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Research Article

Parametric Evaluations of the Rogers-Ramanujan Continued Fraction

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In this paper with the help of the inverse function of the singular moduli we evaluate the Rogers-Ranmanujan continued fraction and its first derivative.

1. Introductory Definitions and Formulas

For $|q| < 1$, the Rogers-Ramanujan continued fraction (RRCF) (see [1]) is defined as

$$R(q) := \frac{q^{1/5}}{1+}\frac{q^1}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots. \tag{1.1}$$

We also define

$$(a;q)_n := \prod_{k=0}^{n-1}\left(1 - aq^k\right),$$
$$f(-q) := \prod_{n=1}^{\infty}(1 - q^n) = (q;q)_\infty. \tag{1.2}$$

Ramanujan give the following relations which are very useful:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f\left(-q^{1/5}\right)}{q^{1/5}f\left(-q^5\right)}, \tag{1.3}$$

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{1.4}$$

From the theory of elliptic functions (see [1–3]),

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 \sin(t)^2}} dt, \tag{1.5}$$

is the elliptic integral of the first kind. It is known that the inverse elliptic nome $k = k_r$, $k_r'^2 = 1 - k_r^2$ is the solution of the equation

$$\frac{K(k_r')}{K(k)} = \sqrt{r}, \tag{1.6}$$

where $r \in \mathbf{R}_+^*$. When r is rational then the k_r are algebraic numbers.

We can also write the function f using elliptic functions. It holds (see [3])

$$f(-q)^8 = \frac{2^{8/3}}{\pi^4} q^{-1/3} (k_r)^{2/3} (k_r')^{8/3} K(k_r)^4, \tag{1.7}$$

and also holds

$$f\left(-q^2\right)^6 = \frac{2k_r k_r' K(k_r)^3}{\pi^3 q^{1/2}}. \tag{1.8}$$

From [4] it is known that

$$R'(q) = \frac{1}{5} q^{-5/6} f(-q)^4 R(q) \sqrt[6]{R(q)^{-5} - 11 - R(q)^5}. \tag{1.9}$$

Consider now for every $0 < x < 1$ the equation

$$x = k_r, \tag{1.10}$$

which has solution

$$r = k^{(-1)}(x). \tag{1.11}$$

Hence for example

$$k^{(-1)}\left(\frac{1}{\sqrt{2}}\right) = 1. \tag{1.12}$$

With the help of $k^{(-1)}$ function we evaluate the Rogers Ramanujan continued fraction.

2. Propositions

The relation between k_{25r} and k_r is (see [1] page 280)

$$k_r k_{25r} + k'_r k'_{25r} + 2 \cdot 4^{1/3} \left(k_r k_{25r} k'_r k'_{25r}\right)^{1/3} = 1. \tag{2.1}$$

For to solve (2.1) we give the following.

Proposition 2.1. *The solution of the equation*

$$x^6 + x^3\left(-16 + 10x^2\right)w + 15x^4w^2 - 20x^3w^3 + 15x^2w^4 + x\left(10 - 16x^2\right)w^5 + w^6 = 0. \tag{2.2}$$

when one knows w is given by

$$\begin{aligned} \frac{y^{1/2}}{w^{1/2}} &= \frac{w^{1/2}}{x^{1/2}} \\ &= \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right)^2} + \frac{1}{2}\sqrt{\frac{2}{3}}\left(\frac{L^{1/6}}{M^{1/6}} - 4\frac{M^{1/6}}{L^{1/6}}\right), \end{aligned} \tag{2.3}$$

where

$$w = \sqrt{\frac{L(18+L)}{6(64+3L)}} < 1, \tag{2.4}$$

$$M = \frac{18+L}{64+3L}. \tag{2.5}$$

If it happens that $x = k_r$ and $y = k_{25r}$, then $r = k^{(-1)}(x)$ and $w^2 = k_{25r}k_r$, $(w')^2 = k'_{25r}k'_r$.

Proof. The relation (2.3) can be found using Mathematica. See also [5]. □

Proposition 2.2. *If $q = e^{-\pi\sqrt{r}}$ and*

$$a = a_r = \left(\frac{k'_r}{k'_{25r}}\right)^2 \sqrt{\frac{k_r}{k_{25r}}} M_5(r)^{-3}, \tag{2.6}$$

then

$$a_r = R(q)^{-5} - 11 - R^5(q), \tag{2.7}$$

where $M_5(r)$ is root of $(5x-1)^5(1-x) = 256(k_r)^2(k'_r)^2x$.

Proof. Suppose that $N = n^2\mu$, where n is positive integer and μ is positive real then it holds that

$$K\left[n^2\mu\right] = M_n(\mu)K[\mu], \tag{2.8}$$

where $K[\mu] = K(k_\mu)$

The following formula for $M_5(r)$ is known:

$$(5M_5(r) - 1)^5(1 - M_5(r)) = 256(k_r)^2(k_r')^2 M_5(r). \tag{2.9}$$

Thus, if we use (1.4) and (1.7) and the above consequence of the theory of elliptic functions, we get:

$$R^{-5}(q) - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)} = a = a_r = \left(\frac{k_r'}{k_{25r}'}\right)^2\sqrt{\frac{k_r}{k_{25r}}}M_5(r)^{-3}. \tag{2.10}$$

See also [4, 5]. □

3. The Main Theorem

From Proposition 2.2 and relation $w^2 = k_{25r}k_r$ we get

$$w^5 - k_r^2 w = \frac{k_r^3(k_r^2 - 1)}{a_r M_5(r)^3}. \tag{3.1}$$

Combining (2.2) and (3.1), we get

$$\begin{aligned}&\left[-10k_r^4 + 26k_r^6 + a_r M_5(r)^3 k_r^6 - 16k_r^8\right] + \left[-k_r^3 - 6a_r M_5(r)^3 k_r^3 + k_r^5 - 6a_r M_5(r)^3 k_r^5\right]w \\ &\quad + \left[a_r M_5(r)^3 k_r^2 + 15a_r M_5(r)k_r^4\right]w^2 - 20a_r M_5(r)^3 k_r^3 w^3 + 15a_r M_5(r)^3 k_r^2 w^4 = 0.\end{aligned} \tag{3.2}$$

Solving with respect to $a_r M_5(r)^3$, we get

$$a_r M_5(r)^3 = \frac{16k_r^6 - 26k_r^4 - wk_r^3 + 10k_r^2 + wk_r}{k_r^4 - 6k_r^3 w - 20k_r^3 w + 15w^2k_r^2 - 6k_r w + 15w^4 + w^2}. \tag{3.3}$$

Also we have

$$\frac{K(k_{25r})}{K(k_r)} = M_5(r) = \frac{1}{m} = \left(\sqrt{\frac{k_{25r}}{k_r}} + \sqrt{\frac{k_{25r}'}{k_r'}} - \sqrt{\frac{k_{25r}k_{25r}'}{k_r k_r'}}\right)^{-1} = \left(\frac{w}{k_r} + \frac{w'}{k_r'} - \frac{ww'}{k_r k_r'}\right)^{-1}. \tag{3.4}$$

The above equalities follow from [1] page 280 Entry 13-xii and the definition of w. Note that m is the multiplier.

Hence for given $0 < w < 1$, we find $L \in \mathbf{R}$ and we get the following parametric evaluation for the Rogers Ramanujan continued fraction

$$R\left(e^{-\pi\sqrt{r(L)}}\right)^{-5} - 11 - R\left(e^{-\pi\sqrt{r(L)}}\right)^{5} = a_r = \frac{16k_r^6 - 26k_r^4 - wk_r^3 + 10k_r^2 + wk_r}{k_r^4 - 6k_r^3 w - 20k_r w^3 + 15w^2k_r^2 - 6k_r w + 15w^4 + w^2}\left(\frac{w}{k_r} + \frac{w'}{k_r'} - \frac{ww'}{k_r k_r'}\right)^3. \tag{3.5}$$

Thus for a given w we find L and M from (2.4) and (2.5). Setting the values of M, L, w in (2.3) we get the values of x and y (see Proposition 2.1). Hence from (3.5) if we find $k^{(-1)}(x) = r$ we know $R(e^{-\pi\sqrt{r}})$. The clearer result is as follows.

Main Theorem. *When w is a given real number, one can find x from* (2.3). *Then for the Rogers-Ramanujan continued fraction the following holds:*

$$\begin{aligned} R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{-5} &- 11 - R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{5} \\ &= a_r = \frac{16x^6 - 26x^4 - wx^3 + 10x^2 + wx}{x^4 - 6x^3w - 20xw^3 + 15w^2x^2 - 6xw + 15w^4 + w^2} \\ &\quad \times \left(\frac{w}{x} + \frac{w'}{\sqrt{1-x^2}} - \frac{ww'}{x\sqrt{1-x^2}}\right)^3. \end{aligned} \tag{3.6}$$

Theorem 3.1. *(the first derivative). One has*

$$\begin{aligned} R'\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right) &= \frac{2^{4/3}x^{1/2}(1-x^2)}{5w^{1/6}w'^{2/3}}\left(\frac{w}{x} + \frac{w'}{\sqrt{1-x^2}} - \frac{ww'}{x\sqrt{1-x^2}}\right)^{1/2} \\ &\quad \times R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)\frac{K^2(x)e^{\pi\sqrt{k^{(-1)}(x)}}}{\pi^2}. \end{aligned} \tag{3.7}$$

Proof. Combining (1.7) and (1.9) and Proposition 2.2 we get the proof. □

We will see now how the function $k^{(-1)}(x)$ plays the same role in other continued fractions. Here we consider also the Ramanujan's Cubic fraction (see [5]), which is completely solvable using k_r.

Define the function

$$G(x) = \frac{x}{\sqrt{2\sqrt{x} - 3x + 2x^{3/2} - 2\sqrt{x}\sqrt{1 - 3\sqrt{x} + 4x - 3x^{3/2} + x^2}}}. \tag{3.8}$$

Set for a given $0 < w_3 < 1$

$$x = G(w_3). \tag{3.9}$$

Then as in Main Theorem, for the Cubic continued fraction $V(q)$, the following holds (see [5]):

$$t = V\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right) = \frac{\left(1-x^2\right)^{1/3} w_3^{1/4}}{2^{1/3} x^{1/3}\left(1-\sqrt{w_3}\right)}. \tag{3.10}$$

Observe here that again we only have to know $k^{(-1)}(x)$.

If $x = k_r$, for a certain r, then

$$k_{9r} = \frac{w_3}{k_r}, \tag{3.11}$$

and if we set

$$T = \sqrt{1 - 8V(q)^3}, \tag{3.12}$$

then the follwing holds:

$$(k_r)^2 = x^2 = \frac{(1-T)(3+T)^3}{(1+T)(3-T)^3}, \tag{3.13}$$

which is solvable always in radicals quartic equation. When we know w_3 we can find $k_r = x$ from $x = G(w_3)$ and hence t.

The inverse also holds: if we know $t = V(q)$ we can find T and hence $k_r = x$. The w_3 can be found by the degree 3 modular equation which is always solvable in radicals:

$$\sqrt{k_r k_r'} + \sqrt{k_{9r} k_{9r}'} = 1. \tag{3.14}$$

Let now

$$V(q) = z \Longleftrightarrow q = V^{(-1)}(z), \tag{3.15}$$

if

$$V_i(t) := \sqrt{\frac{1-\sqrt{1-8t^3}}{1+\sqrt{1-8t^3}}\left(\frac{3+\sqrt{1-8t^3}}{3-\sqrt{1-8t^3}}\right)^3}, \tag{3.16}$$

then

$$V_i\left(V\left(e^{-\pi\sqrt{x}}\right)\right) = k_x, \tag{3.17}$$

or

$$V\left(e^{-\pi\sqrt{r}}\right) = V_i^{(-1)}(k_r),$$
$$V\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right) = V_i^{(-1)}(x), \tag{3.18}$$

or

$$e^{-\pi\sqrt{k^{(-1)}(x)}} = V^{(-1)}\left(V_i^{(-1)}(x)\right) = (V_i \circ V)^{(-1)}(x),$$
$$k^{(-1)}(V_i(V(q))) = \frac{1}{\pi^2}\log(q)^2 = r. \tag{3.19}$$

Setting now values into (3.19) we get values for $k^{(-1)}(\cdot)$. The function $V_i(\cdot)$ is an algebraic function.

4. Evaluations of the Rogers-Ramanujan Continued Fraction

Note that if $x = k_r$, $r \in \mathbf{Q}_+^*$, then we have the classical evaluations with k_r and k_{25r}.

Evaluations

(1) We have

$$R\left(e^{-2\pi}\right) = \frac{-1}{2} - \frac{\sqrt{5}}{2} + \sqrt{\frac{5+\sqrt{5}}{2}},$$
$$R'\left(e^{-2\pi}\right) = 8\sqrt{\frac{2}{5}\left(9+5\sqrt{5}-2\sqrt{50+22\sqrt{5}}\right)}\frac{e^{2\pi}}{\pi^3}\Gamma\left(\frac{5}{4}\right)^4. \tag{4.1}$$

(2) Assume that $x = 1/\sqrt{2}$, hence $k^{(-1)}(1/\sqrt{2}) = 1$. From (2.5) which for this x can be solved in radicals, with respect to w, we find

$$w = \frac{\sqrt{2}}{4}\left(\sqrt{5}-1\right) - \frac{1}{2}\sqrt{7\sqrt{5}-15}. \tag{4.2}$$

Hence from

$$w' = \sqrt{\sqrt{1-\frac{w^4}{x^2}}\sqrt{1-x^2}}, \tag{4.3}$$

we get

$$w' = \left(\frac{1 + 21\sqrt{-30 + 14\sqrt{5}} - 9\sqrt{-150 + 70\sqrt{5}}}{\sqrt{2}} \right)^{1/4}. \tag{4.4}$$

Setting these values to (3.6) we get the value of a_r and then $R(q)$ in radicals. The result is

$$\begin{aligned} R(e^{-\pi})^{-5} - 11 - R(e^{-\pi})^5 = -\frac{1}{8}&\left(3 + \sqrt{5} - \sqrt{-30 + 14\sqrt{5}}\right) \\ &\times \left[1 - \sqrt{5} + \sqrt{-30 + 14\sqrt{5}} + 2^{3/8}\left(-3 + \sqrt{5} - \sqrt{-30 + 14\sqrt{5}}\right) \right. \\ &\qquad \left. \times \left(1 + 21\sqrt{-30 + 14\sqrt{5}} - 9\sqrt{-150 + 70\sqrt{5}}\right)^{1/4} \right] \\ &\times \left[\sqrt{-1574 + 704\sqrt{5}} - 655\sqrt{-30 + 14\sqrt{5}} + 293\sqrt{-150 + 70\sqrt{5}} \right]^{-1}. \end{aligned} \tag{4.5}$$

(3) Set $w = 1/64$ and $a = 1359863889$, $b = 36855$, then

$$\begin{aligned} x = 9\left(\sqrt{a}+b\right)^{5/6}\Big[&49152 \cdot 6^{1/3}\left(\sqrt{a}+b\right)^{1/6} - 960\left(\sqrt{a}+b\right)^{5/6} + 2 \cdot 6^{2/3}\left(\sqrt{a}+b\right)^{3/2} \\ &+ 384 \cdot 2^{2/3} \cdot 3^{1/6}\sqrt{\Big[-64\left(\sqrt{a}+b\right) + 3^{1/6} \cdot 2^{2/3}\sqrt{453287963} \cdot \left(b + \sqrt{a}\right)^{2/3}} \\ &\qquad + 8192 \cdot 6^{1/3}\left(\sqrt{a}+b\right)^{1/3} + 12285 \cdot 6^{2/3}\left(\sqrt{a} + \mathrm{b}\right)^{2/3}\Big] \\ &- 2 \cdot 6^{5/6}\left(\sqrt{a} + \mathrm{b}\right)^{1/6}\sqrt{\Big[4096 \cdot 2^{1/3} \cdot 3^{5/6}\sqrt{453287963}\left(\sqrt{a} + \mathrm{b}\right)^{1/3}} \\ &\qquad + 36855 \cdot 2^{2/3} 3^{1/6}\sqrt{453287963}\left(\sqrt{a} + \mathrm{b}\right)^{2/3} \\ &\qquad + 150958080\,6^{1/3}\left(\sqrt{a}+b\right)^{1/3} 453025819 \cdot 6^{2/3}\left(\sqrt{a}+\mathrm{b}\right)^{2/3} \\ &\qquad - 192\left(453025819 + 12285\sqrt{a}\right)\Big]\Big]^{-1}. \end{aligned} \tag{4.6}$$

(4) For

$$w = \sqrt{\frac{277}{108} + \frac{13\sqrt{385}}{108}}, \tag{4.7}$$

we get

$$x = \frac{\sqrt{277/12 + \left(13\sqrt{385}\right)/12}}{4 + \sqrt{7}}. \tag{4.8}$$

Hence

$$\begin{aligned} &R\left(\exp\left[-\pi \cdot k^{(-1)}\left(\frac{\sqrt{277/12 + \left(13\sqrt{385}\right)/12}}{4 + \sqrt{7}}\right)^{1/2}\right]\right) \\ &\quad = \left(-\frac{-8071}{18} + \frac{1075\sqrt{55}}{18} + \frac{1}{18}\sqrt{5(25740148 - 3470530\sqrt{55})}\right)^{1/5}. \end{aligned} \tag{4.9}$$

(5) Set $q = e^{-\pi\sqrt{r_0}}$, then from

$$\begin{aligned} V\left(e^{-\pi\sqrt{r_0}}\right) &= V_i^{(-1)}(k_{r_0}) = V_0, \\ V\left(q^{1/3}\right) &= \sqrt[3]{V(q)\frac{1 - V(q) + V(q)^2}{1 + 2V(q) + 4V(q)^2}}. \end{aligned} \tag{4.10}$$

We can evaluate all

$$V(q_0(n)) = b_0(n) = \text{Algebraic function of } r_0, \tag{4.11}$$

where

$$\begin{aligned} q_0(n) &= e^{-\pi\sqrt{r_0}/3^n}, \\ V_i(V(q_0(n))) &= V_i(b_0(n)) = k_{r_0/9^n}, \end{aligned} \tag{4.12}$$

hence

$$k^{(-1)}(V_i(b_0(n))) = \frac{r_0}{9^n}. \tag{4.13}$$

An example for $r_0 = 2$ is

$$V\left(e^{-\pi\sqrt{2}}\right) = -1 + \sqrt{\frac{3}{2}},$$
$$V\left(e^{-\pi\sqrt{2}/3}\right) = \frac{1}{2^{1/3}}\left(-1 + \sqrt{\frac{3}{2}}\right)^{1/3}, \tag{4.14}$$
$$V\left(e^{-\pi\sqrt{2}/9}\right) = \rho_3^{1/3},$$

where ρ_3 can be evaluated in radicals but for simplicity we give the polynomial form

$$-1 - 72x - 6408x^2 + 50048x^3 + 51264x^4 - 4608x^5 + 512x^6 = 0$$
$$\cdots . \tag{4.15}$$

Then, respectively, we get the values

$$k^{(-1)}\left(-49 + 35\sqrt{2} + 4\sqrt{3\left(99 - 70\sqrt{2}\right)}\right) = \frac{2}{9},$$
$$k^{(-1)}\left(V_i\left(\rho_3^{1/3}\right)\right) = \frac{2}{81}, \tag{4.16}$$
$$\cdots .$$

Hence

$$k^{(-1)}(V_i(b_0(n))) = \frac{r_0}{9^n}. \tag{4.17}$$

Also it holds that

$$R\left(e^{-\pi\sqrt{r_0}/3^n}\right)^{-5} - 11 - R\left(e^{-\pi\sqrt{r_0}/3^n}\right)^5$$
$$= \frac{16x_n^6 - 26x_n^4 - w_n x_n^3 + 10x_n^2 + w_n x_n}{x_n^4 - 6x_n^3 w_n - 20x_n w_n^3 + 15w_n^2 x_n^2 - 6x_n w_n + 15w_n^4 + w_n^2} \tag{4.18}$$
$$\times \left(\frac{w_n}{x_n} + \frac{w_n'}{\sqrt{1 - x_n^2}} - \frac{w_n w_n'}{x_n\sqrt{1 - x_n^2}}\right)^3,$$

where $x_n = V_i(b_0(n)) =$ known. The w_n are given from (2.2) (in this case we do not find a way to evaluate w_n in radicals).

Theorem 4.1. *Set*

$$w = \frac{\sqrt{108 + 144a^4 + 24a^8 + a^{12} - \sqrt{-11664 + \left(108 + 144a^4 + 24a^8 + a^{12}\right)^2}}}{6\sqrt{3}}, \tag{4.19}$$

then

$$x = \frac{\sqrt{108 + a^4\left(12 + a^4\right)^2 - \sqrt{a^4\left(6 + a^4\right)\left(12 + a^4\right)^2\left(36 + 18a^4 + a^8\right)}}}{2\sqrt{3}\left(3 + a^4 - a^2\sqrt{6 + a^4}\right)},$$
$$R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{-5} - 11 - R\left(e^{-\pi\sqrt{k^{(-1)}(x)}}\right)^{5} = A(a), \tag{4.20}$$

where

$$w' = \sqrt{\sqrt{1 - \frac{w^4}{x^2}}\sqrt{1 - x^2}}. \tag{4.21}$$

The $A(a)$ is a known algebraic function of a and can calculated from the Main Theorem.

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