{p(p-1)}{4}\right). \tag{1.11}$$

Then, $y_p \in Tx_{p-1} = \{0\}$; hence, $x_n = 0$ (for all $n \geq p$), from which we deduce that Condition (A$'$) is satisfied.

2. Preliminaries

Let $K \subset X$ be a closed convex and Q a mapping of X onto K, then Q is said to be sunny if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in X$ and $t \geq 0$. A mapping Q of X into X is said to be a retraction if $Q^2 = Q$. A subset K of X is said to be a sunny nonexpansive retract of X if there exists a sunny nonexpansive retraction of X onto K, and it is said to be a nonexpansive retract of X. If $X = H$, the metric projection P is a sunny nonexpansive retraction from H to any closed convex subset of H. The following Lemmas will be useful in this paper.

Lemma 2.1 (see [4]). *Let K be a nonempty convex subset of a smooth Banach space X, let $J : X \longrightarrow X^*$ be the (normalized) duality mapping of X, and let $Q : X \longrightarrow K$ be a retraction, then the following are equivalent:*

(1) $\langle x - Px, j(y - Px)\rangle \leq 0$ *for all $x \in X$ and $y \in K$,*

(2) *Q is both sunny and nonexpansive.*

We note that Lemma 2.1 still holds if the normalized duality map J is replaced with the general duality map J_φ, where φ is a gauge function.

Lemma 2.2 (see [5]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \beta_n, \quad n \geq 0, \tag{2.1}$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\beta_n\}$ *is a real number sequence such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty}(\beta_n/\gamma_n) \le 0$ *or* $\sum_{n=0}^{\infty} |\beta_n| < \infty$,

then $\{a_n\}$ *converges to zero, as* $n \to \infty$.

Lemma 2.3 (see [1]). *Let X be a real Banach space, then for all* $x, y \in X$, *one gets that*

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, j_\varphi(x+y)\rangle, \quad \forall j_\varphi \in J_\varphi. \tag{2.2}$$

Lemma 2.4 (see [6]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X such that*

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n) y_n, \quad n \ge 0, \tag{2.3}$$

where $\{\gamma_n\}$ *is a sequence in* $[0,1]$ *such that*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1. \tag{2.4}$$

Assume that $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$, *then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

3. Main Results

Theorem 3.1. *Let X be a reflexive Banach space with weakly sequentially continuous duality mapping* J_φ *for some gauge* φ, *let K be a nonempty closed convex subset of X, and let* $T_n : K \to C(K)$, $n = 0, 1, 2\ldots$, *be a family of multivalued nonexpansive mappings with* $F \neq \emptyset$ *which is sunny nonexpansive retract of K with Q a nonexpansive retraction. Furthermore,* $T_n(p) = \{p\}$ *for any fixed-point* $p \in F$, $\{x_n\}$ *is defined by* (1.6), *and* $\alpha_n \in (0,1)$ *satisfies the following conditions:*

(1) $\alpha_n \to 0$ *as* $n \to \infty$,

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(3) $\{x_n\}$ *satisfies Condition* (A').

Then, $\{x_n\}$ *converges strongly to a common fixed-point* $\overline{x} = Q(f(\overline{x}))$ *of a family* T_n, $n = 0, 1, 2\ldots$, *as* $n \to \infty$. *Moreover,* $\overline{x}$ *is the unique solution in F to the variational inequality*

$$\langle f(\overline{x}) - \overline{x}, j_\varphi(y - \overline{x})\rangle \le 0, \quad \forall y \in F. \tag{3.1}$$

Proof. First, we show the uniqueness of the solution to the variational inequality (3.1) in X. In fact, let $\overline{y} \in F$ be another solution of (3.1) in F, then we have

$$\langle f(\overline{x}) - \overline{x}, j_\varphi(\overline{y} - \overline{x})\rangle \le 0, \qquad \langle f(\overline{y}) - \overline{y}, j_\varphi(\overline{x} - \overline{y})\rangle \le 0. \tag{3.2}$$

From (3.2), we have that

$$(1-\alpha)\varphi(\|\overline{x} - \overline{y}\|)\|\overline{x} - \overline{y}\| \le 0. \tag{3.3}$$

We must have $\overline{x} = \overline{y}$ and the uniqueness is proved. Let $p \in F$, then, from iteration (1.6), we obtain that

$$\begin{aligned}\|x_{n+1} - p\| &\le \|x_{n+1} - \alpha_n f(p) - (1-\alpha_n)p\| + \|\alpha_n f(p) + (1-\alpha_n)p - p\| \\ &= \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(y_n - p)\| + \alpha_n\|f(p) - p\| \\ &\le \alpha_n\alpha\|x_n - p\| + (1-\alpha_n)H(T_n x_n, T_n p) + \alpha_n\|f(p) - p\| \\ &\le (1-(1-\alpha)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\|.\end{aligned} \tag{3.4}$$

Using an induction, we obtain $\|x_n - p\| \le \max\{\|x_0 - p\|, (1/(1-\alpha))\|f(p) - p\|\}$, for all integers n, thus, $\{x_n\}$ is bounded and so are $\{T_n x_n\}$ and $\{f(x_n)\}$. This implies that

$$d(x_{n+1}, T_n(x_n)) \le \|x_{n+1} - y_n\| = \alpha_n\|f(x_n) - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.5}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle \le 0. \tag{3.6}$$

Since X is reflexive and $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup q$ such that

$$\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle = \limsup_{k\to\infty}\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n_k} - \overline{x})\rangle. \tag{3.7}$$

From (3.5) and $\{x_n\}$ satisfying Condition (A′), we obtain that $q \in F$. On the other hand, we notice that the assumption that the duality mapping J_φ is weakly continuous implies that X is smooth; from Lemma 2.1, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle &= \limsup_{k\to\infty}\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n_k} - \overline{x})\rangle \\ &= \langle f(\overline{x}) - \overline{x}, j_\varphi(q - \overline{x})\rangle \\ &= \langle Q(\overline{x}) - \overline{x}, j_\varphi(q - \overline{x})\rangle \le 0.\end{aligned} \tag{3.8}$$

Finally, we will show that $x_n \to \overline{x}$ as $n \to \infty$. From iteration (1.6) and Lemma 2.3, we get that

$$\begin{aligned}\Phi(\|x_{n+1} - \overline{x}\|) &\le \Phi(\|\alpha_n(f(x_n) - f(\overline{x})) + (1-\alpha_n)(y_n - \overline{x})\|) + \alpha_n\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle \\ &\le \Phi(\alpha_n\alpha\|x_n - \overline{x}\| + (1-\alpha_n)H(T_n x_n, T_n\overline{x})) + \alpha_n\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle \\ &\le (1-\alpha_n(1-\alpha))\Phi(\|x_n - \overline{x}\|) + \alpha_n\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\rangle.\end{aligned} \tag{3.9}$$

Lemma 2.2 gives that $x_n \to \overline{x}$ as $n \to \infty$. Moreover, $\overline{x}$ satisfying the variational inequality follows from the property of Q. □

Let $f \equiv u \in K$ in iteration (1.6) be a constant mapping, then $\overline{x} = Qu$. In fact, we have the following corollary.

Corollary 3.2. *Let $\{x_n\}$ and T_n be as in Theorem 3.1, $f \equiv u \in K$, then $\{x_n\}$ converges strongly to a common fixed-point $\overline{x} = Q(u)$ of a family T_n, $n = 0, 1, 2 \ldots$, as $n \to \infty$. Moreover, $\overline{x}$ is the unique solution in F to the variational inequality*

$$\langle u - Q(u), j_\varphi(y - Q(u)) \rangle \leq 0, \quad \forall y \in F. \tag{3.10}$$

If $X = H$, then the condition that F is a sunny nonexpansive retract of K in Theorem 3.1 is not necessary, and one has the following Corollary.

Corollary 3.3. *Let H be a Hilbert space with weakly sequentially continuous duality mapping J_φ for some gauge φ, and let $\{x_n\}$ and T_n be as in Theorem 3.1, then $\{x_n\}$ converges strongly to a common fixed-point $\overline{x} = P_F f(\overline{x})$ of a family of T_n, $n = 0, 1, 2 \ldots$, where P_F is the metric projection from K onto F.*

Proof. It is well known that H is reflexive; by Propositions 2.3 and 2.6(ii) of [7], we get that F is closed and convex, and hence the projection mapping P_F is sunny nonexpansive retraction mapping, and the result follows from Theorem 3.1. □

Corollary 3.4. *Let X be a real smooth Banach space, let K be a nonempty compact subset of X, and let T_n and $\{x_n\}$ be as in Theorem 3.1, then $\{x_n\}$ converges strongly to a common fixed-point $\overline{x} = Q(f(\overline{x}))$ of a family of T_n, $n = 0, 1, 2 \ldots$, as $n \to \infty$. Moreover, $\overline{x}$ is the unique solution in F to the variational inequality*

$$\langle f(\overline{x}) - \overline{x}, j_\varphi(y - \overline{x}) \rangle \leq 0, \quad \forall y \in F. \tag{3.11}$$

Proof. Following the method of the proof of Theorem 3.1, we get that

$$d(x_{n+1}, T_n(x_n)) \leq \|x_{n+1} - y_n\| = \alpha_n \|f(x_n) - y_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.12}$$

Next, we will show that

$$\limsup_{n \to \infty} \langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x}) \rangle \leq 0. \tag{3.13}$$

Since K is compact and $\{x_n\}$ is bounded, we can assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in K$,

$$\limsup_{n \to \infty} \langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x}) \rangle = \lim_{k \to \infty} \langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n_k} - \overline{x}) \rangle. \tag{3.14}$$

From (3.12) and $\{x_n\}$ satisfying Condition (A′), we obtain that $q \in F$. On the other hand, from the fact that X is smooth, the duality being norm to weak* continuous, and the standard characterization of retraction on F, we obtain that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(\overline{x})-\overline{x}, j_\varphi(x_{n+1}-\overline{x})\rangle &= \lim_{k\to\infty}\langle f(\overline{x})-\overline{x}, j_\varphi(x_{n_k}-\overline{x})\rangle \\
&= \langle f(\overline{x})-\overline{x}, j_\varphi(q-\overline{x})\rangle \\
&= \langle Q(\overline{x})-\overline{x}, j_\varphi(q-\overline{x})\rangle \le 0.
\end{aligned} \tag{3.15}$$

Now, following the method of the proof of Theorem 3.1, we get the required result. □

Theorem 3.5. *Let X be a reflexive Banach space with weakly sequentially continuous duality mapping J_φ for some gauge φ, let K be a nonempty closed convex subset of X, and let $T_n : K \to C(K)$, $n = 0, 1, 2\ldots$, be a family of multivalued nonexpansive mappings with $F \neq \emptyset$ which is sunny nonexpansive retract of K with Q a nonexpansive retraction. $H(T_{n+1}x, T_n y) \le \|x-y\|$ for arbitrary $n \in \mathbb{N}$. Furthermore, $T_n(p) = \{p\}$ for any fixed-point $p \in F$. $\{x_n\}$ is defined by (1.8) and α_n, β_n satisfy the following conditions:*

(i) $\beta_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty}\beta_n = \infty$,

(iii) $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$.

If $\{x_n\}$ satisfies Condition (A'), then $\{x_n\}$ converges strongly to a common fixed-point $\overline{x} = Q(f(\overline{x}))$ of a family of T_n, $n = 0, 1, 2\ldots$, as $n \to \infty$. Moreover, $\overline{x}$ is the unique solution in F to the variational inequality

$$\langle f(\overline{x})-\overline{x}, j_\varphi(y-\overline{x})\rangle \le 0, \quad \forall y \in F. \tag{3.16}$$

Proof. We first show that the sequence $\{x_n\}$ defined by (1.8) is bounded. In fact, take $p \in F$, noting that $T_n(p) = \{p\}$, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_n\|x_n-p\| + \beta_n\|f(x_n)-p\| \\
&= (1-\alpha_n-\beta_n)\|y_n-T_n p\| + \alpha_n\|x_n-p\| + \beta_n\|f(x_n)-p\| \\
&\le (1-\alpha_n-\beta_n)H(T_n x_n, T_n p) + \alpha_n\|x_n-p\| + \beta_n\|f(x_n)-p\| \\
&\le (1-\alpha_n-\beta_n)\|x_n-p\| + \alpha_n\|x_n-p\| + \beta_n\|f(x_n)-p\| \\
&\le (1-\beta_n)\|x_n-p\| + \beta_n(\alpha\|x_n-p\| + \|f(p)-p\|) \\
&\le (1+(\alpha-1)\beta_n)\|x_n-p\| + \beta_n(1-\alpha)\frac{\|f(p)-p\|}{1-\alpha}.
\end{aligned} \tag{3.17}$$

It follows from induction that

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\alpha}\right\}, \tag{3.18}$$

so are $\{y_n\}$ and $\{f(x_n)\}$. Thus, we have that

$$\lim_{n\to\infty} \beta_n \left\|f(x_n) - y_n\right\| = 0. \tag{3.19}$$

Next, we show that

$$\lim_{n\to\infty} d(x_{n+1}, T_n(x_n)) = 0. \tag{3.20}$$

Let $\lambda_n = \beta_n/(1-\alpha_n)$ and $z_n = \lambda_n f(x_n) + (1-\lambda_n)y_n$, then

$$\lim_{n\to\infty} \lambda_n = 0, \qquad x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n. \tag{3.21}$$

Therefore, we have for some appropriate constant $M > 0$ that the following inequality:

$$\begin{aligned}\|z_{n+1} - z_n\| &= \left\|\lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1})y_{n+1} - \left(\lambda_n f(x_n) + (1-\lambda_n)y_n\right)\right\| \\ &\le |\lambda_{n+1} - \lambda_n| \left\|f(x_{n+1}) - f(x_n)\right\| + \left\|y_{n+1} - y_n\right\| + \lambda_n \left\|y_n\right\| + \lambda_{n+1}\left\|y_{n+1}\right\| \\ &\le |\lambda_{n+1} - \lambda_n| \left\|f(x_{n+1}) - f(x_n)\right\| + H(T_{n+1}x_{n+1}, T_n x_n) + (\lambda_n + \lambda_{n+1})M \\ &\le |\lambda_{n+1} - \lambda_n| \left\|f(x_{n+1}) - f(x_n)\right\| + \|x_{n+1} - x_n\| + (\lambda_n + \lambda_{n+1})M\end{aligned} \tag{3.22}$$

holds. Thus, $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le \lim_{n\to\infty}(|\lambda_{n+1} - \lambda_n|\|f(x_{n+1}) - f(x_n)\| + (\lambda_n + \lambda_{n+1})M) = 0$. By Lemma 2.4, we obtain

$$\begin{gathered}\lim_{n\to\infty} \|x_n - z_n\| = 0, \\ \left\|x_n - y_n\right\| \le \|x_n - z_n\| + \left\|z_n - y_n\right\| = \|x_n - z_n\| + \lambda_n \left\|f(x_n) - y_n\right\| \longrightarrow 0.\end{gathered} \tag{3.23}$$

Therefore, we have

$$d(x_{n+1}, T_n(x_n)) \le \left\|x_{n+1} - y_n\right\| \le \beta_n \left\|f(x_n) - y_n\right\| + \alpha_n \left\|x_n - y_n\right\| \longrightarrow 0. \tag{3.24}$$

Using (3.20) and $\{x_n\}$ satisfying Condition (A′), we can use the same argumentation as Theorem 3.1 proves that $\overline{x} \in F$ and

$$\limsup_{n\to\infty} \left\langle f(\overline{x}) - \overline{x}, j_\varphi(x_{n+1} - \overline{x})\right\rangle \le 0. \tag{3.25}$$

Finally, we show that $x_n \to \overline{x}$ as $n \to \infty$. In fact, from iteration (1.8) and Lemma 2.3, we have

$$\begin{aligned}\Phi(\|x_{n+1}-\overline{x}\|) &= \Phi\left(\left\|\beta_n f(x_n)+\alpha_n x_n+(1-\alpha_n-\beta_n)y_n-\overline{x}\right\|\right)\\ &= \Phi\left(\left\|\alpha_n(x_n-\overline{x})+(1-\alpha_n-\beta_n)(y_n-\overline{x})+\beta_n\left(f(x_n)-f(\overline{x})\right)+\beta_n\left(f(\overline{x})-\overline{x}\right)\right\|\right)\\ &\le \Phi\left(\|\alpha_n(x_n-\overline{x})\|+(1-\alpha_n-\beta_n)H(T_n x_n,T_n\overline{x})+\alpha\beta_n\|x_n-\overline{x}\|\right)\\ &\quad+\beta_n\left\langle f(\overline{x})-\overline{x},j_\varphi(x_{n+1}-\overline{x})\right\rangle\\ &\le \left[1-(1-\alpha)\beta_n\right]\Phi(\|x_n-\overline{x}\|)+\beta_n\left\langle f(\overline{x})-\overline{x},j_\varphi(x_{n+1}-\overline{x})\right\rangle.\end{aligned} \tag{3.26}$$

From (ii) and (3.25), it then follows that

$$\sum_{n=0}^{\infty}(1-\alpha)\beta_n=\infty,\qquad \limsup_n \frac{\left\langle f(\overline{x})-\overline{x},j_\varphi(x_{n+1}-\overline{x})\right\rangle}{1-\alpha}\le 0. \tag{3.27}$$

Apply Lemma 2.2 to conclude that $x_n \to \overline{x}$. □

References

[1] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[2] F. E. Browder, "Convergence theorems for sequences of nonlinear operators in Banach spaces," *Mathematische Zeitschrift*, vol. 100, pp. 201–225, 1967.

[3] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[4] R. E. Bruck Jr., "Nonexpansive projections on subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.

[5] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[6] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 1, pp. 103–123, 2005.

[7] H. H. Bauschke and P. L. Combettes, "A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces," *Mathematics of Operations Research*, vol. 26, no. 2, pp. 248–264, 2001.