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Research Article

The Spectral Scale of a Self-Adjoint Operator in a Semifinite von Neumann Algebra

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We extend Akemann, Anderson, and Weaver's *Spectral Scale* definition to include selfadjoint operators from *semifinite* von Neumann algebras. New illustrations of spectral scales in both the finite and semifinite von Neumann settings are presented. A counterexample to a conjecture made by Akemann concerning normal operators and the geometry of the their perspective spectral scales (in the finite setting) is offered.

1. Introduction

The notion of a "joint spectrum" for an n-tuple of operators was created and developed through the 1970s and work continues today. For an account of the various approaches taken and the achievements made see [1, page 3]. One attempt at a *geometric* interpretation of the "joint spectrum" of an n-tuple of self-adjoint operators is developed in *spectral scale* [2–6].

Definition 1.1. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a finite, faithful, normal, trace τ such that $\tau(1) = 1$, and let $b = b_1 + ib_2 \in \mathcal{M}$ with b_1 and b_2 self-adjoint. Define

$$\Psi : \mathcal{M} \longrightarrow \mathbb{R}^3 \tag{1.1}$$

by $\Psi(c) = (\tau(c), \tau(b_1 c), \tau(b_2 c))$ for all $c \in \mathcal{M}$. The *spectral scale of b with respect to τ* is the set

$$B(b) = \Psi\left(\mathcal{M}_1^+\right) = \{(\tau(c), \tau(b_1 c), \tau(b_2 c)) : 0 \le c \le 1\}. \tag{1.2}$$

The spectral scale of an n-tuple of self-adjoint operators is defined analogously (in the canonical fashion), becoming a subset of $\mathbb{R}^{n+1}$.

The set $B(b)$ has proven useful in gaining insight on problems in operator theory and operator algebras. In 2006, for a self-adjoint operator b in a *finite* factor $\mathcal{M}$ (i.e., $\tau(1) < \infty$ and $\mathcal{M}$ has trivial center), Akemann and Sherman [7] were able to use $B(b)$ to characterize the w^*-closure of the unitary orbit $\mathcal{U}(b) = \{ubu^* : u \text{ is unitary in } \mathcal{M}\}$ of b. More precisely, they showed that $\overline{\mathcal{U}(b)}^{w^*} = \mathcal{C}(b) = \{d \in \mathcal{M} : B(d) \subset B(b)\}$. Unfortunately, the spectral scale has only been developed for *finite* von Neumann algebras. In 2010, Wills [6] was able to extend the definition of $B(b)$ to include the class of unbounded operators.

The purpose of the work that follows to develop $B(b)$ for $b = b^* \in \mathcal{M}$, where $\mathcal{M}$ is a *semifinite* von Neumann algebra, equipped with a semifinite, faithful, normal trace τ such that $\tau(1) = \infty$. We will show that $B(b)$ is an unbounded subset of $\mathbb{R}^2$ which may or may not be closed, and that $B(b)$ gives a complete description of elements in the spectrum $\sigma(b)$ of b that lie outside the open interval determined by the minimum and maximum elements of the essential spectrum $\mathcal{E}$ of b.

We begin with a summary of "how to read" $B(b)$ in the *finite* von Neumann algebra setting.

2. Geometric Interpretations of $B(b)$ When $\mathcal{M}$ Is a Finite von Neumann Algebra

The following is a summary of the main results concerning the geometry of $B(b)$ [2–5].

Theorem 2.1. *Let $\mathcal{M}$, τ, b, Ψ, and $B(b)$ be as given in Definition 1.1. Then the following hold.*

(i) *$B(b)$ is a compact, convex subset of $\mathbb{R}^3$ which is symmetric about the point $(1/2, b_1/2, b_2/2)$. If $b = b^*$, then $B(b)$ can be identified with a subset of $\mathbb{R}^2$.*

(ii) *$B(b)$ gives a complete description of $\sigma(t_1 b_1 + t_2 b_2)$ for all $t_1, t_2 \in \mathbb{R}$.*

(iii) *$\tau(b)$ can be read from the point $\Psi(1)$.*

(iv) *If $b = b^*$, then the set of all slopes of lines tangent to the boundary of $B(b)$ is equal to $\sigma(b)$.*

(v) *If $b = b^*$, then corners in the boundary of $B(b)$ correspond to gaps in $\sigma(b)$.*

(vi) *If $b = b^*$, then the extreme points of $B(b)$ are images under Ψ of spectral projections of b.*

(vii) *If $b \neq b^*$, then the extreme points of $B(b)$ are images under Ψ of spectral projections of the operators $t_1 b_1 + t_2 b_2$ for various $t_1, t_2 \in \mathbb{R}$.*

(viii) *If b is normal, then the set of all complex slopes of 1-dimensional faces of $B(b)$ (i.e., all the 1-dimensional "edges" in the boundary) is equal to $\sigma_p(b)$.*

(ix) *The Numerical Range $W(b)$ is equal to the set of all slopes (complex when $b \neq b^*$) of line segments emanating from the origin to points in $B(b)$.*

(x) *If $\dim(\mathcal{M}) = n$, then for each $1 \leq k \leq n$ the cross-section of $B(b)$ taken at $x = k/n$ is isomorphic to the k-numerical range*

$$W_k(b) = \left\{ \frac{1}{k} \sum_{i=1}^{k} \langle b x_i, x_i \rangle : \{x_i\} \text{ is an orthonormal set} \right\}, \quad 1 \leq k \leq \dim(\mathcal{M}). \tag{2.1}$$

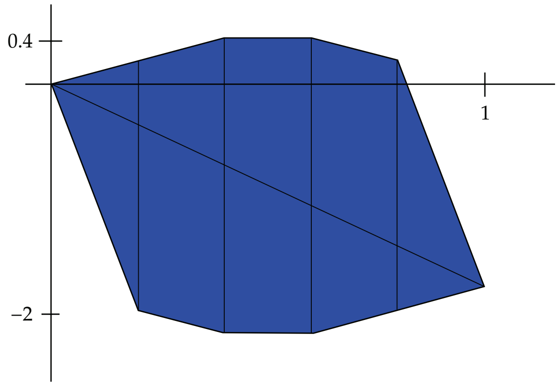

Figure 1: $B(b)$—Example 2.2.

We finish this section with a few examples of spectral scales in the *finite* setting.

Example 2.2. Let $\mathcal{M} = M_5(\mathbb{C})$ with $\tau(x) = (1/5)\,\mathrm{tr}(x)$ for all $x \in \mathcal{M}$, where tr is the canonical trace on $M_5(\mathbb{C})$. Let

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{pmatrix}. \tag{2.2}$$

$B(b)$ is given in Figure 1. The MATLAB program SpecGUI [8] was used to generate Figure 1, and by Theorem 2.1 one can see that $\sigma(b) = \{-10, -1, 0, 1\}$, $\|b\| = 10$, 1 has multiplicity 2, $W(b) = [-10, 1]$, and $\tau(b) = -9/5$.

Example 2.3. Let $\mathcal{M} = L^\infty([0,1], m)$, where m is Lebesgue measure on the interval $[0,1]$, let $\tau(c) = \int_0^1 c\, dm$ for all $c \in \mathcal{M}$, and let $b \in \mathcal{M}$ be defined by $b(x) = x$ for all $x \in [0,1]$. Then b is positive, with no eigenvalues, and $\sigma(b) = [0,1]$. $B(b)$ is given in Figure 2.

Example 2.4. Let $\mathcal{M} = M_3(\mathbb{C})$ with $\tau(c) = (1/3)\,\mathrm{tr}(c)$ for all $c \in \mathcal{M}$, and let

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{pmatrix}. \tag{2.3}$$

$B(b)$ is given in Figure 3 [8].

Since b is normal, by Theorem 2.1 we know that the eigenvalues of b can be read from $B(b)$ as the slopes (complex) of 1-dimensional faces of $B(b)$ (i.e., slopes of 1-dimensional "edges" in the boundary of $B(b)$). Alternatively, if we were given $B(b)$ without knowing

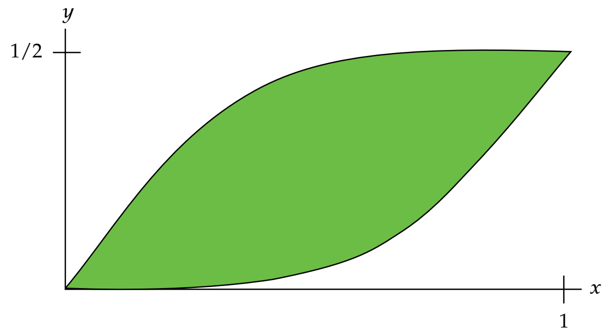

Figure 2: $B(b)$—Example 2.3.

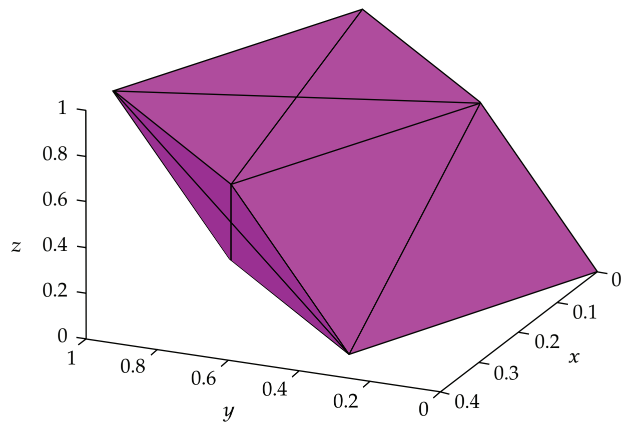

Figure 3: $B(b)$—Example 2.4.

what b actually was, we could conclude by [3, Theorem 6.1] that b is normal. The numerical range $W(b)$ of b can be seen as the slopes (complex) of secant lines between the origin and points in $B(b)$.

Our last example is a counterexample to conjecture made by Akemann in 2005 which stated: b is normal if the cross-section of $B(b)$ taken at $x = 1/2$ is a polygon. The following is a spectral scale that has no 1-dimensional faces, and the cross-section taken $x = 1/2$ is not a polygon, yet the operator it corresponds to is normal.

Example 2.5. Let $\mathcal{C}$ be the unit circle, and let m denote Lebesgue measure on $\mathcal{C}$. Put $\mathcal{M} = L^{\infty}(\mathcal{C})$, and note that $\mathcal{M}$ is a finite von Neumann algebra equipped with the finite, faithful, normal trace $\tau : L^{\infty}(\mathcal{C}) \to \mathbf{R}$ defined by $\tau(c) = (1/2\pi)\int_{\mathcal{C}} c\, dm$ for all $c \in L^{\infty}(\mathcal{C})$.

Consider the function $b(z) = z$ on $\mathcal{C}$. The spectral scale of b with respect to τ is the collection of points

$$B(b) = \left\{ (\tau(c), \tau(bc)) : c \in (L^{\infty}(\mathcal{C}))_1^+ \right\} \tag{2.4}$$

in $\mathbb{R}^3$. From [2, Theorem 2.3] we know that Ψ applied to spectral projections of the operators $b_{\mathbf{t}} = t_1 \operatorname{Re}(b) + t_2 \operatorname{Im}(b)$ for various $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ are extreme points of $B(b)$.

Let $\mu \in [0, 2\pi)$, and let $\theta \in [0, \pi]$. If $\chi_{C_{\mu,\theta}}$ is the characteristic function on the arc $C_{\mu,\theta}$ from $e^{i(\mu-\theta)}$ to $e^{i(\mu+\theta)}$, then,

$$\begin{aligned}\chi_{C_{\mu,\theta}}(z) &= \chi_{[\cos(\theta),\infty)}\left(\operatorname{Re}\left(e^{-i\mu}z\right)\right)\\ &= \chi_{[\cos(\theta),\infty)}((\cos(\mu))x - (\sin(\mu))y)\end{aligned} \tag{2.5}$$

for every $z = x + iy \in \mathbb{C}$ (to see this think of the case where $\mu = 0$). But

$$\begin{aligned}\chi_{[\cos(\theta),\infty)}((\cos(\mu))x - (\sin(\mu))y) &= \chi_{[\cos(\theta),\infty)}(t_1 \operatorname{Re}(b) + t_2 \operatorname{Im}(b))\\ &= \chi_{[\cos(\theta),\infty)}(b_{\mathbf{t}})\end{aligned} \tag{2.6}$$

for $\mathbf{t} = (t_1, t_2) = (\cos(\mu), -\sin(\mu))$. Therefore, every $\chi_{C_{\mu,\theta}}$ is a spectral projection of some $b_{\mathbf{t}}$. Conversely, if $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $s \in \mathbb{R}$, then for if μ is chosen so that $\cos(\mu) = t_1/\sqrt{t_1^2 + t_2^2}$ and $\sin(\mu) = -t_2/\sqrt{t_1^2 + t_2^2}$ and θ is chosen so that $\|\mathbf{t}\| \cos(\theta) = s$, then we have

$$\begin{aligned}\chi_{[s,\infty)}(b_{\mathbf{t}}) &= \chi_{[\|\mathbf{t}\|\cos(\theta),\infty)}(\|\mathbf{t}\| \cos(\mu) \operatorname{Re}(b) - \|\mathbf{t}\| \sin(\mu) \operatorname{Im}(b))\\ &= \chi_{[\cos(\theta),\infty)}(\cos(\mu) \operatorname{Re}(b) - \sin(\mu) \operatorname{Im}(b))\\ &= \chi_{C_{\mu,\theta}}.\end{aligned} \tag{2.7}$$

Since the extreme points of $B(b)$ come from the spectral projections described above, we can calculate the extreme points of $B(b)$ explicitly:

$$\begin{aligned}\left(\tau\left(\chi_{C_{\mu,\theta}}\right), \tau\left(b\chi_{C_{\mu,\theta}}\right)\right) &= \left(\int_{C_{\mu,\theta}} 1 dm, \int_{C_{\mu,\theta}} b dm\right)\\ &= \left(\frac{\theta}{\pi}, \int_{C_{\mu,\theta}} e^{iz} dz\right)\\ &= \left(\frac{\theta}{\pi}, \frac{1}{2\pi i}\left(e^{i(\mu+\theta)} - e^{i(\mu-\theta)}\right)\right)\\ &= \left(\frac{\theta}{\pi}, \frac{1}{\pi} e^{i\mu} \sin(\theta)\right).\end{aligned} \tag{2.8}$$

Since $B(b)$ is the convex hull of its extreme points, we see that $B(b)$ can be recovered by rotating the function $f(x) = (1/\pi)\sin(\pi x)$ $0 \le x \le 1$ about the x-axis (see Figure 4).

3. The Spectral Scale of a Self-Adjoint Operator in a Semifinite von Neumann Algebra

For the remainder of what follows, let $\mathcal{M}$ be a semifinite von Neumann algebra, equipped with a faithful, semifinite, normal trace τ, and let $b \in \mathcal{M}$ be self-adjoint. In order to avoid

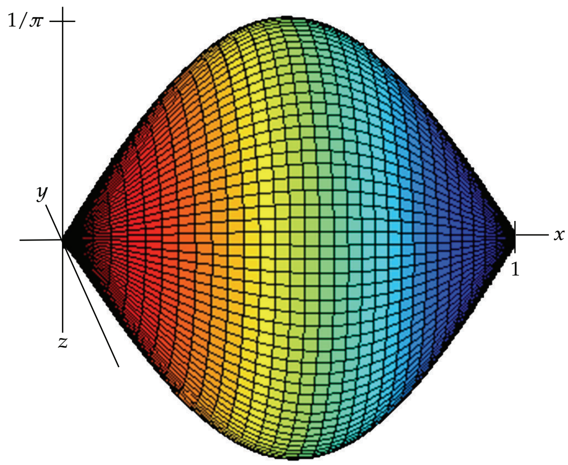

Figure 4: $B(b)$—Example 2.5.

the finite setting (for which the spectral scale has already been developed), we also assume that $\tau(1) = \infty$.

Define $\mathfrak{n}_\tau = \{x \in \mathcal{M} : \tau(x^*x) < +\infty\}$ and $\mathfrak{m}_\tau = \{\sum_{i=1}^n x_i y_i : x_i, y_i \in \mathfrak{n}_\tau\}$. The by [9, Lemma 2.16, page 318], we can define the function $\Psi : \mathfrak{m}_\tau \to \mathbb{R}^2$ by

$$\Psi(c) = (\tau(c), \tau(bc)) \quad \text{for every } c \in \mathfrak{m}_\tau. \tag{3.1}$$

Definition 3.1. The spectral scale of b (with respect to τ) is the set

$$B(b) = \Psi\left((\mathfrak{m}_\tau)_1^+\right) = \left\{(\tau(c), \tau(bc)) : c \in (\mathfrak{m}_\tau)_1^+\right\}. \tag{3.2}$$

Since $(\mathfrak{m}_\tau)_1^+$ is convex, Ψ is linear, and τ is positive, we have that $B(b)$ is a convex subset of $\mathbb{R}^2$ contained in the right half-plane. Since τ is semifinite and $\tau(1) = \infty$, $B(b)$ is an unbounded subset of $\mathbb{R}^2$. We will see that $B(b)$ may or may not be closed.

Note 1. If p and q are projections in $\mathcal{M}$ and $p \leq q$ then we write

$$[p, q] = \{a : p \leq a \leq q\} \tag{3.3}$$

for the order interval determined by p and q. Given a real number s, we let p_s^+ denote the spectral projection of b corresponding to the interval $(-\infty, s]$. Similarly, we let p_s^- denote the spectral projection of b corresponding to the interval $(-\infty, s)$. We use $p_s^\pm$ to indicate either of these projections and we write p_s when $p_s^+ = p_s^-$. We let q_s^+ and q_s^- denote the spectral projections corresponding to the intervals $[s, \infty)$ and (s, ∞), respectively. We also use $q_s^\pm$ to indicate either of these projections and we write q_s when $q_s^+ = q_s^-$. Note that $q_s^\pm = 1 - p_s^\mp$ for every $s \in \mathbb{R}$. The spectral projections $p_s^\pm$ and $q_s^\pm$ will be used throughout the remainder of this text, and will be instrumental in deciphering $B(b)$.

The following lemma was proven in [2, Lemma 1.2] for any type of von Neumann algebra.

Lemma 3.2. *Let b be a self-adjoint element in $\mathcal{M}$, $s \in \mathbb{R}$, $c \in [p_s^-, p_s^+]$, and $a \in \mathcal{M}_1^+$. Then the following statements hold:*

(1) $(b - s1)(1 - c) = (b - s1)(1 - p_s^+) \geq 0$ *and the range projection of* $(b - s1)(1 - p_s^+)$ *is* $1 - p_s^+$*;*

(2) *if* $a^{1/2}(b - s1)(1 - c)a^{1/2} = 0$*, then* $a \leq p_s^+$*;*

(3) $(s1 - b)c = (s1 - b)p_s^- \geq 0$ *and the range projection of* $(s1 - b)p_s^-$ *is* p_s^-*;*

(4) *if* $(1 - a)^{1/2}(s1 - b)c(1 - a)^{1/2} = 0$ *then* $a \geq p_s^-$.

The following two lemmas tell us that points of the form $(\tau(c), \tau(cb))$ lie on the boundary when $c \in [p_s^-, p_s^+]$ (or $c \in [q_s^-, q_s^+]$) for some $s \in \mathbb{R}$. A slight variation of these lemmas was proved in [2, Lemma 1.3].

Lemma 3.3. *If $b \in \mathcal{M}$ is self-adjoint and $s \in \mathbb{R}$, then the following statements hold for each* $c \in [p_s^-, p_s^+] \cap \mathfrak{m}_\tau$ *and* $a \in (\mathfrak{m}_\tau)_1^+$.

(1) *If* $\tau(a) = \tau(c)$*, then* $\tau(ba) \geq \tau(bc)$.

(2) *If* $\tau(a) = \tau(c)$ *and* $\tau(ba) = \tau(bc)$*, then* $a \in [p_s^-, p_s^+]$*; moreover, if* $= p_s^\pm$*, then equality in the second equation obtains only for* $= c$.

Proof. (1) From the previous lemma we have that

$$(b - s1)(1 - c) = (b - s1)\left(1 - p_s^+\right) \geq 0, \quad \text{so that } b(1 - c) \geq s(1 - c). \tag{3.4}$$

Therefore

$$a^{1/2}b(1 - c)a^{1/2} \geq a^{1/2}s(1 - c)a^{1/2}. \tag{3.5}$$

Note that $\tau(b(1 - c)a) < \infty$ since $a \in \mathfrak{m}_\tau$ and $\mathfrak{m}_\tau$ is an ideal. Since $a^{1/2} \in \mathfrak{n}_\tau$ and τ is order preserving on $\mathfrak{m}_\tau$ [6], we have that

$$\begin{aligned} \tau(b(1 - c)a) &= \tau\left(a^{1/2}b(1 - c)a^{1/2}\right) \\ &\geq \tau\left(a^{1/2}s(1 - c)a^{1/2}\right) \\ &= \tau(s(1 - c)a). \end{aligned} \tag{$*$}$$

Since $c \in \mathfrak{m}_\tau$, an identical argument yields

$$\tau(bc(1 - a)) \leq \tau(sc(1 - a)). \tag{$**$}$$

It follows from (∗), (∗∗), and our hypothesis that

$$\begin{aligned}\tau(ba)-\tau(bc) &= \tau(ba-bca-bc+bca)\\ &= \tau(b(1-c)a)-\tau(bc(1-a))\\ &\geq \tau(s(1-c)a)-\tau(sc(1-a))\\ &= \tau(sa)-\tau(sc)\\ &= s(\tau(a)-\tau(c))\\ &= 0,\end{aligned} \tag{3.6}$$

and so $\tau(ba) \geq \tau(bc)$.

To prove (2), suppose that $\tau(a)=\tau(c)$ and $\tau(ba)=\tau(bc)$. Then

$$\begin{aligned}\tau((1-c)a) &= \tau(a-ca)\\ &= \tau(c-ca)\\ &= \tau(c(1-a)),\\ \tau(b(1-c)a) &= \tau(ba-bca)\\ &= \tau(bc-bca)\\ &= \tau(bc(1-a)).\end{aligned} \tag{3.7}$$

Combining these two results with (∗) and (∗∗) we get that

$$\tau(bc(1-a)) = \tau(b(1-c)a) \geq s\tau((1-c)a) = s\tau(c(1-a)) \geq \tau(bc(1-a)). \tag{3.8}$$

Therefore, all the above terms must be equal, and so

$$\begin{aligned}\tau(b(1-c)a) &= s\tau((1-c)a),\\ \tau(bc(1-a)) &= s\tau(c(1-a)).\end{aligned} \tag{3.9}$$

From this it follows that

$$\begin{aligned}\tau\left(a^{1/2}(b-s1)(1-c)a^{1/2}\right) &= 0,\\ \tau\left((1-a)^{1/2}(b-s1)c(1-a)^{1/2}\right) &= 0.\end{aligned} \tag{3.10}$$

Since τ is faithful we get that

$$\begin{aligned}a^{1/2}(b-s1)(1-c)a^{1/2} &= 0,\\ (1-a)^{1/2}(b-s1)c(1-a)^{1/2} &= 0.\end{aligned} \tag{3.11}$$

By parts (2) and (4) of the previous lemma, $a \in [p_s^-, p_s^+]$.

If it is the case that $c = p_s^\pm$, $\tau(a) = \tau(p_s^\pm)$, and $\tau(ba) = \tau(bc)$ then by what we just showed, $a \in [p_s^-, p_s^+]$, so $\tau(p_s^\pm - a) = 0$, which implies that $a = p_s^\pm$ since τ is faithful. □

Remark 3.4. Notice the importance of our choice of a and c as elements of ideal $\mathfrak{m}_\tau$ in the above proof. Without this assumption, many of the equalities and inequalities above would have been either trivial or undefined because τ is not necessarily finite on all of $\mathcal{M}$.

Lemma 3.5. *If $b \in \mathcal{M}$ is self-adjoint and $t \in \mathbb{R}$, then the following statements hold for each $d \in [q_t^-, q_t^+] \cap \mathfrak{m}_\tau$ and $a \in (\mathfrak{m}_\tau)_1^+$.*

(1) *If $\tau(a) = \tau(d)$, then $\tau(ba) \le \tau(bd)$.*

(2) *If $\tau(a) = \tau(d)$ and $\tau(ba) = \tau(bd)$, then $a \in [q_t^-, q_t^+]$; moreover, if $= q_t^\pm$, then equality in the second equation obtains only for $a = d$.*

Proof. (1) Similarly, by Lemma 3.2 (since $1 - d \in [p_t^-, p_t^+]$), we have that

$$\begin{aligned} bd &\ge td, \\ t(1-d) &\ge b(1-d). \end{aligned} \tag{3.12}$$

Therefore

$$(1-a)^{1/2} bd (1-a)^{1/2} \ge (1-a)^{1/2} td (1-a)^{1/2}, \tag{$*$}$$

$$a^{1/2} t(1-d) a^{1/2} \ge a^{1/2} b(1-d) a^{1/2}. \tag{$**$}$$

Using the fact that the trace is faithful and finite on $\mathfrak{m}_\tau$, we get that

$$\begin{aligned} \tau(bd(1-a)) = \tau\left((1-a)^{1/2} bd (1-a)^{1/2}\right) \ge \tau\left((1-a)^{1/2} td (1-a)^{1/2}\right) = \tau(td(1-a)), \\ \tau(t(1-d)a) = \tau\left(a^{1/2} t(1-d) a^{1/2}\right) \ge \tau\left(a^{1/2} b(1-d) a^{1/2}\right) = \tau(b(1-d)a). \end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned} \tau(bd) - \tau(ba) &= \tau(bd(1-a)) - \tau(b(1-d)a) \\ &\ge \tau(td(1-a)) - \tau(t(1-d)a) \\ &= \tau(td) - \tau(ta) \\ &= 0. \end{aligned} \tag{3.14}$$

To prove (2), suppose that $\tau(a)=\tau(d)$ and $\tau(ba)=\tau(bd)$. Then

$$\begin{aligned}\tau((1-d)a)&=\tau(a-da)\\&=\tau(d-da)\\&=\tau(d(1-a)),\\\tau(b(1-d)a)&=\tau(ba-bda)\\&=\tau(bd-bda)\\&=\tau(bd(1-a)).\end{aligned}\tag{3.15}$$

From these two observations, $(*)$, and $(**)$ we get that

$$\tau(bd(1-a))=\tau(b(1-d)a)\leq t\tau((1-d)a)=t\tau(d(1-a))\leq\tau(bd(1-a)).\tag{3.16}$$

But then all these terms must be equal. Therefore $\tau(b(1-d)a)=t\tau((1-d)a)$ and $\tau(bd(1-a))=t\tau(d(1-a))$. It follows that

$$\begin{aligned}\tau\left(a^{1/2}(b-t1)(1-d)a^{1/2}\right)&=0,\\\tau\left((1-a)^{1/2}(b-t1)d(1-a)^{1/2}\right)&=0.\end{aligned}\tag{3.17}$$

But since $(b-t1)(1-d)\geq 0$ and $(b-t1)d\geq 0$ we get

$$\begin{aligned}a^{1/2}(b-t1)(1-d)a^{1/2}&=0,\\(1-a)^{1/2}(b-t1)d(1-a)^{1/2}&=0.\end{aligned}\tag{3.18}$$

By Lemma 3.2 we get $a\in[q_t^-,q_t^+]$. If $d=q_t^{\pm}$, then since τ is faithful we must have $a=d=q_t^{\pm}$. □

Notice that this is one major difference between $B(b)$ in the case where $\mathcal{M}$ is *finite*, and $B(b)$ when $\mathcal{M}$ is semifinite. In the *finite* setting $B(b)$ was convex and compact. Next, we define the boundary of $B(b)$, and then study some of its properties.

Definition 3.6. Let $\overline{B(b)}$ denote the closure of $B(b)$ in $\mathbb{R}^2$. The *lower boundary* of $B(b)$ is defined as

$$\partial_{\mathcal{L}}B(b)=\left\{(x,y)\in\overline{B(b)}:(x,y')\in B(b)\Longrightarrow y'\geq y\right\}.\tag{3.19}$$

The *upper boundary* of $B(b)$ is defined analogously as

$$\partial_{\mathcal{U}}B(b)=\left\{(x,y)\in\overline{B(b)}:(x,y')\in B(b)\Longrightarrow y'\leq y\right\}.\tag{3.20}$$

Since $\overline{B(b)}$ is a closed convex set, and τ takes on every value in $[0,\infty]$, we let

$$f : [0,\infty) \longrightarrow \mathbb{R} \tag{3.21}$$

defined by $f(x) = y$ for each $(x,y) \in \partial_{\mathcal{L}} B(b)$ denote the *lower boundary function determined by* b. Similarly, let

$$g : [0,\infty) \longrightarrow \mathbb{R} \tag{3.22}$$

defined by $g(x) = y$ for all $(x,y) \in \partial_{\mathcal{U}} B(b)$ denote the *upper boundary function determined by* b.

It is clear that f is a convex function and g is a concave function since $\overline{B(b)}$ is convex.

Theorems 3.7 and 3.8 describe the possibilities for the boundary of $B(b)$. These theorems describe the extreme points $B(b)$, the line segments in $\partial_{\mathcal{L}} B(b) \cap B(b)$ and $\partial_{\mathcal{U}} B(b) \cap B(b)$, respectively, and the rays in $\partial_{\mathcal{L}} B(b) \cap B(b)$ and $\partial_{\mathcal{U}} B(b) \cap B(b)$ (resp.). Rays contained in $\partial_{\mathcal{L}} B(b) \setminus B(b)$ and $\partial_{\mathcal{U}} B(b) \setminus B(b)$ will be described in Theorems 3.9 and 3.10 (resp.). Differentiability of the functions f and g will be discussed in Theorems 3.12 and 3.13 (resp.).

Note 2. For a convex set $\mathcal{C}$ we let $\operatorname{Ext}(\mathcal{C})$ denote the set of extreme points of $\mathcal{C}$.

Theorem 3.7. *Let $a \in (\mathfrak{m}_\tau)_1^+$. Then one has the following.*

(1) *If $s \in \sigma(b)$ with $p_s^\pm \in \mathfrak{m}_\tau$, then $\Psi(p_s^\pm) \in \operatorname{Ext}(B(b)) \cap \partial_{\mathcal{L}} B(b)$.*

(2) *If $s \in \sigma_p(b)$ with $p_s^+ \in \mathfrak{m}_\tau$, then $\Psi([p_s^-, p_s^+])$ is a line segment in $\partial_{\mathcal{L}} B(b) \cap B(b)$ having slope s.*

(3) *If $\Psi(a) \in \partial_{\mathcal{L}} B(b) \cap B(b)$ and $\Psi(a) \neq \Psi(p_s^\pm)$ for all $s \in \mathbb{R}$, then $a \in [p_s^-, p_s^+]$ for some $s \in \sigma(b)$ with $\tau(p_s^-) < \infty$.*

(4) *If $\Psi(a) \in \operatorname{Ext}(B(b)) \cap \partial_{\mathcal{L}} B(b)$, then $a = p_s^\pm$ for some $s \in \sigma_p(b)$ with $p_s^\pm \in \mathfrak{m}_\tau$.*

(5) *$\mathcal{R}$ is a ray in $\partial_{\mathcal{L}} B(b) \cap B(b)$ having slope s if and only if $\mathcal{R} = \Psi([p_s^-, p_s^+])$ and $s \in \sigma_p(b)$ with $\tau(p_s^-) < \tau(p_s^+) = \infty$.*

Proof. (1) Fix $s \in \sigma(b)$ such that $\tau(p_s^\pm) < \infty$. Then by part (1) of Lemma 3.3 we have that

$$\Psi(p_s^\pm) \in \partial_{\mathcal{L}} B(b). \tag{3.23}$$

By part (2) of Lemma 3.3 we have that

$$\Psi^{-1}\left(\Psi(p_s^\pm)\right) \cap (\mathfrak{m}_\tau)_1^+ = \{p_s^\pm\}. \tag{3.24}$$

To see that $\Psi(p_s^\pm)$ are extreme in $B(b)$, suppose that

$$\Psi(p_s^\pm) = \lambda \Psi(a_1) + (1-\lambda)\Psi(a_2) \tag{3.25}$$

for some $a_1, a_2 \in (\mathfrak{m}_\tau)_1^+$ and $0 < \lambda < 1$. But then

$$\Psi(p_s^\pm) = \Psi(\lambda a_1 + (1-\lambda) a_2), \tag{3.26}$$

and by part (2) of Lemma 3.3 we have that $p_s^\pm = \lambda a_1 + (1-\lambda)a_2$. Since projections are extreme points of $\mathcal{M}_1^+$, we must have $a_1 = a_2 = p_s^\pm$. Therefore, $\Psi(p_s^\pm) \in \mathrm{Ext}(B(b)) \cap \partial_{\mathcal{L}} B(b)$.

(2) Fix $s \in \sigma_p(b)$ so that $p_s^+ \in \mathfrak{m}_\tau$. Then we have that $p_s^- < p_s^+$, $\Psi(p_s^-) \neq \Psi(p_s^+)$, and $b(p_s^+ - p_s^-) = s(p_s^+ - p_s^-)$. Note that if $c \in [p_s^-, p_s^+]$, then $\Psi(c) \in \partial_{\mathcal{L}} B(b)$ by part (1) of Lemma 3.3, so

$$\Psi([p_s^-, p_s^+]) \subset \partial_{\mathcal{L}} B(b) \cap B(b). \tag{3.27}$$

For each $\lambda \in (0,1)$ let $p_\lambda = \lambda p_s^- + (1-\lambda)p_s^+$. Then, each p_λ is in $[p_s^-, p_s^+]$, and so $\Psi(p_\lambda) \in \partial B_{\mathcal{L}}(b) \cap B(b)$. Furthermore, $\Psi(p_\lambda)$ is a typical point on the line segment between $\Psi(p_s^-)$ and $\Psi(p_s^+)$ and the slope of this line segment is

$$\frac{\tau(bp_s^+) - \tau(bp_s^-)}{\tau(p_s^+) - \tau(p_s^-)} = \frac{\tau(b(p_s^+ - bp_s^-))}{\tau(p_s^+ - p_s^-)} = \frac{\tau(s(p_s^+ - p_s^-))}{\tau(p_s^+ - p_s^-)} = s. \tag{3.28}$$

Therefore $\Psi([p_s^-, p_s^+])$ is a line segment in $\partial_{\mathcal{L}} B(b) \cap B(b)$ with slope s. Note that since $\Psi(p_s^\pm)$ are extreme points of $B(b)$ contained in $\partial_{\mathcal{L}} B(b)$, we know that the endpoints of any line segment containing $\Psi([p_s^-, p_s^+])$ must be $\Psi(p_s^-)$ and $\Psi(p_s^+)$.

(3) Suppose $\Psi(a) \in \partial_{\mathcal{L}} B(b) \cap B(b)$ and $\Psi(a) \neq \Psi(p_s^\pm)$ for all $s \in \sigma(b)$. We will show that $p_s^- < a < p_s^+$ for some $s \in \sigma(b)$ with $\tau(p_s^-) < \infty$. Let

$$\begin{aligned} r_1 &= \sup\{s \in \sigma(b) : \tau(p_s^-) \le \tau(a)\}, \\ r_2 &= \inf\{s \in \sigma(b) : \tau(a) \le \tau(p_s^+)\}. \end{aligned} \tag{3.29}$$

We want to show that $r_1 = r_2$. Note that r_1 and r_2 are in $\sigma(b)$ since $\sigma(b)$ is closed, and $\tau(p_{r_1}^-) \le \tau(a)$ by lower semicontinuity of τ.

We first show that $r_1 \le r_2$. If $s, t \in \sigma(b)$ with $\tau(p_s^-) \le \tau(a) \le \tau(p_t^+)$, and $t < s$, then by the positivity of τ we must also have $\tau(p_t^+) \le \tau(p_s^-)$, so $\tau(p_s^-) = \tau(a) = \tau(p_t^+)$. This fact, along with Lemma 3.3, contradicts our assumption about $\Psi(a)$. Thus, everything in $\{s \in \sigma(b) : \tau(p_s^-) \le \tau(a)\}$ is less than or equal to everything in $\{s \in \sigma(b) : \tau(a) \le \tau(p_s^+)\}$, hence $r_1 \le r_2$.

Now, either $\tau(a) \le \tau(p_{r_2}^+)$ or not. Suppose $\tau(p_{r_2}^+) < \tau(a)$. Since $\tau(p_{r_2}^-) \le \tau(p_{r_2}^+) < \tau(a)$, and since r_1 is an upper bound for $\{s \in \sigma(b) : \tau(p_s^-) \le \tau(a)\}$, we must have $r_2 \le r_1$. But $r_1 \le r_2$. Therefore $r_1 = r_2$ when $\tau(p_{r_2}^+) < \tau(a)$.

Assume that $\tau(a) \le \tau(p_{r_2}^+)$. Then, $\tau(p_{r_1}^-) \le \tau(a) \le \tau(p_{r_2}^+)$. But we are assuming that $\Psi(a) \neq \Psi(p_s^\pm)$ for all $s \in \sigma(b)$, and that $\Psi(a) \in \partial_{\mathcal{L}} B(b) \cap B(b)$. Therefore, by Lemma 3.3 we get

$$\tau(p_{r_1}^-) < \tau(a) < \tau(p_{r_2}^+). \tag{3.30}$$

Observe that, since r_2 is a lower bound for the set $\{s \in \sigma(b) : \tau(a) \le \tau(p_s^+)\}$, if $s < r_2$ and $s \in \sigma(b)$ then

$$\tau(p_s^+) < \tau(a) < \tau(p_{r_2}^+). \tag{3.31}$$

Similarly, if $s \in \sigma(b)$ with $r_1 < s$, then

$$\tau(p_{r_1}^-) < \tau(a) < \tau(p_s^-). \tag{3.32}$$

Suppose that $r_1 < r_2$. Then since $r_1, r_2 \in \sigma(b)$ we have that

$$\tau(p_{r_1}^+) < \tau(a) < \tau(p_{r_2}^-). \tag{3.33}$$

On the other hand if $s \in (r_1, r_2) \cap \sigma(b)$ then by (3.32) and (3.33) we would have $\tau(p_s^+) < \tau(a) < \tau(p_s^-)$. This is not possible, so it must be that $(r_1, r_2) \cap \sigma(b) = \emptyset$. Therefore $p_{r_1}^+ = p_{r_2}^-$, and we have $\tau(p_{r_1}^+) = \tau(p_{r_2}^-)$, contradicting (3.33). Hence, it must be that $r_1 = r_2 = r$ and so $\tau(p_r^-) < \tau(a) < \tau(p_r^+)$. Therefore $r \in \sigma_p(b)$ and $a \in [p_r^-, p_r^+]$ by Lemma 3.3.

(4) Since $\overline{B(b)}$ is convex, every point in the boundary of $B(b)$ must either be an extreme point, or lie on a flat spot (face). Therefore, (1), and (2), (3), imply (4).

(5) ($\Leftarrow$) Suppose $\tau(p_s^-) < \tau(p_s^+) = \infty$ for some $s \in \sigma_p(b)$. We claim that $\mathcal{R} = \Psi([p_s^-, p_s^+])$ is a ray in $\partial_{\mathcal{L}} B(b) \cap B(b)$ with slope s, emanating from the point $\Psi(p_s^-) = (\tau(p_s^-), \tau(bp_s^-))$. We will first show that every point in $\Psi([p_s^-, p_s^+])$ lies on the ray through $\Psi(p_s^-)$ with slope s.

Let $c \in [p_s^-, p_s^+] \cap \mathfrak{m}_\tau$, and write $c = p_s^- + d$ for some $d \in (p_s^+ - p_s^-)\mathcal{M}(p_s^+ - p_s^-)$. Then $b(c - p_s^-) = bd = sd = s(c - p_s^-)$, so $\tau(b(c - p_s^-)) = \tau(s(c - p_s^-))$. This implies that

$$\tau(bc) = \tau(s(c - p_s^-)) + \tau(bp_s^-) = s \cdot \tau(c) + \tau((b - s1)p_s^-). \tag{3.34}$$

Therefore $\Psi(c)$ lies on the ray with slope s passing through the point $(\tau(p_s^-), \tau(bp_s^-))$. Furthermore, by the convexity of $B(b)$, we know there can only be one ray having slope s in the lower boundary of $B(b)$, and so if $\Psi(x)$ is any other point in $B(b)$ that lies on a ray with slope s in the lower boundary, then we must have $\Psi(x) \in \mathcal{R} = \Psi([p_s^-, p_s^+])$.

(5) ($\Rightarrow$) If $\mathcal{R}$ is a ray in $\partial_{\mathcal{L}} B(b) \cap B(b)$ with slope s, emanating from $\Psi(x)$, then $\Psi(x)$ is an extreme point of $B(b)$ (if not, then $\mathcal{R}$ does not emanate from $\Psi(x)$ since $\Psi(x)$ would then lie on a flat portion of $\partial_{\mathcal{L}} B(b) \cap B(b)$), so we must have $x = p_r^\pm \in \mathfrak{m}_\tau$ for some $r \in \sigma(b)$ by part (4). But $\mathcal{R}$ is a ray, so $\mathcal{R}$ cannot contain any extreme points of $B(b)$. So by part (3) we know that if $z \in (\mathfrak{m}_\tau)_1^+$ with $\Psi(z) \in \mathcal{R}$, then $z \in [p_{s_z}^-, p_{s_z}^+]$ for some $s_z \in \sigma_p(b)$ with $\tau(p_{s_z}^-) < \infty$. Furthermore, since any such $\Psi([p_{s_z}^-, p_{s_z}^+])$ is a line segment (or ray) with slope s_z, and since $\Psi([p_{s_z}^-, p_{s_z}^+]) \subset \mathcal{R}$, we must have $s_z = s$. Since there can only be one ray with slope s in the lower boundary of $B(b)$, we may conclude that $r = s$ and $\mathcal{R} = \Psi([p_s^-, p_s^+])$. □

Theorem 3.8. *Let $a \in (\mathfrak{m}_\tau)_1^+$. Then one has the following.*

(1) *If $t \in \sigma(b)$ with $q_t^\pm \in \mathfrak{m}_\tau$, then $\Psi(q_t^\pm) \in \operatorname{Ext}(B(b)) \cap \partial_{\mathcal{U}} B(b)$.*

(2) *If $t \in \sigma_p(b)$ with $q_t^+ \in \mathfrak{m}_\tau$, then $\Psi([q_t^-, q_t^+])$ is a line segment in $\partial_{\mathcal{U}} B(b) \cap B(b)$ having slope t.*

(3) *If $\Psi(a) \in \partial_{\mathcal{U}} B(b) \cap B(b)$ and $\Psi(a) \neq \Psi(q_t^\pm)$ for all $t \in \mathbb{R}$, then $a \in [q_t^-, q_t^+]$ for some $t \in \sigma(b)$ with $\tau(q_t^-) < \infty$.*

(4) *If $\Psi(a) \in \operatorname{Ext}(B(b)) \cap \partial_{\mathcal{U}} B(b)$ then $a = q_t^\pm$ for some $t \in \sigma_p(b)$ with $q_t^\pm \in \mathfrak{m}_\tau$.*

(5) *$\mathcal{R}$ is a ray in $\partial_{\mathcal{U}} B(b) \cap B(b)$ having slope t if and only if $\mathcal{R} = \Psi([q_t^-, q_t^+])$ and $t \in \sigma_p(b)$ with $\tau(q_t^-) < \tau(q_t^+) = \infty$.*

Proof. Note that for each $s \in \mathbb{R}$ we have that

$$q_s^- = \chi_{(s,\infty)}(b) = \chi_{(-\infty,-s)}(-b). \tag{3.35}$$

Since $-b$ is self-adjoint, the previous theorem applies. Alternatively, substitute Lemma 3.5 for Lemma 3.3 in the proof of Theorem 3.7, the claim follows immediately. □

Until this point we have focused on points that lie in $B(b)$. As mentioned earlier, it may be the case that $B(b)$ is not closed. The next four results describe this phenomenon, and tell us exactly when $B(b)$ is closed, when it is not, and, when we get a "dotted" (open) portion in the boundary of $B(b)$.

Theorem 3.9. *Let $\mathcal{E}_{\mathcal{L}} = \{s \in \sigma(b) : \tau(p_s^+) = \infty\}$ and let $\lambda = \inf \mathcal{E}_{\mathcal{L}}$. Then $\partial_{\mathcal{L}} B(b) \cap B(b) = \partial_{\mathcal{L}} B(b)$ if and only if $\tau(p_\lambda^+) = \infty$. Furthermore, if $\tau(p_\lambda^+) < \infty$ then*

$$y = \lambda x + \tau\left((b-\lambda)p_\lambda^+\right), \quad x > \tau\left(p_\lambda^+\right) \tag{3.36}$$

is a ray in $\partial_{\mathcal{L}} B(b) \setminus B(b)$ emanating from $\Psi(p_\lambda^+)$.

Proof. Suppose that $\partial_{\mathcal{L}} B(b) \cap B(b) = \partial_{\mathcal{L}} B(b)$. Theorem 3.7 tells us that every $\Psi(c) \in \partial_{\mathcal{L}} B(b)$ has the property that either $c \in [p_s^-, p_s^+]$ for some $s \in \sigma_p(b)$ with $\tau(p_s^-) < \infty$, or $c = p_s^\pm$ for some $s \in \sigma(b)$ with $p_s^\pm \in \mathfrak{m}_\tau$. Note that by the definition of λ, $\tau(p_s^+) = \infty$ for all $s > \lambda$. Furthermore, for every $s > \lambda$ we can choose $\lambda < s' < s$, so $\infty = \tau(p_{s'}^+) \leq \tau(p_s^-)$. Therefore, for every $s > \lambda$ we have $\tau(p_s^-) = \infty$. Suppose, for the sake of contradiction, that $\tau(p_\lambda^+) < \infty$. Then, if $d \in (\mathfrak{m}_\tau)_1^+$ with $\Psi(d) \in \partial_{\mathcal{L}} B(b)$ and $\tau(d) > \tau(p_\lambda^+)$, then $d \in [p_r^-, p_r^+]$ for some $r \in \sigma(b)$ with $\tau(p_r^-) < \infty$. Therefore it must be that $r \leq \lambda$. Clearly this is a contradiction since $\tau(d) > \tau(p_\lambda^+)$. Hence, it must be that $\tau(p_\lambda^+) = \infty$.

Suppose $\tau(p_\lambda^+) = \infty$, and let $(x, y) \in \partial_{\mathcal{L}} B(b)$. We want to show that $(x, y) \in B(b)$. Now, either $\tau(p_\lambda^-) = \infty$ or not. If $\tau(p_\lambda^-) = \infty$ then since $\tau(p_s^+) < \infty$ for all $s < \lambda$, we get that $\tau(\chi_{(\lambda-\epsilon,\lambda)}(b)) = \infty$ for every $\epsilon > 0$. For each $n \in \mathbb{N}$ let $\lambda_n = \lambda - 1/n$. Then $\tau(p_{\lambda_n}^-) \leq \tau(p_{\lambda_n}^+) < \infty$ for every n, and $\tau(p_{\lambda_n}^+) \to \infty$ as $n \to \infty$. Therefore, we can find $s \in \sigma(b)$ such that $\tau(p_s^+) > x$. Let

$$\begin{aligned} r_1 &= \sup\left\{s \in \sigma(b) : \tau\left(p_s^-\right) \leq x\right\}, \\ r_2 &= \inf\left\{s \in \sigma(b) : x \leq \tau\left(p_s^+\right)\right\}. \end{aligned} \tag{3.37}$$

To show that $(x, y) \in B(b)$, we will show that $r_1 = r_2$, which will force $(x, y) = \Psi(c)$ for some $c \in [p_{r_1}^-, p_{r_1}^+]$ by Theorem 3.7. Note that $\tau(p_{r_1}^-) \leq x$ by lower semicontinuity of τ. We will first show that $r_1 \leq r_2$.

If $s, t \in \sigma(b)$ with $\tau(p_s^-) \leq x \leq \tau(p_t^+)$, and $t < s$, then by the positivity of τ we must also have $\tau(p_t^+) \leq \tau(p_s^-)$, which implies $\tau(p_s^-) = \tau(p_t^+)$. Thus, everything in $\{s \in \sigma(b) : \tau(p_s^-) \leq x\}$ is less than or equal to everything in $\{s \in \sigma(b) : x \leq \tau(p_s^+)\}$, hence $r_1 \leq r_2$.

Suppose $r_1 < r_2$. Either $x \leq \tau(p_{r_2}^+)$ or not. Suppose $\tau(p_{r_2}^+) < x$. Since $\tau(p_{r_2}^-) \leq \tau(p_{r_2}^+) < x$, and since r_1 is an upper bound for $\{s \in \sigma(b) : \tau(p_s^-) \leq x\}$, we get $r_2 \leq r_1$. But this can not be, so it must be that $x \leq \tau(p_{r_2}^+)$. This implies $\tau(p_{r_1}^-) \leq x \leq \tau(p_{r_2}^+)$. If equality holds on either side, then we would get $(x, y) \in B(b)$ (again, by Theorem 3.7).

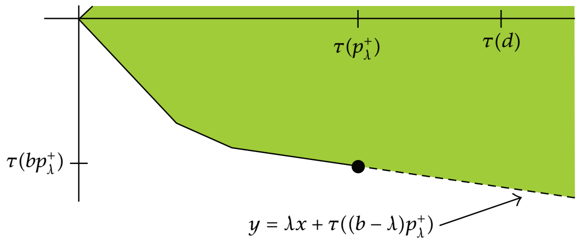

Figure 5: Ray in $\partial_{\mathcal{L}}B(b) \setminus B(b)$—Theorem 3.9.

Suppose $\tau(p_{r_1}^-) < x < \tau(p_{r_2}^+)$. Observe that, since r_2 is a lower bound for the set $\{s \in \sigma(b) : x \leq \tau(p_s^+)\}$, if $s < r_2$ and $s \in \sigma(b)$ then

$$\tau\left(p_s^+\right) < x < \tau\left(p_{r_2}^+\right). \tag{3.38}$$

Similarly, if $s \in \sigma(b)$ with $r_1 < s$ then

$$\tau\left(p_{r_1}^-\right) < x < \tau\left(p_s^-\right). \tag{3.39}$$

Since $r_1 < r_2$, we have

$$\tau\left(p_{r_1}^+\right) < x < \tau\left(p_{r_2}^-\right). \tag{3.40}$$

Thus, if $s \in (r_1, r_2) \cap \sigma(b)$ then by (3.32) and (3.33) we would have $\tau(p_s^+) < x < \tau(p_s^-)$. This is not possible, so it must be that $(r_1, r_2) \cap \sigma(b) = \emptyset$. Therefore $p_{r_1}^+ = p_{r_2}^-$, and we have $\tau(p_{r_1}^+) = \tau(p_{r_2}^-)$, contradicting (3.33). Hence, it must be that $r_1 = r_2 = r$ and so $\tau(p_r^-) < x < \tau(p_r^+)$, and our desired result holds. Therefore, in this case we have $\partial_{\mathcal{L}}B(b) \cap B(b) = \partial_{\mathcal{L}}B(b)$.

If $\tau(p_\lambda^-) < \infty$ then by Theorem 3.7 $\Psi([p_\lambda^-, p_\lambda^+])$ will be a ray in $\partial_{\mathcal{L}}B(b) \cap B(b)$. In both cases we have $\partial_{\mathcal{L}}B(b) \cap B(b) = \partial_{\mathcal{L}}B(b)$.

For the second conclusion of the theorem, suppose $\tau(p_\lambda^+) < \infty$. Plugging $x = \tau(p_\lambda^+)$ into $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$ we get $y = \tau(bp_\lambda^+)$, so the point $\Psi(p_\lambda^+)$ lies on the line $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$. We will now show that, for $x > \tau(p_\lambda^+)$, the ray $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$ forms the lower boundary of $B(b)$, but no point on the ray lies in $B(b)$.

Suppose $d \in (\mathfrak{m}_\tau)_1^+$ with $\tau(d) > \tau(p_\lambda^+)$ (see Figure 5).

We will show that $\Psi(d)$ lies above the line $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$, and, for any $\epsilon > 0$ there exists $d' \in (\mathfrak{m}_\tau)_1^+$ such that $\tau(d') = \tau(d)$ and $\tau(bd')$ is within ϵ of $y = \lambda\tau(d') + \tau((b-\lambda)p_\lambda^+)$.

We will first show that $\Psi(d)$ lies above $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$ when $\lambda = 0$, then prove the general case ($\lambda \neq 0$). Suppose $\lambda = 0$. We want to show that

$$\tau(bd) > \lambda\tau(d) + \tau\left((b-\lambda)p_\lambda^+\right) = \tau\left(bp_0^+\right). \tag{3.41}$$

Suppose for the sake of contradiction that $\tau(bd) \leq \tau(bp_0^+)$. Since $B(b)$ is convex there must be $d' \in (\mathfrak{m}_\tau)_1^+$ such that $\tau(d') = \tau(d)$ and $\tau(bd') = \tau(bp_0^+)$ (see Figure 6).

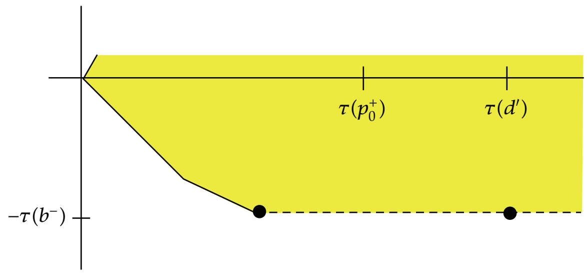

Figure 6: Ray with slope 0 in $\partial_{\mathcal{L}}B(b) \setminus B(b)$ —Theorem 3.9.

Write $b = b^+ - b^-$ with b^+ and b^- positive and $b^+b^- = 0$. Note that $\tau(bp_0^+) = -\tau(b^-)$, so we have $\tau(bd') = -\tau(b^-)$. On the other hand we have that

$$\begin{aligned}
-\tau(b^-) &= \tau(bd') \\
&= \tau(b^+d') - \tau(b^-d') \\
&= \tau(b^+d') - \tau(p_0^-b^-p_0^-d') \\
&= \tau(b^+d') - \tau\left((b^-)^{1/2}p_0^-d'p_0^-(b^-)^{1/2}\right) \\
&\geq \tau(b^+d') - \tau\left((b^-)^{1/2}p_0^-(b^-)^{1/2}\right) \\
&= \tau(b^+d') - \tau(b^-).
\end{aligned} \tag{3.42}$$

Therefore, adding $\tau(b^-)$ to both sides we get $\tau(b^+d') \leq 0$. This implies $0 = \tau(b^+d') = \tau((d')^{1/2}b^+(d')^{1/2})$. Therefore, since τ is faithful,

$$0 = (d')^{1/2}b^+(d')^{1/2} = (d')^{1/2}b(1 - p_0^+)(d')^{1/2}. \tag{3.43}$$

By part (2) of Lemma 3.2 we have

$$d' \leq p_0^+. \tag{3.44}$$

But our assumption is that $\tau(d') = \tau(d) > \tau(p_\lambda^+) = \tau(p_0^+)$. Therefore, we have reached a contradiction to our assumption that $\tau(bd) \leq \tau(bp_0^+)$, so it must be that $\tau(bd) > \tau(bp_0^+)$.

Now suppose that $\lambda \neq 0$. As before, we want to show that

$$\tau(bd) > \lambda\tau(d) + \tau\left((b - \lambda)p_\lambda^+\right). \tag{3.45}$$

Suppose that is not. Then as before we can find $d' \in (\mathfrak{m}_\tau)_1^+$ such that

$$\tau(bd') = \lambda\tau(d') + \tau\left((b - \lambda)p_\lambda^+\right). \tag{3.46}$$

Note that for every $s \in \mathbb{R}$ we have that

$$p_s^+ = \chi_{(-\infty,s]}(b) = \chi_{(-\infty,0]}(b - s1). \tag{3.47}$$

Let $a = b - \lambda 1$ and consider $B(a)$. From (3.46) and (3.47) we have that

$$\tau(ad') = \tau(a\chi_{(-\infty,0]}(a)). \tag{3.48}$$

Using (3.47) and the argument concluded at (3.44) we get $d' \le \chi_{(-\infty,0]}(a) = p_\lambda^+$. But this is a contradiction since $\tau(d') = \tau(d) > \tau(p_\lambda^+)$. Therefore $\tau(bd) > \lambda\tau(d) + \tau((b-\lambda)p_\lambda^+)$.

To prove $B(b)$ comes arbitrarily close to the line $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$, let $\epsilon > 0$. Using the definition of λ and our assumption that $\tau(p_\lambda^+) < \infty$ we have $\tau(p_{\lambda+\epsilon/2}^+ - p_\lambda^+) = \infty$. Since τ is semifinite we can find a nonzero, positive $e < p_{\lambda+\epsilon/2}^+ - p_\lambda^+$ such that $\tau(e) < \infty$. Furthermore, we may scale e so that $\tau(p_\lambda^+ + e) = \tau(d)$. Let $d' = p_\lambda^+ + e$. Then since $e < p_{\lambda+\epsilon/2}^+ - p_\lambda^+ \le 1 - p_\lambda^+ = q_\lambda^-$, we have that

$$\begin{aligned}
\tau(bd') - (\lambda\tau(d') + \tau((b-\lambda)p_\lambda^+)) &= \tau(bd') - \lambda\tau(d') - \tau((b-\lambda)p_\lambda^+) \\
&= \tau(b(p_\lambda^+ + e)) - \lambda\tau(p_\lambda^+ + e) - \tau((b-\lambda)p_\lambda^+) \\
&= \tau((b-\lambda)e) \\
&= \tau((b-\lambda)q_\lambda^- e q_\lambda^-) \\
&= \tau(q_\lambda^-(b-\lambda)q_\lambda^- e) \\
&= \tau\left(e^{1/2}q_\lambda^-(b-\lambda)q_\lambda^- e^{1/2}\right) \\
&\le \tau\left(e^{1/2}q_\lambda^- e^{1/2}\right) \\
&= \tau(q_\lambda^- e) \\
&= \tau(e) \\
&< \frac{\epsilon}{2}.
\end{aligned} \tag{3.49}$$

□

Theorem 3.10. *Let $\mathcal{E}_U = \{t \in \sigma(b) : \tau(q_t^+) = \infty\}$ and let $\rho = \sup \mathcal{E}_{\mathcal{U}}$. Then $\partial_{\mathcal{U}} B(b) \cap B(b) = \partial_{\mathcal{U}} B(b)$ if and only if $\tau(q_\rho^+) = \infty$. Furthermore, if $\tau(q_\rho^+) < \infty$ then*

$$y = \rho x + \tau\left((b-\rho)q_\rho^+\right), \quad x > \tau\left(q_\rho^+\right) \tag{3.50}$$

is a ray in $\partial_{\mathcal{U}} B(b) \setminus B(b)$ emanating from $\Psi(q_\rho^+)$.

Proof. Substitute Theorem 3.8 for Theorem 3.7 in the proof of Theorem 3.9. □

Corollary 3.11. *Let λ and ρ be as in Theorems 3.9 and 3.10 (resp.). Then $B(b) = \overline{B(b)}$ if and only if $\tau(p_\lambda^+) = \tau(q_\rho^+) = \infty$.*

Proof. Note that $\tau(1) = \infty$ if and only if $\mathcal{E}_{\mathcal{U}} \neq \emptyset$ if and only if $\mathcal{E}_{\mathcal{L}} \neq \emptyset$ (since $q_s^{\pm} = 1 - p_s^{\mp}$ for each $s \in \mathbb{R}$). Thus, the claim follows immediately from Theorems 3.9 and 3.10. □

Note that Theorems 3.7, 3.8, 3.9, and 3.10 tell us that the boundary of $B(b)$ (which contains information about $\sigma(b)$) is completely determined by spectral projections of the form $p_s^{\pm}$ and $q_t^{\pm} = 1 - p_t^{\mp}$ where $s, t \in \mathbb{R}$. To summarize, the following information is contained in the boundary of $B(b)$.

(i) Extreme points contained in the lower boundary are of the form $\Psi(p_s^{\pm})$ with $s \in \mathbb{R}$.

(ii) Line segments having slope s in the lower boundary are of the form $\Psi([p_s^-, p_s^+])$ with $s \in \mathbb{R}$ and $\tau(p_s^+) < \infty$.

(iii) Rays having slope s in the lower boundary are of the form $\Psi([p_s^-, p_s^+])$ with $s \in \mathbb{R}$ and $\tau(p_s^-) < \infty$ and $\tau(p_s^+) = \infty$.

(iv) A "dotted" ray in the lower boundary occurs if $\tau(p_\lambda^+) < \infty$. Moreover, this ray emanates from $\Psi(p_\lambda^+)$ and has as its equation $y = \lambda x + \tau((b-\lambda)p_\lambda^+)$.

(v) Extreme points contained in the upper boundary are of the form $\Psi(q_t^{\pm})$ with $t \in \mathbb{R}$.

(vi) Line segments having slope t in the upper boundary are sets of the form $\Psi([q_t^-, q_t^+])$ with $t \in \mathbb{R}$ and $\tau(q_t^+) < \infty$.

(vii) Rays having slope t in the upper boundary are sets of the form $\Psi([q_t^-, q_t^+])$ with $t \in \mathbb{R}$ and $\tau(q_t^-) < \infty$ and $\tau(q_t^+) = \infty$.

(viii) A "dotted" ray in the upper boundary occurs if $\tau(q_\rho^+) < \infty$. Moreover, this ray emanates from $\Psi(q_\rho^+)$ and has as its equation $y = \rho x + \tau((b-\rho)q_\rho^+)$.

Recall that for each $(x, y) \in \partial_{\mathcal{L}} B(b)$ we let $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = y$ denote the lower boundary function generated by b. Similarly, g denotes the upper boundary function generated by b. For each $x \in [0, \infty)$, let $f'_-(x)$ ($g'_-(x)$) and $f'_+(x)$ ($g'_+(x)$) denote the left-sided and right-sided, (respectively) derivatives of f and g at x. Note that by Theorem 3.7, every $\Psi(c)$ in $\partial_{\mathcal{L}} \cap B(b)$ has the property that $\Psi(c) \in \Psi([p_s^-, p_s^+])$ for some $s \in \sigma(b)$ with $\tau(p_s^-) < \infty$.

Theorem 3.12. *Let $s \in \sigma(b)$, and let $c \in [p_s^-, p_s^+] \cap (\mathfrak{m}_\tau)_1^+$. Then*

(1) *$f'_-(\tau(c))$ and $f'_+(\tau(c))$ exist,*

(2) *$f'_-(\tau(p_s^-)) = \sup\{s' \in \sigma(b) : s' < s\}$,*

(3) *$f'_+(\tau(p_s^+)) = \inf\{s' \in \sigma(b) : s < s'\}$,*

(4) *f is not differentiable at $\tau(p_s^{\pm})$ if and only if $(s - \epsilon, s) \cap \sigma(b) = \emptyset$ (or $(s, s+\epsilon) \cap \sigma(b) = \emptyset$) for some $\epsilon > 0$.*

Proof. (1) Since $\overline{B(b)}$ is convex, f is a convex function, and so for any $x \in (0, +\infty)$

$$\frac{f(x) - f(x-h)}{h} \tag{3.51}$$

increases as h decreases to zero. Furthermore, if $x = \tau(c)$ with $c \in [p_s^-, p_s^+] \cap \mathfrak{m}_\tau$ for some $s \in \sigma(b)$, then since $\Psi([p_s^-, p_s^+])$ is a line segment (or ray) in the lower boundary with slope equal to s (by Theorem 3.7), we know $(f(x) - f(x-h))/h$ is bounded above by s. Since bounded increasing sequences converge, $f'_-(x)$ exists at each $x = \tau(c)$ with $c \in [p_s^-, p_s^+] \cap \mathfrak{m}_\tau$ for some $s \in \sigma(b)$. Using a similar argument considering the sequence $(f(x+h) - f(x))/h$ for some fixed $x = \tau(c)$ as h decreases to 0, and Theorem 3.7, we get that $f'_+(x)$ also exists at every $x \in [0, +\infty)$.

(2) Fix $s \in \sigma(b)$ with $s_{\min} < s$, where $s_{\min}$ is the minimum of $\sigma(b)$, and suppose $x = \tau(p_s^-) < \infty$. Let

$$r = \sup\{s' \in \sigma(b) : s' < s\}. \tag{3.52}$$

Since $\sigma(b)$ is closed, $r \in \sigma(b)$. We want to show that $r = f'_-(\tau(p_s^-))$. Clearly $r \leq s$.

First suppose that $(r - \epsilon, r) \cap \sigma(b) \neq \emptyset$ for some $\epsilon > 0$. If we had $r = s$ then we would have $(s - \epsilon, s) \cap \sigma(b) \neq \emptyset$, which contradicts the definition of r. Therefore, in this case we must have $r < s$. But then again, by the definition of r, we have $(r, s) \cap \sigma(b) \neq \emptyset$. Therefore $r \in \sigma_p(b)$, and by Theorem 3.7 we know that $\Psi(p_s^-) = \Psi(p_r^+)$ is the right endpoint of a line segment in $\partial_{\mathcal{L}} B(b) \cap B(b)$ with slope equal to r. Thus, $r = f'_-(\tau(p_s^-))$.

If $(r - \epsilon, r) \cap \sigma(b) = \emptyset$ for every $\epsilon > 0$, then we can choose an increasing sequence (s_n) of distinct points in $\sigma(b)$ that converge to r. But then since τ is normal, we know that $\alpha_n = \tau(p_{s_n}^-)$ converges in the w^*-topology to $\alpha = \tau(p_r^-)$. Since all the elements under consideration are distinct spectral projections of b we have

$$s_n\left(p_r^- - p_{s_n}^-\right) \leq b\left(p_r^- - p_{s_n}^-\right) \leq r\left(p_r^- - p_{s_n}^-\right) \tag{3.53}$$

for each $n \in \mathbb{N}$. Applying τ to both sides of this we get

$$s_n(\alpha - \alpha_n) \leq f(\alpha) - f(\alpha_n) \leq r(\alpha - \alpha_n) \tag{3.54}$$

for every $n \in \mathbb{N}$. Therefore

$$s_n \leq \frac{f(\alpha) - f(\alpha_n)}{\alpha - \alpha_n} \leq r \tag{3.55}$$

for every n. Since (s_n) converges to r, we get

$$\lim_n \frac{f(\alpha) - f(\alpha_n)}{\alpha - \alpha_n} = r. \tag{3.56}$$

Since the function defined by

$$h(x) = \frac{f(\alpha) - f(x)}{\alpha - x} \quad \forall x \in [0, \infty] \tag{3.57}$$

is an increasing function. From this and (3.55) above, we see that if (r_n) is any sequence in $\mathbb{R}$ increasing to $\tau(p_s^-)$, then

$$\lim_n \frac{f(\alpha) - f(r_n)}{\alpha - r_n} = r. \tag{3.58}$$

Thus, $f'_-(\tau(p_s^-)) = r$.

(3) Let $r = \inf\{s' \in \sigma(b) : s < s'\}$. Clearly, by the definition of r, we have $s \leq r$. Using the same argument as in (2), we get

$$f'_+(\tau(p_r^+)) = r = \inf\{s' \in \sigma(b) : s < s'\}. \tag{3.59}$$

(4) From parts (2) and (3) above, we have

$$f'_-(\tau(p_s^-)) = \sup\{s' \in \sigma(b) : s' < s\}, \qquad f'_+(\tau(p_s^+)) = \inf\{s' \in \sigma(b) : s < s'\}. \tag{3.60}$$

But then, $(s, s+\epsilon) \cap \sigma(b) = \emptyset$ for some $\epsilon > 0$ if and only if $x < y$ whenever $x \in \{s' \in \sigma(b) : s' < s\}$ and $y \in \{s' \in \sigma(b) : s' > s\}$ if and only if $f'_-(\tau(p_s^-) < f'_+(\tau(p_s^+)$ if and only if f is not differentiable at $\tau(p_s^-)$ or f is not differentiable at $\tau(p_s^+)$. An identical argument shows that $(s-\epsilon, s) \cap \sigma(b) = \emptyset$ for some $\epsilon > 0$ if and only if f is not differentiable at $\tau(p_s^-)$ or f is not differentiable at $\tau(p_s^+)$. □

Theorem 3.13. *Let $t \in \sigma(b)$, and let $c \in [q_t^-, q_t^+] \cap (\mathfrak{m}_\tau)_1^+$. Then*

(1) *$g'_-(\tau(c))$ and $g'_+(\tau(c))$ exist,*

(2) *$g'_-(\tau(q_t^-)) = \sup\{t' \in \sigma(b) : t' < t\}$,*

(3) *$g'_+(\tau(q_t^+)) = \inf\{t' \in \sigma(b) : t < t'\}$,*

(4) *g is not differentiable at $\tau(q_t^\pm)$ if and only if $(t-\epsilon, t) \cap \sigma(b) = \emptyset$ (or $(t, t+\epsilon) \cap \sigma(b) = \emptyset$) for some $\epsilon > 0$.*

Proof. Substitute (3.34) for (3.33) into the proof of Theorem 3.12. □

Theorems 3.12 and 3.13 tell us that if $p_s^+ = p_s^-$ and s does not lie in a gap of the spectrum, then f is differentiable at $\tau(p_s)$ and $f'(\tau(p_s)) \in \sigma(b)$. Part (4) of these theorems also confirm that corners in the boundary of $B(b)$ correspond to gaps in $\sigma(b)$.

We finish with a few examples of spectral scales in the semifinite von Neumann algebra setting. Recall from Theorems 3.9 and 3.10 that we let

$$\begin{aligned} \mathcal{E}_{\mathcal{L}} &= \{s \in \sigma(b) : \tau(p_s^+) = \infty\}, \quad \text{with } \lambda = \inf \mathcal{E}_{\mathcal{L}}, \\ \mathcal{E}_{\mathcal{U}} &= \{t \in \sigma(b) : \tau(q_t^+) = \infty\}, \quad \text{with } \rho = \sup \mathcal{E}_{\mathcal{U}}. \end{aligned} \tag{3.61}$$

We also let

$$\mathcal{E} = \{\lambda \in \mathbb{C} : \tau(\chi_{(\lambda-\epsilon, \lambda+\epsilon)}(b)) = \infty \ \forall \epsilon > 0\} \tag{3.62}$$

denote the *essential spectrum* of b. We will see in the next section that $B(b)$ describes the minimum and maximum elements of $\mathcal{E}$. Note that all of the figures that follow are (*supposed to resemble*) unbounded subsets of $\mathbb{R}^2$.

Example 3.14. Let $p \in \mathcal{M}$ be a projection. Then either $\tau(p) < \infty$ and $\tau(1-p) = \infty$, or $\tau(p) = \infty$ and $\tau(1-p) < \infty$, or $\tau(p) = \infty$ and $\tau(1-p) = \infty$. In any case we have that $\tau(p_\lambda^+) = \tau(q_\rho^+) = \infty$. Therefore by Corollary 3.11 we have $\overline{B(p)} = B(p)$. Furthermore, $B(p)$ must be one of the following (see Figure 7).

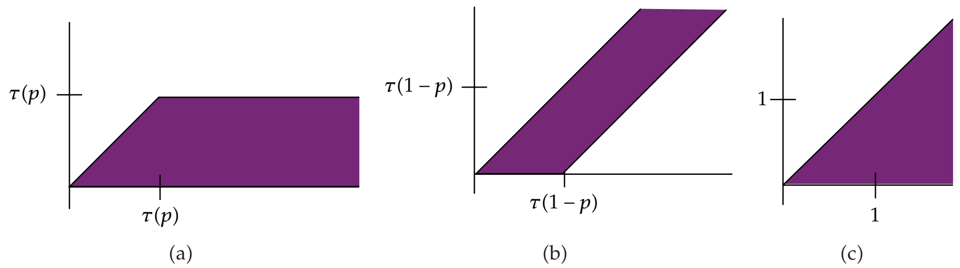

Figure 7: $B(p)$—Example 3.14.

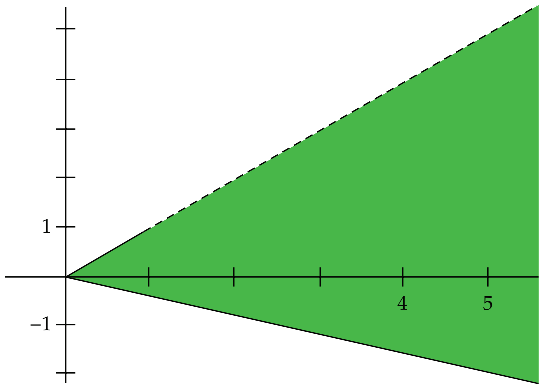

Figure 8: $B(b)$—Example 3.15.

Example 3.15. Let $\mathcal{M} = B(\mathcal{H})$ for some infinite-dimensional separable Hilbert Space $\mathcal{H}$ with τ the standard canonical trace on $\mathcal{M}$, and let

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\dfrac{1}{3} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \dfrac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\dfrac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \dfrac{2}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\dfrac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.63}$$

$B(b)$ is pictured in Figure 8.

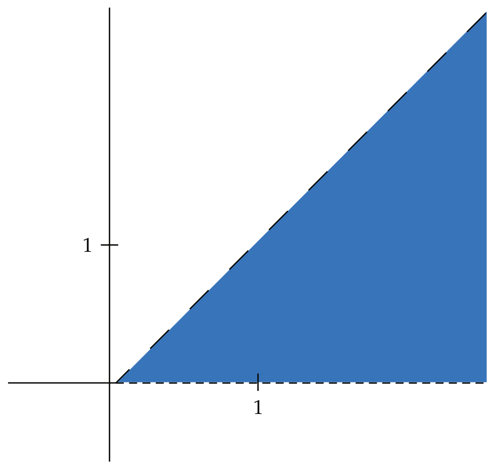

Figure 9: $B(b)$—Example 3.16.

Notice that in this case we have $\lambda = -1/3$ and $\rho = 1$ are the minimum and maximum of the essential spectrum $\mathcal{E}$ of b. Moreover, $\tau(p_\lambda^+) = \infty$ and $\tau(q_\rho^+) < \infty$. By Theorem 3.9, $\partial_{\mathcal{L}} B(b) \cap B(b) = \partial_{\mathcal{L}} B(b)$, and by Theorem 3.10, $\partial_{\mathcal{U}} B(b) \cap B(b) \neq \partial_{\mathcal{U}} B(b)$. Corollary 3.11 then tells us that $\overline{B(b)} \neq B(b)$. Note the closed portion of $\partial_{\mathcal{U}} B(b)$ between $x = 0$ and $x = 1$, which tells us that $\rho = 1$ is an eigenvalue with multiplicity equal to 1.

Example 3.16. Suppose $\mathcal{M} = B(\mathcal{H})$ for some infinite-dimensional separable Hilbert Space $\mathcal{H}$, with τ the standard canonical trace, and let $b \in \mathcal{M}$ be the diagonal operator with $\mathrm{diag}(b) = \mathbb{Q} \cap (0,1)$. In this case we have $\lambda = 0$ and $\rho = 1$. Furthermore, $\lambda \notin \mathcal{E}_{\mathcal{L}}$ and $\rho \notin \mathcal{E}_{\mathcal{U}}$, so $\partial_{\mathcal{L}} B(b) \cap B(b) \neq \partial_{\mathcal{L}} B(b)$ and $\partial_{\mathcal{U}} B(b) \cap B(b) \neq \partial_{\mathcal{U}} B(b)$. Again, by Corollary 3.11 $\overline{B(b)} \neq B(b)$. $B(b)$ is pictured in Figure 9.

Example 3.17. Let $\mathcal{M} = L^\infty([0,\infty), m)$ with $\tau(c) = \int_0^\infty c \, dm$ for all $c \in \mathcal{M}$, where m is Lebesgue measure on $[0,\infty)$. Let $b \in \mathcal{M}$ be defined by $b(x) = e^{-x}$ on $[0,\infty)$. Then $\sigma(b)$ is equal to the essential range of b, which equals $[0,1]$. For each $s \in [0,1]$ we have

$$\begin{aligned} p_s^- = p_s^+ &= \chi_{(-\infty,s]}(b) = \chi_{[-\ln(s),\infty)}, \\ q_s^+ = q_s^- = 1 - p_s^+ &= 1 - \chi_{[-\ln(s),\infty)} = \chi_{[0,-\ln(s))}. \end{aligned} \tag{3.64}$$

Since points of the form $(\tau(p_s^\pm), \tau(bp_s^\pm))$ and $(\tau(q_s^\pm), \tau(bq_s^\pm))$ determine the extreme points of $B(b)$ (see Theorems 3.7 and 3.8), by performing a couple quick integrals we find g and f (the upper and lower boundary curves (resp.) generated by b):

$$g(x) = 1 - e^{-x}, \qquad f(x) = 0. \tag{3.65}$$

Note that we have $\lambda = 0 \notin \mathcal{E}_{\mathcal{L}}$ and $\rho = 0 \in \mathcal{E}_{\mathcal{U}}$. Therefore $\partial_{\mathcal{L}} B(b) \cap B(b) \neq B(b)$ and $\partial_{\mathcal{U}} B(b) \cap B(b) = B(b)$. Thus $\overline{B(b)} \neq B(b)$ by Corollary 3.11. $B(b)$ is pictured in Figure 10.

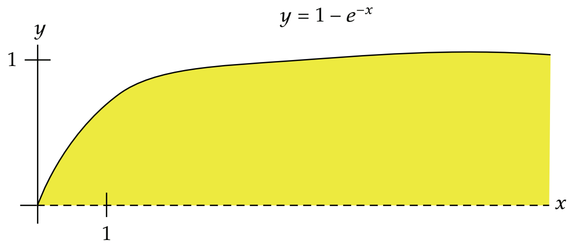

Figure 10: $B(b)$—Example 3.17.

Since $g(x)$ has a horizontal asymptote at $y = 1$, we know that $\tau(b) < \infty$. The spectrum of b can be read from Figure 10 as slopes of lines tangent to g and f. Since $\lambda = 0 \in \sigma(b) \setminus \sigma_p(b)$ and $\tau(p_0^+) < \infty$, $f(x)$ is "dotted" (or "open").

Example 3.18. Suppose $\mathscr{M} = B(\mathscr{H})$ for some infinite-dimensional separable Hilbert Space $\mathscr{H}$, with τ the standard canonical trace, and let

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.66}$$

$B(b)$ is pictured in Figure 11 (the black dots in the boundary are meant to emphasize the extreme points).

From Figure 11 we can see that the eigenvalue 1 has multiplicity three by noticing that the line segment $\Psi([p_1^-, p_1^+])$ extends from $x = 0$ to $x = 3$. We can also see that b is trace class since the upper and lower boundary curves g and f (resp.) have asymptotes. In fact, if we write $b = b^+ - b^-$ with $b^+ \geq 0$ and $b^- \geq 0$, we see that the asymptote for g is $y = \tau(b^+)$ and the asymptote for f is $y = -\tau(b^-)$. Also note that we have $\lambda = 0 \in \mathcal{E}_{\mathcal{L}}$ and $\rho = 0 \in \mathcal{E}_{\mathcal{U}}$, so by Corollary 3.11 $B(b)$ is closed. Since the slopes of the line segments in the boundary of $B(b)$ tend to 0 as $x \to \infty$, we know that b is compact.

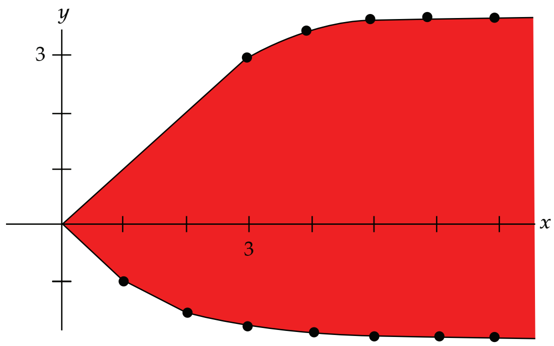

Figure 11: $B(b)$—Example 3.18.

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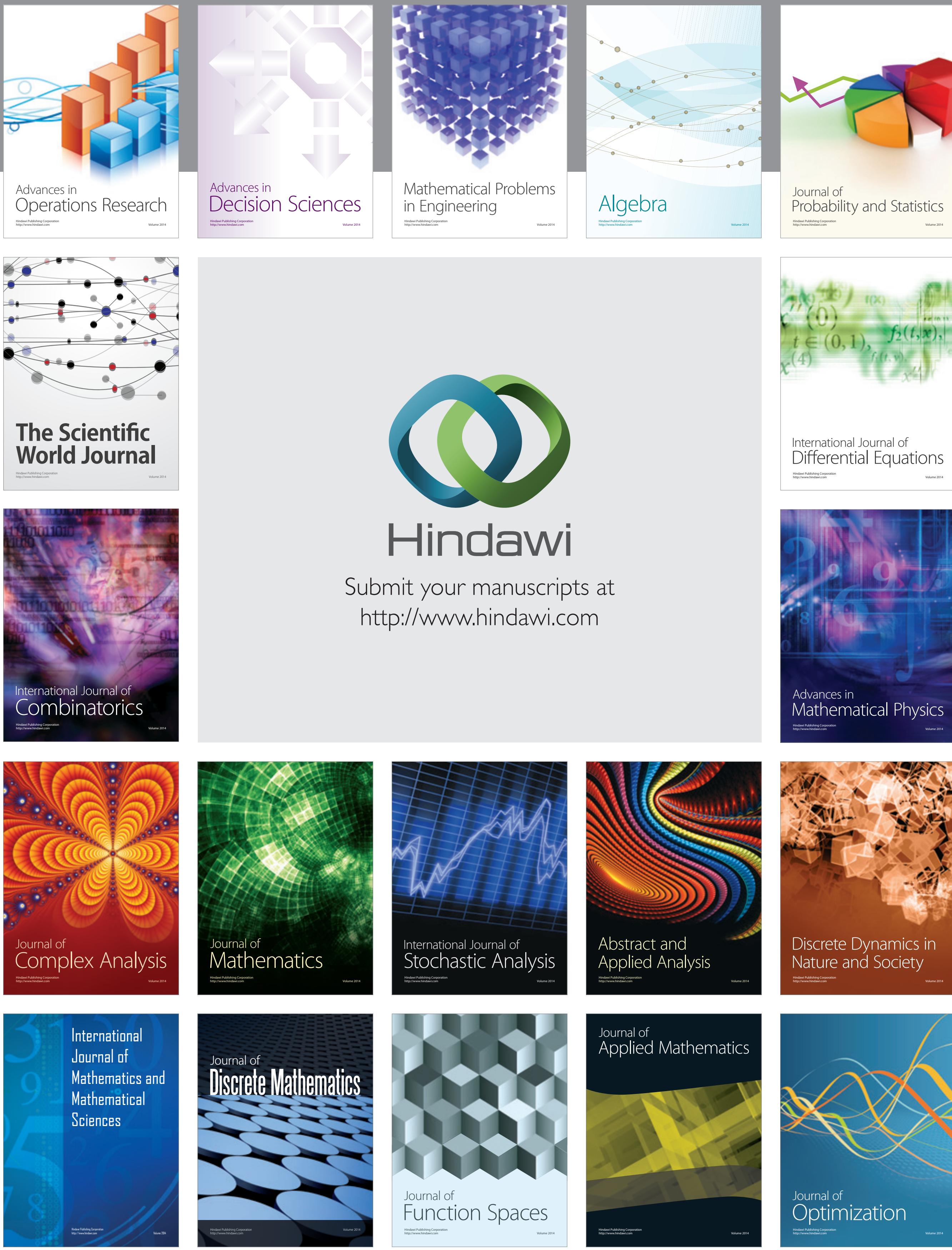
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