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Research Article

Homeomorphisms of Compact Sets in Certain Hausdorff Spaces

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We construct a class of Hausdorff spaces (compact and noncompact) with the property that nonempty compact subsets of these spaces that have the same cardinality are homeomorphic. Also, it is shown that these spaces contain compact subsets that are infinite.

1. Introduction

In this paper, we construct a class of Hausdorff spaces with the property that nonempty compact subsets of these spaces that have the same cardinality are homeomorphic (Theorem 3.7). Conditions are given for these spaces to be compact (Corollary 2.10). Also, it is shown that these spaces contain compact subsets that are infinite (Corollary 2.10).

This paper uses the Zermelo-Fraenkel axioms of set theory with the axiom of choice (see [1–3]). We let ω denote the finite ordinals (i.e., the natural numbers) and $\mathbb{N}$ denotes the counting numbers (i.e., $\mathbb{N} = \omega \setminus \{0\}$). Also, for a given set X, we denote the collection of all subsets of X by $\mathcal{P}(X)$, and we denote the cardinality of X by $|X|$. In other words, $|X|$ is the smallest ordinal number for which a bijection of $|X|$ onto X exists.

In this paper, we will only consider compact topologies that are Hausdorff. A topology τ on a set X is compact if and only if $\mathcal{A} \subseteq \tau$ and $X \subseteq \bigcup \mathcal{A}$ imply $X \subseteq \bigcup_{j=1}^{n} U_j$ for some $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \mathcal{A}$. Therefore, compact topologies need not be Hausdorff.

2. A Class of Hausdorff Spaces

Let V, W, and x_0 be sets such that W is infinite and the collection

$$\{V, W, \{x_0\}\} \tag{2.1}$$

is pairwise disjoint. For example, let $V = \{(\nu, 0) \mid \nu \in \omega\}$, $W = \{(\mu, 1) \mid \mu \in 2^{\omega}\}$, and $x_0 = 2^{2^{\omega}}$. Unless otherwise stated, we let

$$X = \{x_0\} \cup V \cup W. \tag{2.2}$$

Recall that for set Y and $G \subseteq Y$, we have

$$Y \setminus G = \{y \in Y \mid y \notin G\}. \tag{2.3}$$

Definition 2.1. Let A be an infinite set. Define

$$\mathcal{F}r(A) = \{F \in \mathcal{P}(A) \mid |A \setminus F| \in \omega\}. \tag{2.4}$$

We call $\mathcal{F}r(A)$ the Fréchet filter on A.

Note that A being infinite implies that $\mathcal{F}r(A)$ is a filter (see [4, Definition 3.1, page 48]).

Definition 2.2. Consider the collection $\mathcal{B}_1 \subseteq \mathcal{P}(X)$ defined as follows:

$$\mathcal{B}_1 = \mathcal{P}(X \setminus \{x_0\}) \cup \{F \cup \{x_0\} \mid F \in \mathcal{F}r(W)\}. \tag{2.5}$$

Proposition 2.3. *The collection $\mathcal{B}_1$ generates a Hausdorff topology τ on X.*

Proof. Clearly, $\mathcal{B}_1$ is a basis for a topology τ (see [5, Section 13]).

Let $v, w \in X$ such that $v \neq w$. If $v, w \in X \setminus \{x_0\}$, then $\{v\}, \{w\} \in \mathcal{B}_1$, $v \in \{v\}$, $w \in \{w\}$, and

$$\{v\} \cap \{w\} = \emptyset. \tag{2.6}$$

If $v = x_0$, then either $w \in V$ or $w \in W$. Assume that $w \in V$, and let $F \in \mathcal{F}r(W)$. Since $V \cap W = \emptyset$, $\{w\} \in \mathcal{P}(V) \subseteq \mathcal{B}_1$ and $F \cup \{x_0\} \in \mathcal{B}_1$, we have

$$\{w\} \cap [F \cup \{x_0\}] = \emptyset. \tag{2.7}$$

Assume that $w \in W$. Note that $\{w\} \in \mathcal{B}_1$. Also, $W \setminus \{w\} \in \mathcal{F}r(W)$, which implies $[W \setminus \{w\}] \cup \{x_0\} \in \mathcal{B}_1$ and

$$v = x_0 \in [W \setminus \{w\}] \cup \{x_0\}. \tag{2.8}$$

Observe that,

$$\{w\} \cap ([W \setminus \{w\}] \cup \{x_0\}) = \emptyset. \tag{2.9}$$

We infer that τ is Hausdorff. □

Proposition 2.4. *If $A \subseteq X \setminus \{x_0\}$, then A is compact in (X, τ) if and only if A is a finite set.*

Proof. Note that finite sets are compact in any topological space. So, assume that A is an infinite, and let

$$\mathcal{A} = \{\{a\} \mid a \in A\} \subseteq \mathcal{P}(X \setminus \{x_0\}) \subseteq \tau, \tag{2.10}$$

which implies

$$A = \bigcup \mathcal{A}. \tag{2.11}$$

Let $\mathcal{U} \subseteq \mathcal{A}$ be a nonempty, *finite* subcollection of $\mathcal{A}$. Therefore, there exists $\{a_i\}_{i=1}^{m} \subseteq A$, for some $m \in \mathbb{N}$, such that

$$\mathcal{U} = \{\{a_i\}\}_{i=1}^{m}, \tag{2.12}$$

which implies

$$\bigcup \mathcal{U} = \{a_i\}_{i=1}^{m}. \tag{2.13}$$

If $A \subseteq \bigcup \mathcal{U}$, then we would have $A \subseteq \{a_i\}_{i=1}^{m}$, contradicting A being an infinite set. Consequently, infinite subsets of $X \setminus \{x_0\}$ are not compact in the topological space (X, τ). □

Corollary 2.5. *The set W is not compact in (X, τ).*

Corollary 2.6. *The set V is compact in (X, τ) if and only if V is finite.*

Proposition 2.7. *Let $A \subseteq X \setminus \{x_0\}$. The set $A \cup \{x_0\}$ is compact in (X, τ) if and only if $A \cap V$ is a finite set.*

Proof. The topology τ on X is generated by $\mathcal{B}_1$ (see Proposition 2.3).

Assume that $A \cap V$ is an infinite set. Let $F \in \mathcal{F}r(W)$, and let $Q = F \cup \{x_0\}$. Hence,

$$x_0 \in Q, \qquad Q \in \tau. \tag{2.14}$$

Let $\mathcal{A}_A = \{\{a\} \mid a \in A\}$, and let $\mathcal{A} = \{Q\} \cup \mathcal{A}_A$. Note that $\mathcal{A}_A \subseteq \tau$, $\mathcal{A} \subseteq \tau$,

$$A = \bigcup \mathcal{A}_A, \qquad A \cup \{x_0\} \subseteq \bigcup \mathcal{A}. \tag{2.15}$$

Suppose that $\mathcal{U} \subseteq \mathcal{A}_A$ is a finite subcollection such that

$$A \cup \{x_0\} \subseteq \bigcup \mathcal{U}. \tag{2.16}$$

It can be assumed, without loss of generality, that $Q \in \mathcal{U}$ and $\mathcal{U} = \{U_i\}_{i=0}^m$ for $m+1 = |\mathcal{U}| \in \mathbb{N}$ such that $U_0 = Q$. So, by definition of $\mathcal{A}$, there exists $\{a_i\}_{i=1}^m \subseteq A$ such that $U_i = \{a_i\}$ for $i = 1, \ldots, m$. Note that $Q = F \cup \{x_0\} \subseteq W$, which implies

$$Q \cap (A \cap V) = A \cap (Q \cap V) = \emptyset. \tag{2.17}$$

So, expressions (2.16) and (2.17) would imply

$$A \cap V \subseteq \bigcup_{i=1}^{m} U_i = \{a_1, \ldots, a_m\}, \tag{2.18}$$

contradicting $A \cap V$ being an infinite set. We infer that $A \cup \{x_0\}$ is not compact in (X, τ).

Conversely, assume that $A \cap V$ is a finite set. Let $\mathcal{A} \subseteq \tau$ such that

$$A \cup \{x_0\} \subseteq \bigcup \mathcal{A}. \tag{2.19}$$

Hence, there exists $U \in \mathcal{A}$ such that $x_0 \in U$. Since $x_0 \notin E$ for $E \in \mathcal{P}(X \setminus \{x_0\})$, there exists $F \in \mathcal{F}r(W)$ such that

$$x_0 \in F \cup \{x_0\} \subseteq U. \tag{2.20}$$

Assume that $A \cap V = \{v_i\}_{i=1}^m$ for some $m \in \omega$ [we define $\{v_i\}_{i=1}^0 = \emptyset$]. Observe that,

$$A = A \cap [V \cup W] = ([A \cap F] \cup [A \cap W \setminus F]) \cup \{v_i\}_{i=1}^m. \tag{2.21}$$

Since $W \setminus F$ is finite, we can assume that

$$A \cap W \setminus F = \{a_q\}_{q=1}^k, \tag{2.22}$$

for some $k \in \omega$ (we define $\{a_q\}_{q=1}^0 = \emptyset$). Hence, there exists $\{U_i\}_{i=1}^m \subseteq \mathcal{A}$ and $\{U_q\}_{q=1}^k \subseteq \mathcal{A}$ such that

$$v_i \in U_i, \qquad a_q \in U_q, \tag{2.23}$$

for $i = 1, \ldots, m$ and $q = 1, \ldots, k$ (again, we define $\bigcup_{i=1}^0 U_i = \bigcup_{q=1}^0 U_q = \emptyset$). Consequently, from expressions (2.19), (2.20), (2.21), (2.22), and (2.23), we have

$$A \cup \{x_0\} \subseteq U \cup \left[\bigcup_{i=1}^{m} U_i\right] \cup \left[\bigcup_{q=1}^{k} U_q\right]. \tag{2.24}$$

We infer $A \cup \{x_0\}$ is compact in (X, τ). □

Corollary 2.8. *The set $W \cup \{x_0\}$ is compact in (X, τ).*

Proof. Note that $W \subseteq X \setminus \{x_0\}$. Also, $W \cap V = \emptyset$ implies that $W \cap V$ is finite. Therefore, $W \cup \{x_0\}$ is compact in (X, τ) by Proposition 2.7. □

Corollary 2.9. *The set $V \cup \{x_0\}$ is compact in (X, τ) if and only if V is finite.*

Proof. Note that $V \subseteq X \setminus \{x_0\}$ and $V = V \cap V$. Therefore, the corollary follows from Proposition 2.7. □

Corollary 2.10. *The topological space (X, τ) is compact if and only if V is finite.*

Proof. Observe that $X = (X \setminus \{x_0\}) \cup \{x_0\}$ and $V = (X \setminus \{x_0\}) \cap V$. Therefore, the corollary follows from Proposition 2.7. □

Proposition 2.11. *If $K \subseteq X$ is an infinite, compact set (in (X, τ)), then $K = A \cup \{x_0\}$ for some infinite set $A \subseteq X \setminus \{x_0\}$ such that $A \cap V$ is a finite set.*

Proof. If $x_0 \notin K$, then we would have $K \subseteq X \setminus \{x_0\}$, contradicting Proposition 2.4, since K is an infinite, compact set. Hence, $K = (K \setminus \{x_0\}) \cup \{x_0\}$ and $(K \setminus \{x_0\}) \cap V$ is a finite set by Proposition 2.7, since $(K \setminus \{x_0\}) \cup \{x_0\}$ is compact. Also, note that K being an infinite set implies $K \setminus \{x_0\}$ is an infinite set. Let $A = K \setminus \{x_0\}$. □

Notation 2.12. *Let Z be a nonempty set, and let θ be a Hausdorff topology on Z. For $z \in Z$, we let $\mathcal{N}_\theta(z)$ denote the filter of θ-neighborhoods of z; that is, $U \in \mathcal{N}_\theta(z)$ if and only if $z \in O \subseteq U$ for some $O \in \theta$.*

Recall that for $A \subseteq X$ and $x \in X$, x is an *accumulation point* of A if and only if for $U \in \mathcal{N}_\tau(x)$, there exists $a \in A$ such that $a \neq x$ and $a \in U$.

Remark 2.13. If $A \subseteq X$ and $x \in X \setminus \{x_0\}$, then x is not an accumulation point of A.

Indeed, $x \in X \setminus \{x_0\}$ implies $\{x\} \in \mathcal{N}_\tau(x)$. So, if $a \in A$ such that $a \neq x$, then $a \notin \{x\}$.

Remark 2.14. The element x_0 is an accumulation point of $X \setminus \{x_0\}$ *in* (X, τ).

Indeed, let $U \in \mathcal{N}_\tau(x_0)$. Hence, $\{x_0\} \cup F \subseteq U$ for some $F \in \mathcal{F}r(W)$. Since W is infinite and $x_0 \notin W$, we have that $F \setminus \{x_0\} = F \neq \emptyset$. Let $y \in F$. Hence, $y \in X \setminus \{x_0\}$ and $y \in U$.

Consequently, $X \setminus \{x_0\}$ is not closed [in (X, τ)], which implies $\{x_0\}$ is not open; that is, $\{x_0\} \notin \tau$. In fact, x_0 is the only element of X such that $\{x_0\} \notin \tau$.

3. Homeomorphisms of Compact Sets in (X, τ)

The following proposition is obvious and is stated without proof.

Proposition 3.1. *Let κ be an infinite cardinal. If D and G are sets such that $x_0 \notin D \cup G$ and*

$$|D| = \kappa = |G|, \tag{3.1}$$

then there exists a map $\zeta : D \cup \{x_0\} \to G \cup \{x_0\}$ such that $\zeta(x_0) = x_0$ and ζ is a bijection.

Remark 3.2. If $F \in \mathcal{F}r(W)$ and $Z \subseteq X$ is a finite set, then $F \setminus Z \in \mathcal{F}r(W)$.

Lemma 3.3. *Let J and K be nonempty subsets of $X \setminus \{x_0\}$ such that $J \cap V$ and $K \cap V$ are finite sets. Let $\varphi : J \cup \{x_0\} \to K \cup \{x_0\}$ be a bijection such that $\varphi(x_0) = x_0$. If $F \in \mathcal{F}r(W)$, then*

$$(K \cup \{x_0\}) \cap (E \cup \{x_0\}) \subseteq \varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \tag{3.2}$$

for some $E \in \mathcal{F}r(W)$.

Proof. Let $F \in \mathcal{F}r(W)$. Note that

$$J \cup \{x_0\} = [(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \bigcup (J \cap (W \setminus F)) \bigcup (J \cap V). \tag{3.3}$$

Also, the sets $J \cap (W \setminus F)$ and $J \cap V$ are finite.

Let

$$Z = \varphi\Big[(J \cap (W \setminus F)) \bigcup (J \cap V)\Big]. \tag{3.4}$$

Consequently, $Z \subseteq X \setminus \{x_0\}$ and Z is a *finite set*.

Let

$$E = F \setminus Z. \tag{3.5}$$

Therefore, $E \in \mathcal{F}r(W)$ by Remark 3.2. Note that

$$(F \cup \{x_0\}) \setminus Z = E \cup \{x_0\}. \tag{3.6}$$

So,

$$(K \cup \{x_0\}) \bigcap (E \cup \{x_0\}) = \Big[(K \cup \{x_0\}) \bigcap (F \cup \{x_0\})\Big] \setminus Z. \tag{3.7}$$

Observe that,

$$\begin{aligned} \varphi[J \cup \{x_0\}] &= \varphi\Big[[(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \bigcup (J \cap (W \setminus F)) \bigcup (J \cap V)\Big] \\ &= \varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \bigcup \varphi\Big[(J \cap (W \setminus F)) \bigcup (J \cap V)\Big] \\ &= \varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \bigcup Z. \end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned} (\varphi[J \cup \{x_0\}]) \setminus Z &= \Big(\varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})] \bigcup Z\Big) \setminus Z \\ &= (\varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})]) \setminus Z \\ &\subseteq \varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})]. \end{aligned} \tag{3.9}$$

Consequently,

$$\begin{aligned}(K \cup \{x_0\}) \cap (E \cup \{x_0\}) &= [(K \cup \{x_0\}) \cap (F \cup \{x_0\})] \setminus Z \\ &\subseteq (K \cup \{x_0\}) \setminus Z \\ &= (\varphi[J \cup \{x_0\}]) \setminus Z \\ &\subseteq \varphi[(J \cup \{x_0\}) \cap (F \cup \{x_0\})].\end{aligned} \tag{3.10}$$

□

Proposition 3.4. *Let τ be the Hausdorff topology on X generated by $\mathcal{B}_1$. Let A and B be infinite subsets of $X \setminus \{x_0\}$ such that $A \cap V$ and $B \cap V$ are finite sets. If A and B have the same cardinality (i.e., $|A| = |B|$), then any bijection of $A \cup \{x_0\}$ onto $B \cup \{x_0\}$ that has x_0 as a fixed point is a homeomorphism.*

Proof. Let $\zeta : A \cup \{x_0\} \to B \cup \{x_0\}$ be a bijection with $\zeta(x_0) = x_0$. Note that the existence of such a bijection is established by Proposition 3.1. Let $\zeta^{-1} : B \cup \{x_0\} \to A \cup \{x_0\}$ denote the inverse map of ζ.

Let λ be the topology on $A \cup \{x_0\}$ induced by τ, and let ρ be the topology induced on $B \cup \{x_0\}$ by τ.

Let $U \in \lambda$. We will show that $\zeta[U] \in \rho$.

Let $b \in \zeta[U]$. Either $b = x_0$ or $b \in B$. □

Case 1. Assume that $b = x_0$. Hence, $x_0 = \zeta^{-1}(x_0) = \zeta^{-1}(b) \in U$. So, there exists $Q \in \mathcal{B}_1$ such that $x_0 \in (A \cup \{x_0\}) \cap Q \subseteq U$. Since $x_0 \notin D$ for any $D \in \mathcal{P}(X \setminus \{x_0\})$, we have that $Q = F \cup \{x_0\}$ for some $F \in \mathcal{F}r(W)$. Therefore,

$$(A \cup \{x_0\}) \cap Q = (A \cup \{x_0\}) \cap (F \cup \{x_0\}), \tag{3.11}$$

which implies

$$x_0 \in (A \cup \{x_0\}) \cap (F \cup \{x_0\}) \subseteq U, \tag{3.12}$$

which implies

$$b = \zeta(x_0) \in \zeta[(A \cup \{x_0\}) \cap (F \cup \{x_0\})] \subseteq \zeta[U]. \tag{3.13}$$

By Lemma 3.3, there exists $E \in \mathcal{F}r(W)$ such that

$$(B \cup \{x_0\}) \cap (E \cup \{x_0\}) \subseteq \zeta[(A \cup \{x_0\}) \cap (F \cup \{x_0\})]. \tag{3.14}$$

Let $G = (B \cup \{x_0\}) \cap (E \cup \{x_0\})$. Note that $E \cup \{x_0\} \in \tau$. Hence, $G \in \rho$ and

$$b = x_0 \in G \subseteq \zeta[(A \cup \{x_0\}) \cap (F \cup \{x_0\})] \subseteq \zeta[U]. \tag{3.15}$$

Case 2. Assume that $b \in B$. Consequently, $\{b\} \in \mathcal{P}(X \setminus \{x_0\})$, which implies $\{b\} \in \tau$, which implies $\{b\} \in \rho$ (since $\{b\} = \{b\} \cap [B \cup \{x_0\}]$). Therefore, $b \in \zeta[U]$ implies

$$b \in \{b\} \subseteq \zeta[U]. \tag{3.16}$$

From expression (3.15) in Case 1 and expression (3.16) in Case 2, we infer

$$\zeta[U] \in \rho. \tag{3.17}$$

Let $S \in \rho$. We will show that $\zeta^{-1}[S] \in \lambda$.
Let $a \in \zeta^{-1}[S]$. Either $a = x_0$ or $a \in A$.

Case 3. Assume that $a = x_0$. Hence, $x_0 = \zeta(x_0) = \zeta(a) \in S$. So, there exists $M \in \mathcal{B}_1$ such that $x_0 \in (B \cup \{x_0\}) \cap M \subseteq S$. Since $x_0 \notin D$ for any $D \in \mathcal{P}(X \setminus \{x_0\})$, we have that $M = H \cup \{x_0\}$ for some $H \in \mathcal{F}r(W)$. Therefore,

$$(B \cup \{x_0\}) \cap M = (B \cup \{x_0\}) \cap (H \cup \{x_0\}), \tag{3.18}$$

which implies

$$x_0 \in (B \cup \{x_0\}) \cap (H \cup \{x_0\}) \subseteq S, \tag{3.19}$$

which implies

$$a = \zeta^{-1}(x_0) \in \zeta^{-1}[(B \cup \{x_0\}) \cap (H \cup \{x_0\})] \subseteq \zeta^{-1}[S]. \tag{3.20}$$

By Lemma 3.3, there exists $C \in \mathcal{F}r(W)$ such that

$$(A \cup \{x_0\}) \cap (C \cup \{x_0\}) \subseteq \zeta^{-1}[(B \cup \{x_0\}) \cap (H \cup \{x_0\})]. \tag{3.21}$$

Let $T = (A \cup \{x_0\}) \cap (C \cup \{x_0\})$. Note that $C \cup \{x_0\} \in \tau$. Hence, $T \in \lambda$ and

$$a = x_0 \in T \subseteq \zeta^{-1}[(B \cup \{x_0\}) \cap (H \cup \{x_0\})] \subseteq \zeta^{-1}[S]. \tag{3.22}$$

Case 4. Assume that $a \in A$. Consequently, $\{a\} \in \mathcal{P}(X \setminus \{x_0\})$, which implies $\{a\} \in \tau$, which implies $\{a\} \in \lambda$ (since $\{a\} = \{a\} \cap [A \cup \{x_0\}]$). Therefore, $a \in \zeta^{-1}[S]$ implies

$$a \in \{a\} \subseteq \zeta^{-1}[S]. \tag{3.23}$$

From expression (3.22) in Case 3 and expression (3.23) in Case 4, we infer that

$$\zeta^{-1}[S] \in \lambda. \tag{3.24}$$

Consequently, from expression (3.17) and (3.24), we infer that ζ is a homeomorphism of $A \cup \{x_0\}$ onto $B \cup \{x_0\}$.

Proposition 3.5. *Let τ be the Hausdorff topology on X generated by $\mathcal{B}_1$. Let $K \subseteq X$ be a compact set. If $W \subseteq K$, then $W \cup \{x_0\}$ is homeomorphic to K.*

Proof. Note that $W \subsetneq K$ implies $|W| \leq |K|$, which implies K in an infinite set (W is an infinite set), which implies $K = A \cup \{x_0\}$ for some $A \subseteq X \setminus \{x_0\}$ such that $A \cap V$ is a finite set by Proposition 2.11. Consequently, $W \subsetneq K$ and $x_0 \notin W$ imply $W \subseteq A$. So,

$$A = A \cap (W \cup V) = W \cup (A \cap V), \tag{3.25}$$

which implies

$$|A| = |W| + |A \cap V| = |W|, \tag{3.26}$$

(see [2, Corollary 2.3, page 162]). Therefore, $W \cup \{x_0\}$ is homeomorphic to $A \cup \{x_0\} = K$ by Proposition 3.4. □

Remark 3.6. Let Z be a nonempty set and let θ be a Hausdorff topology on Z. If $A \subseteq Z$ is a nonempty finite set, then θ induces the discrete topology on A.

Theorem 3.7. *Let τ be the Hausdorff topology on X generated by $\mathcal{B}_1$, and let K_1 and K_2 be nonempty compact subsets of X. If K_1 and K_2 have the same cardinality (i.e., $|K_1| = |K_2|$), then there exists a homeomorphism of K_1 onto K_2.*

Proof. Let K_1 and K_2 be nonempty compact subsets of X such that $|K_1| = |K_2|$. If K_1 is a finite set, then K_2 is a finite set. Hence, τ induces the discrete topology on K_1 and K_2 (see Remark 3.6). Consequently, any bijection of K_1 onto K_2 is a homeomorphism.

Assume that K_1 is an infinite set. Hence, K_2 is an infinite set. So, $K_1 = A \cup \{x_0\}$ and $K_2 = B \cup \{x_0\}$ such that $A \subseteq X \setminus \{x_0\}$, $B \subseteq X \setminus \{x_0\}$, $A \cap V$ is a finite set and $B \cap V$ is a finite set by Proposition 2.11. Observe that K_1 and K_2 being infinite sets imply A and B are infinite subsets of $X \setminus \{x_0\}$ such that $|A| = |B|$. Therefore, there exists a homeomorphism of K_1 onto K_2 by Proposition 3.4. □

4. Examples

Example 4.1. Let $W = \{1/n\}_{n=1}^{\infty}$, $V = \emptyset$, and $x_0 = 0$. Let

$$X = V \cup W \cup \{x_0\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}, \tag{4.1}$$

and let

$$\mathcal{B}_1 = \mathcal{P}\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right) \cup \left\{F \cup \{x_0\} \mid F \in \mathcal{F}r\left(\left\{\frac{1}{n}\right\}_{n=1}^{\infty}\right)\right\}. \tag{4.2}$$

Consider the Hausdorff space (X, τ), where the topology τ is generated by $\mathcal{B}_1$. Observe that (X, τ) is compact (Corollary 2.10, since $V = \emptyset$) and $W = \{1/n\}_{n=1}^{\infty}$ is not compact by Corollary 2.5. If $K \subseteq X$ is an infinite compact set, then

$$|K| = \omega = \left|\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}\right|, \tag{4.3}$$

therefore, K is homeomorphic to $\{1/n\}_{n=1}^{\infty} \cup \{0\}$ (Theorem 3.7). In other words, all infinite, compact subsets of $\{1/n\}_{n=1}^{\infty} \cup \{0\}$ are homeomorphic. Note that topology τ on X is induced by the standard euclidean topology on $\mathbb{R}$.

Example 4.2. Let B be a set such that $|B| = \omega$. Let $\varphi : \omega \to B$ be a bijection. For $n \in \omega$, we will denote $\varphi(n)$ by x_n, that is, $x_n = \varphi(n)$. Therefore, $B = \{x_n\}_{n\in\omega}$, where $m, n \in \omega$ and $m \neq n$ imply $x_m \neq x_n$. Let $W = \{x_{2j}\}_{j\in\mathbb{N}}$ and let $V = \{x_{2j+1}\}_{j\in\omega}$. Note that the maps $j \to x_{2j}$ and $j \to x_{2j+1}$ are bijections of $\mathbb{N}$ onto W and ω onto V, respectively. Also, $x_0 \notin W \cup V$, $W \cap V = \emptyset$ and $B = \{x_0\} \cup V \cup W$; consequently, $V = [B \setminus \{x_0\}] \setminus W$. We can write $\mathcal{B}_1 \subseteq \mathcal{P}(B)$ (see Definition 2.2) as follows.

$$\mathcal{B}_1 = \mathcal{P}(\{x_n\}_{n=1}^{\infty}) \cup \{F \cup \{x_0\} \mid F \in \mathcal{F}r(W)\}. \tag{4.4}$$

The collection $\mathcal{B}_1$ generates a Hausdorff topology τ on B (see Proposition 2.3). Note that (B, τ) is not compact (Corollary 2.10), sets V and W are not compact (Corollaries 2.6 and 2.5, resp.), $V \cup \{x_0\}$ is not compact (Corollary 2.9), and $W \cup \{x_0\}$ is compact (Corollary 2.8). Also, if $K \subseteq B$ is an infinite compact set, then

$$|K| = \omega = |W \cup \{x_0\}|, \tag{4.5}$$

therefore, K is homeomorphic to $W \cup \{x_0\} = \{x_{2j}\}_{j\in\omega}$ (Theorem 3.7). In other words, all infinite, compact subsets of $B = \{x_n\}_{n\in\omega}$ are homeomorphic.

Example 4.3. Let θ be an infinite cardinal and consider the collection $\{\kappa_n\}_{n\in\omega}$ of infinite cardinals defined as follows. Let $\kappa_0 = \omega$ and for $n \in \omega$, let $\kappa_{n+1} = 2^{\kappa_n}$. Also, we denote $\kappa_\omega = \bigcup_{n\in\omega} \kappa_n$. Hence, κ_ω is the cardinal number, that is, the supremum of $\{\kappa_n\}_{n\in\omega}$ and $\kappa_n < \kappa_\omega$ for each $n \in \omega$ (see the Alephs section in [3], page 29). For $n \in \omega$, define

$$W_n = \{(\mu, n) \mid \mu \in \kappa_n\}, \qquad W_\omega = \{(\mu, \omega) \mid \mu \in \kappa_\omega\}. \tag{4.6}$$

Observe that $|W_\omega| = \kappa_\omega$, $|W_n| = \kappa_n$ for each $n \in \omega$, and the collection

$$\{W_n\}_{n\in\omega} \cup \{W_\omega\} \tag{4.7}$$

is pairwise disjoint. Let

$$W = W_\omega \cup \left[\bigcup_{n\in\omega} W_n\right], \qquad V = \{(\lambda, \omega + 1) \mid \lambda \in \theta\}, \qquad x_0 = (2^{\kappa_\omega}, \omega + 2). \tag{4.8}$$

Note that the collection $\{\{x_0\}, W, V\}$ is pairwise disjoint. Let

$$X = \{x_0\} \cup V \cup W \tag{4.9}$$

and consider the noncompact, Hausdorff space (X, τ), where topology τ is generated by $\mathcal{B}_1$ (see Definition 2.2 and Corollary 2.10). For $\eta \in \omega + 1$, define

$$\mathcal{K}_\eta = \{K \in \mathcal{P}(X) \mid K \text{ is compact}, |K| = \kappa_\eta\}. \tag{4.10}$$

Observe, $W_\eta \cup \{x_0\} \in \mathcal{K}_\eta$ for $\eta \in \omega + 1$ by Proposition 2.7 (since $W_\eta \cap V = \emptyset$) and the fact that $|W_\eta \cup \{x_0\}| = \kappa_\eta$. Also, the collection $\{\mathcal{K}_\eta\}_{\eta \in \omega+1}$ is pairwise disjoint and Theorem 3.7 implies that all of the sets in $\mathcal{K}_\eta$ are homeomorphic to each other. Since $\kappa_n < \kappa_\omega$ for each $n \in \omega$, we have

$$|W| = \left| W_\omega \cup \left[\bigcup_{n \in \omega} W_n \right] \right| = \kappa_\omega + \kappa_\omega = \kappa_\omega, \tag{4.11}$$

(see [2, Corollary 2.3, page 162]), which implies $|W \cup \{x_0\}| = \kappa_\omega$. Consequently, $W \cup \{x_0\} \in \mathcal{K}_\omega$ (see Corollary 2.8), which implies $W \cup \{x_0\}$ is homeomorphic to $W_\omega \cup \{x_0\}$. If we let $[V]^{<\omega}$ denote the set of all finite subsets of V, then

$$(W_\eta \cup E) \cup \{x_0\} \in \mathcal{K}_\eta \quad \text{for } E \in [V]^{<\omega},\ \eta \in \omega + 1, \tag{4.12}$$

(see Proposition 2.7). Consequently, the size of θ can affect the cardinality of $\mathcal{K}_\eta$ for $\eta \in \omega + 1$ (e.g., let θ be a Mahlo cardinal ([3, Chapter 8, page 95])). If the generalized continuum hypothesis is assumed, then the collection $\{\mathcal{K}_\eta\}_{\eta \in \omega+1}$ is a partition of the collection of all infinite, compact subsets of X.

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