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Research Article

On a Gauss-Kuzmin-Type Problem For a Generalized Gauss-Kuzmin Operator

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A generalized limit probability measure associated with a random system with complete connections for a generalized Gauss-Kuzmin operator, only for a special case, is defined, and its behaviour is investigated. As a consequence a specific version of Gauss-Kuzmin-type problem for the above generalized operator is obtained.

1. Introduction

Let $Y = C([0,1])$ be the Banach space of complex-valued continuous functions on $[0,1]$ under the supremum norm, and let $N^* = \{1,2,\ldots\}$, $N = \{0,1,2,\ldots\}$. Then for every $f \in Y$ and for every $\alpha \geq 1$ the function $G_\alpha f$ introduced by Fluch [1] and defined by

$$(G_\alpha f)(w) = \sum_{x\in N^*} \frac{\alpha^2}{(\alpha x + w)\cdot(\alpha x + \alpha - 1 + w)} \cdot f\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \tag{1.1}$$

for all $w \in [0,1]$, is called a *generalized Gauss-Kuzmin operator*.

The present paper arises as an attempt to determine a generalized limit probability measure, only for a special case, associated with a random system with complete connections for the above generalized Gauss-Kuzmin operator obtained in Ganatsiou [2], for every $\alpha > 2$. This will give us the possibility to obtain a specific variant of Gauss-Kuzmin-type problem for the above operator.

Our approach is given in the context of the theory of dependence with complete connections (see Iosifescu and Grigorescu [3]). For a more detailed study of the theory and applications of dependence with complete connections to the metrical problems and other interesting aspects of number theory we refer the reader to [4–9] and others.

The paper is organized as follows. In Section 2, we present all the necessary results regarding the ergodic behaviour of a random system with complete connections associated with the generalized Gauss-Kuzmin operator G_α obtained in [2], in order to make more comprehensible the presentation of the paper. In Section 3, we introduce the determination of a limit probability measure associated with the above random system with complete connections, only for a special case, for every $\alpha > 2$, which will give us the possibility to study in Section 4 a specific version of the associated Gauss-Kuzmin type problem.

2. Auxiliary Results

For every $\alpha \geq 1$, we consider the function ρ_α defined by

$$\rho_\alpha(w) = \frac{\alpha}{\alpha + w}, \quad w \in [0,1], \tag{2.1}$$

and set

$$g_n = \frac{G_\alpha^n f}{\rho_\alpha}, \quad n \in N, \tag{2.2}$$

where $G_\alpha^{n+1} f = G_\alpha(G_\alpha^n f)$, for every $n \in N$ and for every $f \in Y$.

Then we obtain the following statement which gives a relation deriving from an analogous of the Gauss- Kuzmin type equation.

Proposition 2.1. *The function g_n satisfies*

$$g_{n+1}(w) = \sum_{x \in N^*} \frac{\alpha \cdot (\alpha + w)}{(\alpha x + w) \cdot (\alpha x + \alpha + w)} \cdot g_n\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \tag{2.3}$$

for any $n \in N$ and $w \in [0,1]$.

Furthermore we obtain the following.

Proposition 2.2. *For every $\alpha \geq 1$, the function*

$$P_\alpha(w, x) = \frac{\alpha \cdot (\alpha + w)}{(\alpha x + w) \cdot (\alpha x + \alpha + w)}, \quad w \in [0,1], \ x \in N^*, \tag{2.4}$$

defines a transition probability function from $([0,1], \ B_{[0,1]})$ to $(X, P(X))$, where $X = N^*$ and $P(X)$ the power set of X.

Equation (2.3) and Proposition 2.2 lead to the consideration of a family of random systems with complete connections (RSCCs)

$$\{(W, W)(X, X), u_\alpha, P_\alpha\}, \quad \alpha \geq 1, \tag{2.5}$$

where

$$W = [0,1]\ , \qquad W = B_{[0,1]}, \qquad X = N^*, \qquad X = P(X),$$
$$u_\alpha(w,x) = \frac{\alpha}{\alpha x + \alpha - 1 + w}, \quad P_\alpha(w,x) = \frac{\alpha \cdot (\alpha + w)}{(\alpha x + w)\cdot(\alpha x + \alpha + w)}, \quad w \in W,\ x \in X. \tag{2.6}$$

In the next, we consider the transition probability function Q_α, $\alpha \geq 1$, of the Markov chain associated with the family of the RSCCs (2.5) and the corresponding Markov operator U_α, $\alpha \geq 1$, defined by

$$U_\alpha f(w) = \sum_{x \in N^*} \frac{\alpha \cdot (\alpha + w)}{(\alpha x + w)\cdot(\alpha x + \alpha + w)} \cdot f\left(\frac{\alpha}{\alpha x + \alpha - 1 + w}\right), \tag{2.7}$$

for all complex-valued measurable bounded functions f on $[0,1]$.

This gives us the possibility of obtain the following.

Proposition 2.3. *The family of RSCCs* (2.5) *is with contraction. Moreover, its associated Markov operator* U_α *given by* (2.7) *is regular with respect to* $L([0,1])$*, the Banach space of all real-valued bounded Lipschitz functions on* $[0,1]$.

On the contrary the RSCC associated with a concrete piecewise fractional linear map (see Ganatsiou [10]) is not an RSCC with contraction since $r_1 = 1$ and its associated Markov chain is not compact and regular with respect to the set $L([0,1])$, even though there exists a point $y^* \in (0,1)$, such that

$$\lim_{n\to\infty}\left|\sum_n (y) - y^*\right| = 0, \tag{2.8}$$

for all $y \in (0,1)$. This corrects the escape of [10] gives an RSCC associated with a concrete piecewise fractional linear map which is not uniformly ergodic (a special case of [4]).

By virtue of Proposition 2.3, it follows from [3, Theorem 3.4.5] that the family of RSCCs (2.5) is uniformly ergodic. Furthermore, Theorem 3.1.24 in [3] implies that, for every $\alpha \geq 1$, there exists a unique probability measure γ_α on $B_{[0\ ,1]}$, which is stationary for the kernel Q_α, such that

$$\lim_{n\to\infty} U_\alpha^n f = \int_0^1 f\, d\gamma_\alpha, \quad f \in L([0,1]). \tag{2.9}$$

This means that

$$\gamma_\alpha(B) = \int_0^1 Q_\alpha(w,B)\gamma_\alpha(dw), \tag{2.10}$$

where

$$Q_\alpha(w,B) = \sum_{X \in B_w} P_\alpha(w,x), \tag{2.11}$$

with

$$B_w = \{x \in N^* \mid u_\alpha(w,\ x) \in \mathrm{B}\}, \quad \text{for every } B \in W,\ w \in [0,1]. \tag{2.12}$$

Moreover, for some c > 0 and $0 < \theta < 1$, we have

$$\left\| U_\alpha^n f - \int_0^1 f\, d\gamma_\alpha \right\| \leq \mathrm{c} \cdot \theta^n \cdot \|f\|_L, \tag{2.13}$$

for all $n \in \mathrm{N}^*$ and $f \in L([0,1])$, where $\|\cdot\|_L$ denotes the usual norm in $L([0,1])$, where

$$U_\alpha^\infty f = \int_0^1 f(w)\gamma_\alpha(dw). \tag{2.14}$$

In general the form of the limit probability measure associated with the family of random systems with complete connections (2.5) cannot be determined but this is possible only for a special case as we prove in the following section.

For the proofs of the above results we refer the reader to Ganatsiou [2].

3. A Limit Probability Measure Associated with the Family of RSCCs

Now, we are able to determine a limit probability measure associated with the family of RSCCs (2.5) as is shown in the following.

Proposition 3.1. *The probability measure γ_α has the density*

$$\rho_\alpha(w) = \frac{\alpha}{\alpha + w}, \quad \textit{for every} \quad w \in [0,1], \tag{3.1}$$

with constant $1/\alpha \cdot \log(1+\alpha^{-1})$ only for the special case $a \cdot u^{-1} + 1 - a[u^{-1} + a^{-1}] < 1$, for every $a > 2$, $0 < u \leq 1$.

Proof. By virtue of uniqueness of γ_α we have to show that it satisfies relation (2.10). Since the intervals $[0,u)$, $0 < u \leq 1$ generate $B_{[0,1]}$ it is sufficient to verify (2.10) only for $B = [0,u)$, $0 < u \leq 1$.

Suppose that $B = [0,u)$. Then, for every $w \in [0,1]$, we have

$$\begin{aligned} B_w &= \{x \in N^* \mid u_\alpha(w,x) \in [0,u)\} = \left\{x \in N^* \mid \frac{\alpha}{(\alpha x + \alpha - 1 + w)} < u\right\} \\ &= \left\{x \in N^* \mid x \geq \left[u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}\right]\right\}. \end{aligned} \tag{3.2}$$

Hence by (2.11), we have that

$$Q_\alpha(w,[0,u)) = \frac{\alpha + w}{\alpha\left[u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}\right] + w}, \tag{3.3}$$

where

$$\left[u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}\right] = \begin{cases} \left[u^{-1} + \alpha^{-1}\right], & \text{if } 0 \le w < \alpha \cdot u^{-1} + 1 - \alpha \cdot \left[u^{-1} + \alpha^{-1}\right], \\ \left[u^{-1} + \alpha^{-1}\right] - 1, & \text{if } \alpha \cdot u^{-1} + 1 - \alpha \cdot \left[u^{-1} + \alpha^{-1}\right] < w \le 1. \end{cases} \tag{3.4}$$

We consider the case $\alpha u^{-1} + 1 - \alpha \cdot [u^{-1} + \alpha^{-1}] < 1$ or $u^{-1} < [u^{-1} + \alpha^{-1}]$, for every $\alpha > 2,\ 0 < u \le 1$. Consequently, we obtain that

$$\begin{aligned} \int_0^1 Q_\alpha(w, [0, \mathrm{u})) \cdot \rho_\alpha(w) dw &= \frac{1}{\log(1 + \alpha^{-1})} \cdot \int_0^1 \frac{dw}{\alpha \cdot [u^{-1} - w \cdot \alpha^{-1} + \alpha^{-1}] + w} \\ &= \frac{1}{\log(1 + \alpha^{-1})} \cdot \Big[\log\left(\alpha \cdot u^{-1} + 1\right) - \log\left(\alpha \cdot u^{-1} + 1 - \alpha\right) \\ &\quad + \log\left(\alpha \cdot \left[u^{-1} + \alpha^{-1}\right] - \alpha + 1\right) - \log\left(\alpha \cdot \left[u^{-1} + \alpha^{-1}\right]\right)\Big]. \end{aligned} \tag{3.5}$$

In the next we put

$$\begin{aligned} \mathrm{I} &= \log\left(\alpha \cdot \left[u^{-1} + \alpha^{-1}\right] - \alpha + 1\right) - \log\left(\alpha \cdot \left[u^{-1} + \alpha^{-1}\right]\right) \\ &= \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right), \\ \mathrm{II} &= \log\left(\alpha \cdot u^{-1} + 1\right) - \log\left(\alpha \cdot u^{-1} + 1 - \alpha\right) \\ &= \log\left(1 + \frac{u}{\alpha}\right) - \log\left(1 + \frac{u}{\alpha} - u\right) \end{aligned} \tag{3.6}$$

By taking the limit of

$$\begin{aligned} \mathrm{III} &= I - \log\left(1 + \frac{u}{\alpha} - u\right) \\ &= \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right) - \log\left(1 + \frac{u}{\alpha} - u\right) \end{aligned} \tag{3.7}$$

when $u \to 1$ we have that

$$\begin{aligned} \lim_{u \to 1} \log\left(1 + \frac{u}{\alpha} - \mathrm{u}\right) &= \log\left(\frac{1}{\alpha}\right), \\ \lim_{u \to 1} \log\left(1 - \frac{1}{[u^{-1} + \alpha^{-1}]} + \frac{1}{\alpha \cdot [u^{-1} + \alpha^{-1}]}\right) &= \log\left(\frac{1}{\alpha}\right), \quad \text{for every } \alpha > 2. \end{aligned} \tag{3.8}$$

So part III tends to 0 when $u \to 1$. This means that

$$\lim_{u\to 1}\left[\int_0^1 Q_\alpha(w,[0,\mathrm{u}))\cdot\rho_\alpha(w)dw\right]=\frac{1}{\log(1+\alpha^{-1})}\cdot\lim_{u\to 1}\log\left(1+\frac{u}{\alpha}\right) \tag{3.9}$$

which is equal to

$$\begin{aligned}\lim_{u\to 1}\int_0^u\rho_\alpha(w)dw&=\lim_{u\to 1}\int_0^u\frac{1}{\alpha\cdot\log\left(1+\alpha^{-1}\right)}\cdot\frac{\alpha}{\alpha+w}dw\\&=\frac{1}{\log(1+\alpha^{-1})}\lim_{u\to 1}\left[\log(\alpha+u)-\log\alpha\right]\\&=\frac{1}{\log(1+\alpha^{-1})}\lim_{u\to 1}\log\left(\frac{\alpha+u}{\alpha}\right)\\&=\frac{1}{\log(1+\alpha^{-1})}\lim_{u\to 1}\log\left(1+\frac{u}{\alpha}\right)\end{aligned} \tag{3.10}$$

and the proof is complete. □

4. A Version of the Gauss-Kuzmin-Type Problem

Let μ be a nonatomic measure on the σ-algebra $B_{[0,1]}$. Then we may define

$$\begin{aligned}V_o(w)&=\mu([0,w]),\\V_n(w)=V_n(w,\mu)&=\int_0^w G_\alpha^n f(t)dt,\quad n\in N^*,\ w\in[0,1].\end{aligned} \tag{4.1}$$

Suppose that V_0' exists and it is bounded (μ has bounded density). Then by induction we have that V_n' exists and it is bounded for any $n\in N^*$ with

$$V_n'(w)\equiv G_\alpha^n f(w)=\mathrm{G}_\alpha\left[\left(\mathrm{G}_\alpha^{n-1}f\right)(w)\right],\quad f\in L([0,1]),\ n\in\mathrm{N}^*. \tag{4.2}$$

So

$$\int_0^w V_n'(t)dt=\int_0^w G_\alpha^n f(t)dt\ ,\qquad V_n(w)=\int_0^w G_\alpha^n f(t)dt \tag{4.3}$$

while

$$g_n(w)=\frac{G_\alpha^n f(w)}{p_\alpha(w)}\equiv\frac{V_n'(w)}{p_\alpha(w)},\quad n\in N. \tag{4.4}$$

Now, we are able to determine the limit $\lim_{n\to\infty}V_n(1/w)$ and to give the rate of this convergence, that is, a specific version of the associated Gauss-Kuzmin type problem.

Proposition 4.1. (i) *If the density V_0' of μ is a Riemann integrable function, then*

$$\lim_{n\to\infty} V_n\left(\frac{1}{w}\right) = \frac{1}{\log(1+\alpha^{-1})}\cdot\log\left(\frac{\alpha w+1}{\alpha w}\right), \quad w\geq 1,\ \alpha>2,\ n\in N^*. \tag{4.5}$$

(ii) *If the density V_0' of μ is an element of $L([0,1)]$, then there exist two positive constants c and $\theta<1$ such that*

$$\lim_{n\to\infty} V_n\left(\frac{1}{w}\right) = (1+q\theta^n)\cdot\frac{1}{\log(1+\alpha^{-1})}\cdot\log\left(\frac{\alpha w+1}{\alpha w}\right), \tag{4.6}$$

for all $w\geq 1$, $a>2$, $n\in N^*$, where $q=q(\mu,n,w)$with $|q|\leq c$.

Proof. Let $V_0'\in L([0,1])$. Then $g_o\in L([0,1])$, and by using relation (2.14) we have

$$U_\alpha^\infty g_0 \equiv \lim_{n\to\infty} U_\alpha^n g_0 = \int_0^1 g_0(w)\gamma_\alpha(dw) = \int_0^1 V_0'(w)dw = 1. \tag{4.7}$$

According to relation (2.13), there exist two positive constants c and $\theta<1$ such that

$$U_\alpha^n g_0 = U_\alpha^\infty g_0 + T_\alpha^n g_0, \quad n\in N^*,\ \text{with}\ \left\|T_\alpha^n g_0\right\|\leq \mathrm{c}\cdot\theta^n. \tag{4.8}$$

If we consider the Banach space $C([0,1])$ of all real continuous functions defined on $[0,1]$ with the supremum norm, then since $L([0,1])$ is a dense subset of $C([0,1])$ we have

$$\lim_{n\to\infty}\left|T_\alpha^{\mathrm{n}} g_0\right| = 0, \quad \text{for every}\ \ g_o\in C([0,1]). \tag{4.9}$$

This means that it is valid for any measurable function g_o which is γ_α-almost surely continuous, that is, for any Riemann integrable function g_o. Consequently we obtain

$$\begin{aligned}\lim_{n\to\infty} V_n\left(\frac{1}{w}\right) &= \lim_{n\to\infty}\int_0^{1/w} U_\alpha^n g_0(t)\rho_\alpha(t)dt \\ &= \int_0^{1/w}\rho_\alpha(t)dt = \int_0^{1/w}\frac{1}{\log(1+\alpha^{-1})}\cdot\frac{\alpha}{\alpha+\mathrm{t}}dt \\ &= \frac{1}{\log(1+\alpha^{-1})}\cdot\log\left(\frac{\alpha w+1}{\alpha w}\right),\end{aligned} \tag{4.10}$$

that is the solution of the associated Gauss-Kuzmin type problem. □

Remarks. (1) It is notable that for $\alpha = 1$ the RSCC associated with the generalized Gauss-Kuzmin operator is identical to that associated with the ordinary continued fraction expansion (see Iosifescu and Grogorescu [3]). Moreover the corresponding limit probability measure associated with the family of RSCCs (2.5) for $\alpha = 1$ is identical to the limit probability measure associated with the above random system with complete connections for the ordinary continued fraction expansion, that is, identical to the Gauss's measure γ on $B_{[0,1]}$ defined by

$$\gamma(A) = \frac{1}{\log 2}\int_A \frac{dt}{t+1}, \quad A \in B_{[0,1]}. \tag{4.11}$$

(2) It is an open problem the determination of an analogous limit probability measure for the case $\alpha u^{-1} + 1 - \alpha \cdot [u^{-1} + \alpha^{-1}] > 1$.

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