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Research Article

Certain Conditions for Starlikeness of Analytic Functions of Koebe Type

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For $\alpha \geq 0$, $\lambda > 0$, we consider the $M(\alpha,\lambda)_b$ of normalized analytic $\alpha-\lambda$ convex functions defined in the open unit disc $\mathbb{U}$. In this paper, we investigate the class $M(\alpha,\lambda)_b$, that is, $\mathrm{Re}\{(zf_b'(z)/f_b(z))[1-\alpha+\alpha(1-\lambda)(zf_b'(z)/f_b(z))+\alpha\lambda(1+(zf_b''(z)/f_b'(z)))]\} > 0$, with f_b is Koebe type, that is, $f_b(z) := z/(1-z^n)^b$. The subordination result for the aforementioned class will be given. Further, by making use of Jack's Lemma as well as several differential and other inequalities, the authors derived sufficient conditions for starlikeness of the class $M(\alpha,\lambda)_b$ of n-fold symmetric analytic functions of Koebe type. Relevant connections of the results presented here with those given in the earlier works are also indicated.

1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Also, as usual, let

$$S^* = \left\{ f : f \in \mathcal{A},\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ (z \in \mathbb{U}) \right\},$$
$$K = \left\{ f : f \in \mathcal{A},\ \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ (z \in \mathbb{U}) \right\}, \tag{1.2}$$

be the familiar classes of *starlike functions* in $\mathbb{U}$ and *convex functions* in $\mathbb{U}$, respectively.

If the functions f and g are analytic in $\mathbb{U}$, then we say that the function f is *subordinate* to g, or g is *superordinate* to f (written as $f \prec g$) if there exist a function $w(z)$ analytic $\mathbb{U}$, such that $|w(z)| < 1$ and $z \in U$, and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$. If g is univalent in $\mathbb{U}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Next, we let the $M(\alpha)$, that is,

$$M(\alpha) = \left\{ f(z) \in A : \operatorname{Re}\left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0,\ (z \in \mathbb{U})\right\}. \tag{1.3}$$

The class $M(\alpha)$ was first introduced by Mocanu [1], which was then known as the class of α convex (or α-starlike) functions. Later, Miller et al. [2] studied this class and showed that $M(\alpha)$ is a subclass of S^* for any real number α and also that $M(\alpha)$ is a subclass of K for $\alpha \geq 1$. We note that $M(0) = S^*$ and $M(1) = K$. Note also that Mocanu introduced $M(\alpha)$ with $f(z) \cdot f'(z)/0$. But Sakaguchi and Fukui [3] later showed that this condition was not needed.

Motivated essentially by the aforementioned earlier works, we aim here at deriving sufficient conditions for starlikeness of n-fold symmetric function f_b of the Koebe type, defined by

$$f_b(z) := \frac{z}{(1-z^n)^b} \quad (b \geq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.4}$$

which obviously corresponds to the familiar Koebe function when $n = 1$ and $b = 2$.

Definition 1.1. A function $f(z)$ given by (1.1) is said to be in the class $M(\alpha, \lambda)_b$, for $\alpha \geq 0$, $\lambda > 0$, if the following conditions are satisfied:

$$\operatorname{Re}\left\{ \frac{zf_b'(z)}{f_b(z)}\left[1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda\left(1 + \frac{zf_b''(z)}{f_b'(z)}\right)\right]\right\} > 0, \quad (z \in \mathbb{U}). \tag{1.5}$$

In this paper, we consider the class of functions $M(\alpha, \lambda)_b$.

In addition, in this paper, authors investigate the subordination of the class denoted by $M(\alpha, \lambda)_b$.

We have the following inclusion relationships:

(i) $M(0, \lambda)_1 \subset S^*$

(ii) $M(\alpha, 1)_1 \subset \mathscr{H}(\alpha) \subset S^*$, which $\mathscr{H}(\alpha)$ has studied by [4].

The work of Siregar et al. [5] and Bansal and Raina [6] have also motivated us to come to these problems. Look also at [7, 8] for different studies.

The following result (popularly known as Jack's Lemma) will also be required in the derivation of our result (Theorem 4.1 below).

2. Preliminaries

Lemma 2.1 (see [9]). *Let $q(z)$ be univalent in $\mathbb{U}$ and let the function θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = \gamma z q'(z)\phi(q(z)), \quad \gamma > 0,$$
$$h(z) = \theta(q(z)) + Q(z), \tag{2.1}$$

and suppose that

(i) *$Q(z)$ is univalent and starlike in $\mathbb{U}$;*

(ii) $\mathrm{Re}(zh'(z)/Q(z)) = \mathrm{Re}(\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)) > 0,\ z \in \mathbb{U}$.

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$, $p(\mathbb{U}) \subset D$, and

$$\theta(p(z)) + \gamma z p'(z)\phi(p(z)) \prec \theta(q(z)) + \gamma z q'(z)\phi(q(z)) = h(z), \tag{2.2}$$

then

$$p(z) \prec q(z), \tag{2.3}$$

and q is the best dominant.

Lemma 2.2 (see [10]). *Let the (nonconstant) function $w(z)$ be analytic in $\mathbb{U}$ such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_o \in \mathbb{U}$, we have*

$$z_o w'(z_o) = k w(z_o), \tag{2.4}$$

where $k \geq 1$ is a real number.

3. The Subordination Result

Theorem 3.1. *Let $f(z) \in A$ satisfy $f(z) \neq 0$ $(z \in \mathbb{U})$. Also, let the function $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 1$ and $q(z) \neq 0$, for $\lambda > 0$ and $\alpha \geq 0$, such that*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0 \quad (z \in \mathbb{U}),$$
$$\mathrm{Re}\left\{\lambda + 2q(z) + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.1}$$

If

$$\frac{zf_b'(z)}{f_b(z)}\left[1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda\left(1 + \frac{zf_b''(z)}{f_b'(z)}\right)\right] \prec h(z) \quad (z \in \mathbb{U}), \tag{3.2}$$

where

$$h(z) = \alpha\left[q(z)\right]^2 + (1-\alpha)\left(q(z)\right) + \alpha\lambda z q'(z), \tag{3.3}$$

then

$$\frac{z f_b'(z)}{f_b(z)} \prec q(z) \quad (z \in \mathbb{U}), \tag{3.4}$$

and $q(z)$ *is the best dominant of* (3.2).

Proof. We first choose

$$p(z) = \frac{z f_b'(z)}{f_b(z)}, \qquad \theta(w) = w(1-\alpha+\alpha w), \qquad \phi(w) = 1, \tag{3.5}$$

then $\theta(w)$ and $\phi(w)$ are analytic inside the domain $\mathbb{D}^*$, which contains $q(\mathbb{U})$, $q(0) = 1$, and $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$.

Now, if we define the functions $Q(z)$ and $h(z)$ by

$$\begin{gathered} Q(z) = \alpha\lambda z q'(z)\phi\left(q(z)\right) = \alpha\lambda z q'(z), \\ h(z) = \theta\left(q(z)\right) + Q(z) = \alpha\left[q(z)\right]^2 + (1-\alpha)\left(q(z)\right) + \alpha\lambda z q'(z), \end{gathered} \tag{3.6}$$

then it follows from (3.1) that $Q(z)$ is starlike in $\mathbb{U}$ and

$$\operatorname{Re}\left(\frac{z h'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{3.7}$$

We also note that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0) = 1$. Since $0 \notin p(\mathbb{U})$, therefore, $p(\mathbb{U}) \subset \mathbb{D}^*$, $\gamma = \alpha\lambda > 0$ and hence, the hypothesis of Lemma 2.1 are satisfied.

Applying Lemma 2.1, we find that

$$\begin{aligned} &\frac{z f_b'(z)}{f_b(z)}\left[1-\alpha+\alpha(1-\lambda)\frac{z f_b'(z)}{f_b(z)} + \alpha\lambda\left(1+\frac{z f_b''(z)}{f_b'(z)}\right)\right] \\ &\quad = \alpha\left[p(z)\right]^2 + (1-\alpha)\left(p(z)\right) + \alpha\lambda z p'(z) = \theta\left(p(z)\right) + \alpha\lambda z p'(z) \prec h(z) \\ &\quad = \alpha\left[q(z)\right]^2 + (1-\alpha)\left(q(z)\right) + \alpha\lambda z q'(z) = \theta\left(q(z)\right) + \alpha\lambda z q'(z), \quad (z \in \mathbb{U}), \end{aligned} \tag{3.8}$$

which implies that

$$\frac{z f_b'(z)}{f_b(z)} \prec q(z) \quad (z \in \mathbb{U}), \tag{3.9}$$

and $q(z)$ is the best dominant of (3.2). □

4. The Properties of the Class $M(\alpha, \lambda)_b$

We begin by proving a stronger result than what we indicated in the preceding section.

Theorem 4.1. *Let the n-fold symmetric function $f_b(z)$, defined by* (1.4)*, be analytic in U, with*

$$\frac{f_b(z)}{z} \neq 0 \quad (z \in \mathbb{U}). \tag{4.1}$$

If $f_b(z)$ satisfies the inequality:

$$\operatorname{Re}\left\{ \frac{zf_b'(z)}{f_b(z)} \left[1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda\left(1 + \frac{zf_b''(z)}{f_b'(z)}\right)\right]\right\} > \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) - \frac{\alpha\lambda nb}{4}, \tag{4.2}$$

$(z \in \mathbb{U})$, then $f_b(z)$ is starlike in $\mathbb{U}$ for

$$\alpha > 0,\ \lambda > 0, \quad \left(\frac{\alpha\lambda + 2\alpha + 2 - \sqrt{\Delta}}{2\alpha} \leq nb \leq \frac{\alpha\lambda + 2\alpha + 2 + \sqrt{\Delta}}{2\alpha}\right), \quad \left(\Delta := \alpha^2(\lambda+2)^2 + 4\alpha(\lambda-2) + 4\right). \tag{4.3}$$

If $f_b(z)$ satisfies the inequality (4.2) *with $\lambda = 1$, that is, if*

$$\operatorname{Re}\left(\frac{\alpha z^2 f_b''(z)}{f_b(z)} + \frac{zf'(z)}{f(z)}\right) > -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right), \quad (z \in \mathbb{U}) \tag{4.4}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for

$$\alpha > 0, \quad \frac{3\alpha + 2 - \sqrt{\Delta^*}}{2\alpha} \leq nb \leq \frac{3\alpha + 2\sqrt{\Delta^*}}{2\alpha}, \quad \left(\Delta^* := 9\alpha^2 - 4\alpha + 4\right). \tag{4.5}$$

Proof. Let $\alpha > 0$, $\lambda > 0$ and $f_b(z)$ satisfy the hypothesis of Theorem 4.1. We put

$$\frac{zf_b'(z)}{f_b(z)} = \frac{1 + (nb-1)w(z)}{1 - w(z)}, \tag{4.6}$$

where $w(z)$ is analytic in $\mathbb{U}$, with

$$w(0) = 0, \qquad w(z) \neq 1, \quad (z \in \mathbb{U}), \tag{4.7}$$

such that, we can write

$$1 + \frac{zf_b''(z)}{f_b'(z)} = \frac{nbzw'(z)}{[1-w(z)][1+(nb-1)w(z)]} + \frac{1 + (nb-1)w(z)}{1 - w(z)}, \tag{4.8}$$

which, in turn, implies that

$$\begin{aligned}
&\frac{zf_b'(z)}{f_b(z)}\left[1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda\left(1+\frac{zf_b''(z)}{f_b'(z)}\right)\right]\\
&\quad=\left(\frac{1+(nb-1)w(z)}{1-w(z)}\right)\\
&\quad\quad\times\left[1-\alpha+\alpha(1-\lambda)\left(\frac{1+(nb-1)w(z)}{1-w(z)}\right)\right.\\
&\quad\quad\left.+\alpha\lambda\left(\frac{nbzw'(z)}{(1-w(z))[1+(nb-1)w(z)]}+\frac{1+(nb-1)w(z)}{1-w(z)}\right)\right]\\
&\quad=(1-\alpha)\left(\frac{1+(nb-1)w(z)}{1-w(z)}\right)+\alpha\left(\frac{[1+(nb-1)w(z)]^2+\lambda nbzw'(z)}{(1-w(z))^2}\right).
\end{aligned}\tag{4.9}$$

Now, we claim that $|w(z)|<1$ $(z\in\mathbb{U})$. If there exists a z_o in $\mathbb{U}$ such that $|w(z_o)|=1$, then (by Jack's Lemma) Lemma 2.2, we have

$$z_ow'(z_o)=kw(z_o),\tag{4.10}$$

where $k\geq 1$ is a real number.

By setting $w(z_o)=e^{i\theta}$ $(0\leq\theta\leq 2\pi)$, thus, we find that

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{z_of_b'(z_o)}{f_b(z_o)}\left[1-\alpha+\alpha(1-\lambda)\frac{z_of_b'(z_o)}{f_b(z_o)}+\alpha\lambda\left(1+\frac{z_of_b''(z_o)}{f_b'(z_o)}\right)\right]\right\}\\
&\quad=\operatorname{Re}\left\{(1-\alpha)\left(\frac{1+(nb-1)w(z_o)}{1-w(z_o)}\right)+\alpha\left(\frac{\lambda nbz_ow'(z_o)+[1+(nb-1)w(z_o)]^2}{(1-w(z_o))^2}\right)\right\}\\
&\quad=\operatorname{Re}\left\{(1-\alpha)\left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)+\alpha\left(\frac{\lambda nbke^{i\theta}+\left[1+(nb-1)e^{i\theta}\right]^2}{(1-e^{i\theta})^2}\right)\right\}\\
&\quad=\operatorname{Re}\left\{\frac{1+(\alpha\lambda nbk+\alpha nb+nb-2)e^{i\theta}+(nb-1)(\alpha(nb-1)-(1-\alpha))e^{i2\theta}}{(1-e^{i\theta})^2}\right\}\\
&\quad=\frac{[2\alpha nb(\lambda k+2-nb)+4nb-8]\cos\theta+[(nb-1)(\alpha nb-1)+1]\cos 2\theta-\alpha nb(\lambda k+3-2nb)-3nb+6}{2(3-4\cos\theta)+2\cos 2\theta}\\
&\quad\leq\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)-\frac{\alpha\lambda nb}{4},\quad (z\in\mathbb{U}),
\end{aligned}\tag{4.11}$$

since $k\geq 1$.

If we let

$$\begin{aligned}&\operatorname{Re}\left\{\frac{z_o f_b'(z_o)}{f_b(z_o)}\left[1-\alpha+\alpha(1-\lambda)\frac{z_o f_b'(z_o)}{f_b(z_o)}+\alpha\lambda\left(1+\frac{z_o f_b''(z_o)}{f_b'(z_o)}\right)\right]\right\}\\&\quad\le\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)-\frac{\alpha\lambda nb}{4}\\&\quad=\tau(nb),\end{aligned} \tag{4.12}$$

then

$$\tau(nb)\le 0,\quad\left(\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha}\le nb\le\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta};\Delta}{2\alpha}:=\alpha^2(\lambda+2)^2+4\alpha(\lambda-2)+4\right). \tag{4.13}$$

Thus, we have

$$\operatorname{Re}\left\{\frac{zf_b'(z)}{f_b(z)}\left[1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda\left(1+\frac{zf_b''(z)}{f_b'(z)}\right)\right]\right\}\le 0\quad(z\in\mathbb{U}), \tag{4.14}$$

$$\left(\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha}\le nb\le\frac{\alpha\lambda+2\alpha+2\sqrt{\Delta}}{2\alpha};\ \Delta:=\alpha^2(\lambda+2)^2+4\alpha(\lambda-2)+4\right), \tag{4.15}$$

which is a contradiction to the hypotheses of (4.2).

Therefore, $|w(z)|<1$ for all z in $\mathbb{U}$. Hence f_b is starlike in $\mathbb{U}$, then by proving the assertion (i) of Theorem 4.1, this completes the proof of our theorem. □

Next, we arrive to the following remark which was given by Fukui et al. [11], and so we omit the detail here.

Remark 4.2. Let the n-fold symmetric function $f_b(z)$, defined by (1.4), be analytic in U, with

$$\frac{f_b(z)}{z}\neq 0\quad(z\in\mathbb{U}). \tag{4.16}$$

If $f_b(z)$ satisfies the inequality (4.2) with $\alpha=0$, that is, if

$$\operatorname{Re}\left(\frac{zf_b'(z)}{f_b(z)}\right)>1-\frac{nb}{2}\quad(z\in\mathbb{U}), \tag{4.17}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for $0\le nb<2$.

The following remark was obtained by Kamali and Srivastava [12].

Remark 4.3. Let the n-fold symmetric function $f_b(z)$, defined by (1.4), be analytic in U, with

$$\frac{f_b(z)}{z} \neq 0 \quad (z \in \mathbb{U}). \tag{4.18}$$

If $f_b(z)$ satisfies the inequality (4.2) with $\lambda = 1$, that is, if

$$\operatorname{Re}\left(\frac{\alpha z^2 f_b''(z)}{f_b(z)} + \frac{zf'(z)}{f(z)}\right) > -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) \quad (z \in \mathbb{U}), \tag{4.19}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for

$$\alpha > 0, \quad \frac{3\alpha + 2 - \sqrt{\Delta^*}}{2\alpha} \leq nb \leq \frac{3\alpha + 2\sqrt{\Delta^*}}{2\alpha}, \quad \left(\Delta^* := 9\alpha^2 - 4\alpha + 4\right). \tag{4.20}$$

5. Applications of Differential Inequalities

We apply the following result involving differential inequalities with a view to deriving several further sufficient conditions for starlikeness of the n-fold symmetric function f_b defined by (1.4).

Lemma 5.1 (Miller and Mocanu [13]). *Let $\Theta(u, v)$ be a complex-valued function such that*

$$\Theta : \mathbb{D} \longrightarrow \mathbb{C}, \quad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}), \tag{5.1}$$

$\mathbb{C}$ being (as usual) the complex plane, and let

$$u = u_1 + iu_2, \qquad v = v_1 + iv_2. \tag{5.2}$$

Suppose that the functions $\Theta(u, v)$ satisfies each of the following conditions.

(i) *$\Theta(u, v)$ is continuous in $\mathbb{D}$.*

(ii) *$(1, 0) \in \mathbb{D}$ and $\operatorname{Re}(\Theta(1, 0)) > 0$.*

(iii) *$\operatorname{Re}(\Theta(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that*

$$v_1 \leq -\frac{1}{2}\left(1 + u_2^2\right). \tag{5.3}$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{5.4}$$

be analytic (regular) in $\mathbb{U}$ such that

$$\left(p(z), zp'(z)\right) \in \mathbb{D} \quad (z \in \mathbb{U}). \tag{5.5}$$

If

$$\mathrm{Re}\big(\Theta\big(p(z), zp'(z)\big)\big) \in \mathbb{D} \quad (z \in \mathbb{U}), \tag{5.6}$$

then

$$\mathrm{Re}\big(p(z)\big) > 0 \quad (z \in \mathbb{U}). \tag{5.7}$$

Let us now consider the following implication.

Theorem 5.2. *Let n-fold symmetric function f_b, defined by* (1.4) *and analytic in $\mathbb{U}$ with $(f_b(z))/z, \neq 0,\ (z \in \mathbb{U})$, satisfy the following inequality:*

$$\mathrm{Re}\left\{\frac{zf_b'(z)}{f_b(z)}\left[1-\alpha+\alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)}+\alpha\lambda\left(1+\frac{zf_b''(z)}{f_b'(z)}\right)\right]\right\} > \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)-\frac{\alpha\lambda nb}{4}, \tag{5.8}$$

then

$$\mathrm{Re}\left\{\left(\frac{zf_b'(z)}{f_b(z)}\right)^{\mu}\right\} > 0,$$
$$\left(z \in \mathbb{U};\ \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)-\frac{\alpha\lambda nb}{4} < 1;\ \alpha \geq 0,\ \lambda > 0;\ \mu \geq 1\right). \tag{5.9}$$

Proof. If we put

$$p(z) = \left\{\frac{zf_b'(z)}{f_b(z)}\right\}^{\mu}, \tag{5.10}$$

then (5.8) is equivalent to

$$\mathrm{Re}\left\{\frac{\alpha\lambda}{\mu}\{p(z)\}^{(1-\mu)/\mu}zp'(z)+\alpha\{p(z)\}^{2/\mu}+(1-\alpha)p(z)^{1/\mu}\right.$$
$$\left.-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)+\frac{\lambda\alpha nb}{4}\right\} > 0 \tag{5.11}$$
$$\Longrightarrow \mathrm{Re}\big(p(z)\big) > 0 \quad (z \in \mathbb{U}).$$

By setting $p(z) = u$ and $zp'(z) = v$ and letting

$$\Theta(z) = \frac{\alpha\lambda}{\mu}u^{(1-\mu)/\mu}v+\alpha u^{2/\mu}+(1-\alpha)u^{1/\mu}-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)+\frac{\lambda\alpha nb}{4}, \tag{5.12}$$

for $\alpha \geq 0$ and $\mu \geq 1$, we have the following.

(i) $\Theta(u,v)$ is continuous in $\mathbb{D} = (\mathbb{C} \setminus \{0\} \times \mathbb{C})$.

(ii) $(1,0) \in \mathbb{D}$ and

$$\operatorname{Re}(\Theta(1,0)) = \frac{\alpha\lambda nb}{4} + \frac{\alpha nb}{2} + \frac{nb}{2} - \frac{\alpha n^2 b^2}{4} > 0, \tag{5.13}$$

since

$$\left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) - \frac{\lambda\alpha nb}{4} < 1. \tag{5.14}$$

Thus, the conditions (i) and (ii) of Lemma 5.1 are satisfied. Moreover, for $(iu_2, v_1) \in \mathbb{D}$ and $v_1 \leq (-1/2)(1 + u_2^2)$, we obtain

$$\begin{aligned}\operatorname{Re}(\Theta(iu_2, v_1)) &= \frac{\alpha\lambda}{\mu}|u_2|^{(1-\mu)/\mu} v_1 \cos\left(\frac{(1-\mu)\pi}{2\mu}\right) + \alpha|u_2|^{2/\mu}\cos\left(\frac{\pi}{\mu}\right)\\ &\quad + (1-\alpha)|u_2|^{1/\mu}\cos\left(\frac{\pi}{2\mu}\right) - \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) + \frac{\lambda\alpha nb}{4}\\ &\leq -\frac{\alpha\lambda}{2\mu}\left(1 + u_2^2\right)|u_2|^{(1-\mu)/\mu}\sin\left(\frac{\pi}{2\mu}\right) + \alpha|u_2|^{2/\mu}\cos\left(\frac{\pi}{\mu}\right)\\ &\quad + (1-\alpha)|u_2|^{1/\mu}\cos\left(\frac{\pi}{2\mu}\right) - \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) + \frac{\lambda\alpha nb}{4},\end{aligned} \tag{5.15}$$

which, upon putting $|u_2| = \zeta \quad (\zeta > 0)$, yields

$$\operatorname{Re}(\Theta(iu_2, v_1)) \leq \Phi(\zeta), \tag{5.16}$$

where

$$\begin{aligned}\Phi(\zeta) := &-\frac{\alpha\lambda}{2\mu}\left(1 + \zeta^2\right)\zeta^{(1-\mu)/\mu}\sin\left(\frac{\pi}{2\mu}\right) + \alpha\zeta^{2/\mu}\cos\left(\frac{\pi}{\mu}\right) + (1-\alpha)\zeta^{1/\mu}\cos\left(\frac{\pi}{2\mu}\right)\\ &- \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) + \frac{\lambda\alpha nb}{4}.\end{aligned} \tag{5.17}$$

□

Remark 5.3. If, for some choices of the parameters α, λ, μ, and nb, we find that

$$\Phi(\zeta) \leq 0 \quad (\zeta > 0), \tag{5.18}$$

then we can conclude from (5.16) and Lemma 5.1 that the corresponding implication (5.8) holds true.

First of all, for the choice: $\mu = 1$ and $nb = 2$, we have the following.

Theorem 5.4. *If n-fold symmetric function f_b, defined by* (1.4) *and analytic in* $\mathbb{U}$ *with*

$$\frac{f_b(z)}{z} \neq 0, \quad (z \in \mathbb{U}), \tag{5.19}$$

satisfies the following inequality:

$$\operatorname{Re}\left\{ \frac{zf_b'(z)}{f_b(z)} \left[1 - \alpha + \alpha(1-\lambda)\frac{zf_b'(z)}{f_b(z)} + \alpha\lambda\left(1 + \frac{zf_b''(z)}{f_b'(z)}\right)\right]\right\} > -\frac{\alpha\lambda}{2}, \tag{5.20}$$

then $f_b \in S^*$ for any real $\alpha \geq 0$ and $\lambda > 0$.

Proof. For $\mu = 1$, $nb = 2$, we find from (5.17) that

$$\Phi(\zeta) := -\alpha\lambda\left(\frac{1}{2} + s^2\right) - \alpha s^2 \leq 0, \quad (\zeta \in \mathbb{R}), \tag{5.21}$$

which implies Theorem 5.4 in view of the remark. □

Remark 5.5. For $\lambda = 1$, we will obtain the results by Kamali and Srivastava [12].

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