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Research Article

A New Hybrid Algorithm for a Pair of Quasi-ϕ-Asymptotically Nonexpansive Mappings and Generalized Mixed Equilibrium Problems in Banach Spaces

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The purpose of this paper is, by using a new hybrid method, to prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem, the set of solutions for a variational inequality problem, and the set of common fixed points for a pair of quasi-ϕ-asymptotically nonexpansive mappings. Under suitable conditions some strong convergence theorems are established in a uniformly smooth and strictly convex Banach space with Kadec-Klee property. The results presented in the paper improve and extend some recent results.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. We also assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*.

Let $\psi : C \to \mathbb{R}$ be a real-valued function, $\Theta : C \times C \to \mathbb{R}$ a bifunction, and $A : C \to E^*$ a nonlinear mapping. The "so-called" generalized mixed equilibrium problem is to find $u \in C$ such that

$$\Theta(u, y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions for (1.1) is denoted by Ω, that is,

$$\Omega = \{u \in C : \Theta(u, y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) \geq 0, \ \forall y \in C\}. \tag{1.2}$$

Special examples are as follows.

(I) If $\varphi = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C, \tag{1.3}$$

which is called the generalized equilibrium problem. The set of solutions for (1.3) is denoted by GEP.

(II) If $A = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C, \tag{1.4}$$

which is called the mixed equilibrium problem (MEP) [1]. The set of solutions for (1.4) is denoted by MEP.

(III) If $\Theta = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C, \tag{1.5}$$

which is called the mixed variational inequality of Browder type (VI) [2]. The set of solutions for (1.5) is denoted by $\mathrm{VI}(C, A, \varphi)$.

(IV) If $\varphi = 0$ and $A = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) \geq 0, \quad \forall y \in C, \tag{1.6}$$

which is called the equilibrium problem. The set of solutions for (1.6) is denoted by $\mathrm{EP}(\Theta)$.

(V) If $\varphi = 0$ and $\Theta = 0$, the problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u \rangle \geq 0, \quad \forall y \in C, \tag{1.7}$$

which is called the variational inequality of Browder type. The set of solutions for (1.7) is denoted by $\mathrm{VI}(C, A)$.

The problem (1.1) is very general in the sense that numerous problems in physics, optimiztion and economics reduce to finding a solution for (1.1). Some methods have been proposed for solving the generalized equilibrium problem and the equilibrium problem in Hilbert space (see, e.g., [3–6]).

A mapping $S : C \to E$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.8}$$

We denote the fixed point set of S by $F(S)$.

In 2008, S. Takahashi and W. Takahashi [6] proved some strong convergence theorems for finding an element or a common element of EP, $\mathrm{EP}(f) \cap F(S)$ or $\mathrm{VI}(C, A) \cap F(S)$, respectively, in a Hilbert space.

Recently, Takahashi and Zembayashi [7, 8] proved some weak and strong convergence theorems for finding a common element of the set of solutions for equilibrium (1.6) and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

In 2010, Chang et al. [9] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem (1.3) and the set of common fixed points of a pair of relatively nonexpansive mappings in a Banach space.

Motivated and inspired by [4–9], we intend in this paper, by using a new hybrid method, to prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem (1.1) and the set of common fixed points of a pair of quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results.

The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| = \|x^*\|\}, \quad x \in E, \tag{2.1}$$

is called the normalized duality mapping. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$.

In the sequel, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 2.1. The following basic properties can be found in Cioranescu [10].

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is hemicontinuous.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then J is singlevalued, one-to-one and onto.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Next we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. In the sequel, we always use $\phi : E \times E \to \mathbb{R}^{+}$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.4}$$

Following Alber [11], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg\inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.5}$$

Lemma 2.2 (see [11, 12]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then, the following conclusions hold:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$ *for all $x \in C$ and $y \in E$;*

(b) *if $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \Longleftrightarrow \langle z - y, Jx - Jz\rangle \ge 0, \quad \forall y \in C; \tag{2.6}$$

(c) *for $x, y \in E$, $\phi(x, y) = 0$ if and only $x = y$.*

Remark 2.3. If E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$ and Π_C is the metric projection P_C of H onto C.

Let E be a smooth, strictly, convex and reflexive Banach space, C a nonempty closed convex subset of E, $T : C \to C$ a mapping, and $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widetilde{F}(T)$.

Definition 2.4 (see [13]). (1) A mapping $T : C \to C$ is said to be relatively nonexpansive if $F(T) \ne \emptyset$, $F(T) = \widetilde{F}(T)$, and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T). \tag{2.7}$$

(2) A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $Tx = y$.

Definition 2.5 (see [14]). (1) A mapping $T : C \to C$ is said to be quasi-ϕ-nonexpansive if $F(T) \ne \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T). \tag{2.8}$$

(2) A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1,\ x \in C,\ p \in F(T). \tag{2.9}$$

(3) A pair of mappings $T_1, T_2 : C \to C$ is said to be uniformly quasi-ϕ-asymptotically nonexpansive if $F(T_1) \bigcap F(T_2) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that for $i = 1, 2$

$$\phi(p, T_i^n x) \leq k_n \phi(p, x), \quad \forall n \geq 1,\ x \in C,\ p \in F(T_1) \cap F(T_2). \tag{2.10}$$

(4) A mapping $T : C \to C$ is said to be uniformly L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.11}$$

Remark 2.6. (1) From the definition, it is easy to know that each relatively nonexpansive mapping is closed.

(2) The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass, and the class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

Lemma 2.7 (see [15]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0)$ a closed ball of E. Then, for any given subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any positive numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ with $\sum_{i=1}^N \lambda_i = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0,\infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$,*

$$\left\|\sum_{n=1}^N \lambda_n x_n\right\|^2 \leq \sum_{n=1}^N \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.12}$$

Lemma 2.8 (see [15]). *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

For solving the generalized mixed equilibrium problem (1.1), let us assume that the function $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous, the nonlinear mapping $A : C \to E^*$ is continuous and monotone, and the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A_1) $\Theta(x, x) = 0$, for all $x \in C$,

(A_2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$, $\forall x, y \in C$,

(A_3) $\limsup_{t \downarrow 0} \Theta(x + t(z - x), y) \leq \Theta(x, y)$ $\forall x, z, y \in C$,

(A_4) the function $y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.9. *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying the conditions (A_1)–(A_4). Let $r > 0$ and $x \in E$. Then, the followings hold.*

(i) *(Blum and Oettli [3]) there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.13}$$

(ii) *(Takahashi and Zembayashi [8]) Define a mapping $T_r : E \to C$ by*

$$T_r(x) = \left\{ z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}, \quad x \in E. \tag{2.14}$$

Then, the following conclusions hold:

(a) *T_r is single-valued,*

(b) *T_r is a firmly nonexpansive-type mapping, that is, $\forall z, y \in E$,*

$$\langle T_r z - T_r y, JT_r z - JT_r y\rangle \leq \langle T_r z - T_r y, Jz - Jy\rangle, \tag{2.15}$$

(c) $F(T_r) = \mathrm{EP}(\Theta) = \widetilde{F}(T_r)$,

(d) $\mathrm{EP}(\Theta)$ *is closed and convex,*

(e) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x),\ \forall q \in F(T_r)$.

Lemma 2.10 (see [16]). *Let E be a smooth, strictly convex, and reflexive Banach space, and C a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A_1)–(A_4). Let $r > 0$ be any given number and $x \in E$ any given point. Then, the following hold.*

(i) *There exists $u \in C$ such that*

$$\Theta(u, y) + \langle Au, y - u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.16}$$

(ii) *If we define a mapping $K_r : C \to C$ by*

$$\begin{aligned} K_r(x) = \Big\{ u \in C : \Theta(u, y) + \langle Au, y - u\rangle + \psi(y) - \psi(u) \\ + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C \Big\}, \quad \forall x \in C. \end{aligned} \tag{2.17}$$

Then, the mapping K_r has the following properties:

(a) *K_r is single valued,*

(b) *K_r is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r z - K_r y, JK_r z - JK_r y\rangle \leq \langle K_r z - K_r y, Jz - Jy\rangle, \quad \forall z, y \in E, \tag{2.18}$$

(c) $F(K_r) = \Omega = \widetilde{F}(K_r)$,

(d) Ω *is closed and convex,*

(e)

$$\phi(q, K_r z) + \phi(K_r z, z) \leq \phi(q, z), \quad \forall q \in F(K_r),\ z \in E. \tag{2.19}$$

Remark 2.11. It follows from Lemma 2.9 that the mapping K_r is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

3. Main Results

In this section, we will prove a strong convergence theorem for finding a common element of the set of solutions for the generalized mixed equilibrium problem (1.1) and the set of common fixed points for a pair of quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ a lower semicontinuous and convex, function, and $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A_1)–(A_4). Let S, $T : C \to C$ be two closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Suppose that S and T are uniformly L-Lipschitz continuous and that $G = F(T) \bigcap F(S) \bigcap \Omega$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered}
x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\
y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\
u_n \in C \text{ such that}, \quad \forall y \in C, \\
\Theta(u_n, y) + \langle A u_n, y - u_n\rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \\
C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1-\beta_n)\xi_n\}, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.1}$$

where $J : E \to E^*$ *is the normalized duality mapping,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$, $\xi_n = \sup_{u\in G}(k_n - 1)\phi(u, x_n)$. *Suppose that the following conditions are satisfied:*

(i) $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$,

(ii) $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap F(T)\cap\Omega}x_0$, *where* $\Pi_{F(S)\cap F(T)\cap\Omega}$ *is the generalized projection of* E *onto* $F(S) \cap F(T) \cap \Omega$.

Proof. Firstly, we define two functions $H : C \times C \to \mathbb{R}$ and $K_r : C \to C$ by

$$H(x, y) = \Theta(x, y) + \langle Ax, y - x\rangle + \psi(y) - \psi(x), \quad \forall x, y \in C,$$
$$K_r(x) = \left\{u \in C : H(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\}, \quad x \in C. \tag{3.2}$$

By Lemma 2.10, we know that the function H satisfies conditions (A_1)–(A_4) and K_r has properties (a)–(e). Therefore, (3.1) is equivalent to

$$\begin{gathered} x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ u_n \in C \text{ such that, } \quad \forall y \in C, \\ H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \\ C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n\}, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n}x_0, \quad \forall n \geq 0. \end{gathered} \tag{3.3}$$

We divide the proof of Theorem 3.1 into five steps.

(I) First we prove that C_n and Q_n are both closed and convex subsets of C for all $n \geq 0$. In fact, it is obvious that Q_n is closed and convex for all $n \geq 0$. Again we have that

$$\begin{aligned} &\phi(v, z_n) \leq \phi(v, x_n) + \xi_n \Longleftrightarrow 2\langle v, Jx_n - Jz_n\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \xi_n, \\ &\phi(v, u_n) \leq \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n \Longleftrightarrow 2\langle v, Jx_n - Ju_n\rangle \\ &\qquad \leq \|x_n\|^2 - \|u_n\|^2 + (1+k_n)(1-\beta_n)\xi_n. \end{aligned} \tag{3.4}$$

Hence C_n, $\forall n \geq 0$, is closed and convex, and so $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

(II) Next we prove that $F(T)\cap F(S)\cap\Omega\subset C_n\cap Q_n$, $\forall n\geq 0$.

Putting $u_n = K_{r_n}y_n$, $\forall n \geq 0$, by Lemma 2.10 and Remark 2.11, K_{r_n} is relatively nonexpansive. Again since S and T are quasi-ϕ-asymptotically nonexpansive, for any given $u\in F(S)\cap F(T)\cap\Omega$, we have that

$$\begin{aligned}
\phi(u,z_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_n+(1-\alpha_n)JT^n x_n)\Big)\\
&= \|u\|^2-2\langle u,\alpha_n Jx_n+(1-\alpha_n)JT^n x_n\rangle+\|\alpha_n Jx_n+(1-\alpha_n)JT^n x_n\|^2\\
&\leq \|u\|^2-2\alpha_n\langle u,Jx_n\rangle-2(1-\alpha_n)\langle u,JT^n x_n\rangle+\alpha_n\|x_n\|^2\\
&\quad +(1-\alpha_n)\|T^n x_n\|^2-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&= \alpha_n\phi(u,x_n)+(1-\alpha_n)\phi(u,T^n x_n)-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&\leq \alpha_n\phi(u,x_n)+(1-\alpha_n)k_n\phi(u,x_n)-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&\leq k_n\phi(u,x_n)-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&\leq \phi(u,x_n)+\sup_{p\in G}(k_n-1)\phi(p,x_n)-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&= \phi(u,x_n)+\xi_n-\alpha_n(1-\alpha_n)g(\|Jx_n-JT^n x_n\|)\\
&\leq \phi(u,x_n)+\xi_n.
\end{aligned}\tag{3.5}$$

From (3.5) we have that

$$\begin{aligned}
\phi(u,u_n) &= \phi(u,K_{r_n}y_n)\leq\phi(u,y_n)\\
&\leq \phi\Big(u,J^{-1}(\beta_n Jx_n+(1-\beta_n)JS^n z_n)\Big)\\
&= \|u\|^2-2\langle u,\beta_n Jx_n+(1-\beta_n)JS^n z_n\rangle+\big\|\beta_n Jx_n+(1-\beta_n)JS^n z_n\big\|^2\\
&\leq \|u\|^2-2\beta_n\langle u,Jx_n\rangle-2(1-\beta_n)\langle u,JS^n z_n\rangle+\beta_n\|x_n\|^2\\
&\quad +(1-\beta)\|S^n z_n\|^2-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&= \beta_n\phi(u,x_n)+(1-\beta_n)\phi(u,S^n z_n)-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \beta_n\phi(u,x_n)+(1-\beta_n)k_n\phi(u,z_n)-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \beta_n\phi(u,x_n)+(1-\beta_n)k_n(\phi(u,x_n)+\xi_n)-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \beta_n\phi(u,x_n)+(1-\beta_n)(\phi(u,x_n)+\xi_n)+(1-\beta_n)k_n\xi_n-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \phi(u,x_n)+(1-\beta_n)\xi_n+(1-\beta_n)k_n\xi_n-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \phi(u,x_n)+(1+k_n)(1-\beta_n)\xi_n-\beta_n(1-\beta_n)g(\|Jx_n-JS^n z_n\|)\\
&\leq \phi(u,x_n)+(1+k_n)(1-\beta_n)\xi_n\quad\forall n\geq 0.
\end{aligned}\tag{3.6}$$

This implies that $u\in C_n$, $\forall n\geq 0$, and so $F(T)\cap F(S)\cap\Omega\subset C_n$, $\forall n\geq 0$.

Now we prove that $F(T)\cap F(S)\cap\Omega\subset C_n\cap Q_n$, $\forall n\geq 0$.

In fact, from $Q_0=C$, we have that $F(T)\cap F(S)\cap\Omega\subset C_0\cap Q_0$. Suppose that $F(T)\cap F(S)\cap\Omega\subset C_k\cap Q_k$, for some $k\geq 0$. Now we prove that $F(T)\cap F(S)\cap\Omega\subset C_{k+1}\cap Q_{k+1}$. In fact, since $x_{k+1}=\Pi_{C_k\cap Q_k}x_0$, we have that

$$\langle x_{k+1}-z, Jx_0-Jx_{k+1}\rangle\geq 0,\quad \forall z\in C_k\cap Q_k. \tag{3.7}$$

Since $F(T)\cap F(S)\cap\Omega\subset C_k\cap Q_k$, for any $z\in F(T)\cap F(S)\cap\Omega$, we have that

$$\langle x_{k+1}-z, Jx_0-Jx_{k+1}\rangle\geq 0. \tag{3.8}$$

This shows that $z\in Q_{k+1}$, and so $F(T)\cap F(S)\cap\Omega\subset Q_{k+1}$. The conclusion is proved.

(III) Now we prove that $\{x_n\}$ is bounded.

From the definition of Q_n, we have that $x_n=\Pi_{Q_n}x_0$, $\forall n\geq 0$. Hence, from Lemma 2.2(1),

$$\begin{aligned}\phi(x_n,x_0)=\phi(\Pi_{Q_n}x_0,x_0)&\leq\phi(u,x_0)-\phi(u,\Pi_{Q_n}x_0)\\&\leq\phi(u,x_0),\quad\forall u\in F(T)\cap F(S)\cap\Omega\subset Q_n,\ \forall n\geq 0.\end{aligned} \tag{3.9}$$

This implies that $\{\phi(x_n,x_0)\}$ is bounded. By virtue of (2.4), $\{x_n\}$ is bounded. Denote

$$M=\sup_{n\geq 0}\{\|x_n\|\}<\infty. \tag{3.10}$$

Since $x_{n+1}=\Pi_{C_n\cap Q_n}x_0\in C_n\cap Q_n\subset Q_n$ and $x_n=\Pi_{Q_n}x_0$, from the definition of Π_{Q_n}, we have that

$$\phi(x_n,x_0)\leq\phi(x_{n+1},x_0)\leq(M+\|x_0\|)^2,\quad\forall n\geq 0. \tag{3.11}$$

This implies that $\{\phi(x_n,x_0)\}$ is nondecreasing, and so the limit $\lim_{n\to\infty}\phi(x_n,x_0)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty}\phi(x_n,x_0)=r\geq 0. \tag{3.12}$$

By the way, from the definition of $\{\xi_n\}$, (2.4), and (3.10), it is easy to see that

$$\xi_n=\sup_{u\in G}(k_n-1)\phi(u,x_n)\leq\sup_{u\in G}(k_n-1)(\|u\|+M)^2\longrightarrow 0\quad(\text{as } n\longrightarrow\infty). \tag{3.13}$$

(IV) Now, we prove that $\{x_n\}$ converges strongly to some point $p\in G=F(T)\cap F(S)\cap\Omega$.

In fact, since $\{x_n\}$ is bounded in C and E is reflexive, there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\rightharpoonup p$. Again since Q_n is closed and convex for each $n\geq 0$, it is weakly closed, and so $p \in Q_n$ for each $n \geq 0$. Since $x_n = \Pi_{Q_n} x_0$, from the defintion of Π_{Q_n}, we have that

$$\phi(x_{n_i}, x_0) \leq \phi(p, x_0), \quad n \geq 0. \tag{3.14}$$

Since

$$\begin{aligned} \liminf_{n_i \to \infty} \phi(x_{n_i}, x_0) &= \liminf_{n_i \to \infty} \left\{ \|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2 \right\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0 \rangle + \|x_0\|^2 = \phi(p, x_0), \end{aligned} \tag{3.15}$$

we have that

$$\phi(p, x_0) \leq \liminf_{n_i \to \infty} \phi(x_{n_i}, x_0) \leq \limsup_{n_i \to \infty} \phi(x_{n_i}, x_0) \leq \phi(p, x_0). \tag{3.16}$$

This implies that $\lim_{n_i \to \infty} \phi(x_{n_i}, x_0) = \phi(p, x_0)$, that is, $\|x_{n_i}\| \to \|p\|$. In view of the Kadec-Klee property of E, we obtain that $\lim_{n \to \infty} x_{n_i} = p$.

Now we first prove that $x_n \to p$ $(n \to \infty)$. In fact, if there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$, then we have that

$$\begin{aligned} \phi(p, q) &= \lim_{n_i \to \infty, n_j \to \infty} \phi\left(x_{n_i}, x_{n_j}\right) \leq \lim_{n_i \to \infty, n_j \to \infty} \phi(x_{n_i}, x_0) - \phi\left(\Pi_{Q_{n_j}} x_0, x_0\right) \\ &= \lim_{n_i \to \infty, n_j \to \infty} \phi(x_{n_i}, x_0) - \phi\left(x_{n_j}, x_0\right) = 0 \quad (\text{by } (3.12)). \end{aligned} \tag{3.17}$$

Therefore we have that $p = q$. This implies that

$$\lim_{n \to \infty} x_n = p. \tag{3.18}$$

Now we first prove that $p \in F(T) \cap F(S)$. In fact, by the construction of Q_n, we have that $x_n = \Pi_{Q_n} x_0$. Therefore, by Lemma 2.2(a) we have that

$$\begin{aligned} \phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \end{aligned} \tag{3.19}$$

In view of $x_{n+1} \in C_n \cap Q_n \subset C_n$ and noting the construction of C_n we obtain

$$\begin{aligned} \phi(x_{n+1}, z_n) &\leq \phi(x_{n+1}, x_n) + \xi_n, \\ \phi(x_{n+1}, u_n) &\leq \phi(x_{n+1}, x_n) + (1 + k_n)(1 - \beta_n)\xi_n. \end{aligned} \tag{3.20}$$

From (3.13) and (3.19), we have that

$$\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0, \qquad \lim_{n \to \infty} \phi(x_{n+1}, z_n) = 0. \tag{3.21}$$

From (2.4) it yields that $(\|x_{n+1}\|-\|u_n\|)^2 \to 0$ and $(\|x_{n+1}\|-\|z_n\|)^2 \to 0$. Since $\|x_{n+1}\| \to \|p\|$, we have that

$$\|u_n\| \longrightarrow \|p\|, \quad \|z_n\| \longrightarrow \|p\| \quad (\text{as } n \longrightarrow \infty). \tag{3.22}$$

Hence, we have that

$$\|Ju_n\| \longrightarrow \|Jp\|, \quad \|Jz_n\| \longrightarrow \|Jp\| \quad (\text{as } n \longrightarrow \infty). \tag{3.23}$$

This implies that $\{Jz_n\}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, there exists a subsequence $\{Jz_{n_i}\} \subset \{Jz_n\}$ such that $Jz_{n_i} \rightharpoonup p_0 \in E^*$. In view of the reflexiveness of E, we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = p_0$. Since

$$\phi(x_{n_i+1}, z_{n_i}) = \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Jz_{n_i}\rangle + \|z_{n_i}\|^2 = \|x_{n_i+1}\|^2 - 2\langle x_{n_i+1}, Jz_{n_i}\rangle + \|Jz_{n_i}\|^2, \tag{3.24}$$

taking $\liminf_{n\to\infty}$ on both sides of the equality above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned} 0 &\ge \|p\|^2 - 2\langle p, p_0\rangle + \|p_0\|^2 = \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x), \end{aligned} \tag{3.25}$$

that is, $p = x$. This implies that $p_0 = Jp$, and so $Jz_n \rightharpoonup Jp$. It follows from (3.23) and the Kadec-Klee property of E^* that $Jz_{n_i} \to Jp$ (as $n \to \infty$). Noting that $J^{-1} : E^* \to E$ is hemicontinuous, it yields that $z_{n_i} \rightharpoonup p$. It follows from (3.22) and the Kadec-Klee property of E that $\lim_{n_i\to\infty} z_{n_i} = p$.

By the same way as given in the proof of (3.18), we can also prove that

$$\lim_{n\to\infty} z_n = p. \tag{3.26}$$

From (3.18) and (3.26), we have that

$$\|x_n - z_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.27}$$

Since J is uniformly continuous on any bounded subset of E, we have that

$$\|Jx_n - Jz_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.28}$$

For any $u \in F(T) \bigcap F(S) \bigcap \Omega$, it follows from (3.5) that

$$\alpha_n(1-\alpha_n)g(\|Jx_n - T^n x_n\|) \le \phi(u, x_n) - \phi(u, z_n) + \xi_n. \tag{3.29}$$

Since

$$\begin{aligned}\phi(u,x_n)-\phi(u,z_n) &= \|x_n\|^2-\|z_n\|^2-2\langle u, Jx_n-Jz_n\rangle \\ &\le \left|\|x_n\|^2-\|z_n\|^2\right| + 2\|u\|\cdot\|Jx_n-Jz_n\| \\ &\le \|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|u\|\cdot\|Jx_n-Jz_n\|,\end{aligned} \tag{3.30}$$

From (3.27) and (3.28), it follows that

$$\phi(u,x_n)-\phi(u,z_n)\longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{3.31}$$

In view of condition (i) and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we see that

$$g(\|Jx_n-JT^nx_n\|)\longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{3.32}$$

It follows from the property of g that

$$\|Jx_n-JT^nx_n\|\longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{3.33}$$

Since $x_n \to p$ and J is uniformly continuous, it yields that $Jx_n \to Jp$. Hence from (3.33) we have that

$$JT^nx_n\longrightarrow Jp \quad (\text{as } n\longrightarrow\infty). \tag{3.34}$$

Since $J^{-1}: E^* \to E$ is hemicontinuous, it follows that

$$T^nx_n \rightharpoonup p. \tag{3.35}$$

On the other hand, we have that

$$\left|\|T^nx_n\|-\|p\|\right| = \left|\|J(T^nx_n)\|-\|Jp\|\right| \le \|JT^nx_n-Jp\|\longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{3.36}$$

This together with (3.35) shows that

$$T^nx_n\longrightarrow p. \tag{3.37}$$

Furthermore, by the assumption that T is uniformly L-Lipschitz continuous, we have that

$$\begin{aligned}\left\|T^{n+1}x_n-T^nx_n\right\| &\le \left\|T^{n+1}x_n-T^{n+1}x_{n+1}\right\|+\left\|T^{n+1}x_{n+1}-x_{n+1}\right\|+\|x_{n+1}-x_n\|+\|x_n-T^nx_n\| \\ &\le (L+1)\|x_{n+1}-x_n\|+\left\|T^{n+1}x_{n+1}-x_{n+1}\right\|+\|x_n-T^nx_n\|.\end{aligned} \tag{3.38}$$

This together with (3.18) and (3.37), yields $\|T^{n+1}x_n - T^n x_n\| \to 0$ (as $n \to \infty$). Hence from (3.37) we have that $T^{n+1}x_n \to p$, that is, $TT^n x_n \to p$. In view of (3.37) and the closeness of T, it yields that $Tp = p$. This implies that $p \in F(T)$.

By the same way as given in the proof of (3.23) to (3.31), we can also prove that

$$\lim_{n\to\infty} u_n = p, \quad \phi(u, x_n) - \phi(u, u_n) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.39}$$

Since $u_n = K_{r_n} y_n$, from (2.19), (3.6), (3.13), and (3.39), we have that

$$\begin{aligned}\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \le \phi(u, y_n) - \phi(u, u_n) \\ &\le \phi(u, x_n) - \phi(u, u_n) + (1 + k_n)(1 - \beta_n)\xi_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).\end{aligned} \tag{3.40}$$

From (2.4) it yields that $(\|u_n\| - \|y_n\|)^2 \to 0$. Since $\|u_n\| \to \|p\|$, we have that

$$\|y_n\| \longrightarrow \|p\| \quad (\text{as } n \longrightarrow \infty). \tag{3.41}$$

Hence we have that

$$\|Jy_n\| \longrightarrow \|Jp\| \quad (\text{as } n \longrightarrow \infty). \tag{3.42}$$

By the same way as given in the proof of (3.26), we can also prove that

$$\lim_{n\to\infty} y_n = p. \tag{3.43}$$

From (3.39) and (3.43) we have that

$$\|u_n - y_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.44}$$

Since J is uniformly continuous on any bounded subset of E, we have that

$$\|Ju_n - Jy_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.45}$$

For any $u \in F(T) \bigcap F(S) \bigcap \Omega$, it follows from (3.6), (3.13), and (3.39) that

$$\beta_n(1 - \beta_n) g(\|Jx_n - S^n z_n\|) \le \phi(u, x_n) - \phi(u, u_n) + (1 + k_n)(1 - \beta_n)\xi_n \longrightarrow 0. \tag{3.46}$$

In view of condition (ii) and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$, we see that

$$g(\|Jx_n - JS^n z_n\|) \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.47}$$

It follows from the property of g that

$$\|Jx_n - JS^n z_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.48}$$

Since $x_n \to p$ and J is uniformly continuous, it yields, $Jx_n \to Jp$. Hence from (3.48) we have that

$$JS^n z_n \longrightarrow Jp \quad (\text{as } n \longrightarrow \infty). \tag{3.49}$$

Since $J^{-1} : E^* \to E$ is hemicontinuous, it follows that

$$S^n z_n \rightharpoonup p. \tag{3.50}$$

On the other hand, we have that

$$\left| \|S^n z_n\| - \|p\| \right| = \left| \|J(S^n z_n)\| - \|Jp\| \right| \le \|JS^n z_n - Jp\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.51}$$

This together with (3.50) shows that

$$S^n z_n \longrightarrow p. \tag{3.52}$$

Furthermore, by the assumption that S is uniformly L-Lipschitz continuous, we have that

$$\begin{aligned} \left\|S^{n+1} z_n - S^n z_n\right\| &\le \left\|S^{n+1} z_n - S^{n+1} z_{n+1}\right\| + \left\|S^{n+1} z_{n+1} - z_{n+1}\right\| + \|z_{n+1} - z_n\| + \|z_n - S^n z_n\| \\ &\le (L+1)\|z_{n+1} - z_n\| + \left\|S^{n+1} z_{n+1} - z_{n+1}\right\| + \|z_n - S^n z_n\|. \end{aligned} \tag{3.53}$$

This together with (3.26) and (3.52), yields that $\|S^{n+1} z_n - S^n z_n\| \to 0$ (as $n \to \infty$). Hence from (3.52) we have that $S^{n+1} z_n \to p$, that is, $SS^n z_n \to p$. In view of (3.52) and the closeness of T, it yields that $Sp = p$. This implies that $p \in F(S)$.

Next we prove that $p \in \Omega$. From (3.45) and the assumption that $r_n \ge a$, $\forall n \ge 0$, we have that

$$\lim_{n \to \infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.54}$$

Since $u_n = K_{r_n} y_n$, we have that

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \quad \forall y \in C. \tag{3.55}$$

Replacing n by n_k in (3.55), from condition (A_2), we have that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k} \rangle \ge -H(u_{n_k}, y) \ge H(y, u_{n_k}), \quad \forall y \in C. \tag{3.56}$$

By the assumption that $y \mapsto H(x,y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in (3.55), from (3.54) and condition (A_4), we have that $H(y,p) \leq 0, \forall y \in C$.

For $t \in (0,1]$ and $y \in C$, letting $y_t = ty + (1-t)p$, there are $y_t \in C$ and $H(y_t,p) \leq 0$. By conditions (A_1) and (A_4), we have that

$$0 = H(y_t,y_t) \leq tH(y_t,y) + (1-t)H(y_t,p) \leq tH(y_t,y). \tag{3.57}$$

Dividing both sides of the above equation by t, we have that $H(y_t,y) \geq 0, \forall y \in C$. Letting $t \downarrow 0$, from condition (A_3), we have that $H(p,y) \geq 0, \forall y \in C$, that is, $\Theta(p,y) + \langle Ap, y-p\rangle + \varphi(y) - \varphi(p) \geq 0, \forall y \in C$. Therefore $p \in \Omega$, and so $p \in F(T) \bigcap F(S) \bigcap \Omega$.

(V) Finally, we prove that $x_n \to \Pi_{F(T)\bigcap F(S)\bigcap \Omega} x_0$.

Let $w = \Pi_{F(T)\bigcap F(S)\bigcap \Omega} x_0$. From $w \in F(T) \bigcap F(S) \bigcap \Omega \subset C_n \cap Q_n$, and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have that

$$\phi(x_{n+1},x_0) \leq \phi(w,x_0), \quad \forall n \geq 0. \tag{3.58}$$

Since the norm is weakly lower semicontinuous, this implies that

$$\begin{aligned} \phi(p,x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \leq \lim_{n_k\to\infty} \inf\left\{ \|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2 \right\} \\ &\leq \lim_{n_k\to\infty} \inf \phi(x_{n_k},x_0) \leq \lim_{n_k\to\infty} \sup \phi(x_{n_k},x_0) \leq \phi(w,x_0). \end{aligned} \tag{3.59}$$

It follows from the definition of $\Pi_{F(T)\bigcap F(S)\bigcap \Omega} x_0$ and (3.59) that we have $p = w$. Therefore, $x_n \to \Pi_{F(T)\bigcap F(S)\bigcap \Omega} x_0$. This completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 improves and extends the corresponding results in [7–9].

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property(note that each uniformly convex Banach space must have the Kadec-Klee property).

(b) For the mappings, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings, or weak relatively nonexpansive mappings to a pair of quasi-ϕ-asymptotically nonexpansive mappings.

(c) For the equilibrium problem, we extend the generalized equilibrium problem to the generalized mixed equilibrium problem.

The following theorems can be obtained from Theorem 3.1 immediately.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A_1)–(A_4). Let S, T : $C \to C$ be two closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$. Suppose that S and T are uniformly L-Lipschitz continuous and that*

$G = F(T) \bigcap F(S) \bigcap \mathrm{GEP}$ *is a nonempty and bounded subset in* C*. Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{aligned}
&x_0 \in C, \quad C_0 = C, \quad Q_0 = C,\\
&z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n),\\
&y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n),\\
&u_n \in C \text{ such that}, \quad \forall y \in C,\\
&\Theta(u_n, y) + \langle A u_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0,\\
&C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n\},\\
&Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \geq 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.60}$$

where $J : E \to E^*$ *is the normalized duality mapping,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$*, and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$*,* $\xi_n = \sup_{u \in G}(k_n - 1)\phi(u, x_n)$*. If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy conditions (i)-(ii) in Theorem 3.1, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap F(T)\cap \mathrm{GEP}} x_0$*, where* GEP *is the set for the solutions of generalized equilibrium problem* (1.3).

Proof. Putting $\psi = 0$ in Theorem 3.1, the conclusion of Theorem 3.3 can be obtained from Theorem 3.1. □

Theorem 3.4. *Let* E *be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and* C *a nonempty closed convex subset of* E*. Let* $\psi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function and* $\Theta : C \times C \to \mathbb{R}$ *a bifunction satisfying conditions* (A_1)*–*(A_4)*. Let* $S, T : C \to C$ *be two closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$ *and* $k_n \to 1$*. Suppose that* S *and* T *are uniformly* L*-Lipschitz continuous and that* $G = F(T) \bigcap F(S) \bigcap \mathrm{MEP}$ *is a nonempty and bounded subset in* C*. Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{aligned}
&x_0 \in C, \quad C_0 = C, \quad Q_0 = C,\\
&z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n),\\
&y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n),\\
&u_n \in C \text{ such that}, \quad \forall y \in C,\\
&\Theta(u_n, y) + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0,\\
&C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n\},\\
&Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \geq 0\},\\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned} \tag{3.61}$$

where $J : E \to E^*$ *is the normalized duality mapping,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$*, and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$*,* $\xi_n = \sup_{u \in G}(k_n - 1)\phi(u, x_n)$*. If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy conditions* *(i)-(ii) in Theorem 3.1, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap F(T)\cap \mathrm{MEP}}x_0$*, where MEP is the set of solutions for mixed equilibrium problem* (1.4).

Proof. Putting $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.4 can be obtained from Theorem 3.1. □

Theorem 3.5. *Let* E *be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and* C *a nonempty closed convex subset of* E*. Let* $A : C \to E^*$ *be a continuous and monotone mapping and* $\psi : C \to \mathbb{R}$ *a lower semicontinuous and convex function. Let* $S, T : C \to C$ *be two closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1,\infty)$ *and* $k_n \to 1$*. Suppose that* S *and* T *are uniformly L-Lipschitz continuous and that* $G = F(T)\bigcap F(S)\bigcap \mathrm{VI}(C,A,\psi)$ *is a nonempty and bounded subset in* C*. Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{gathered}
x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\
y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\
u_n \in C \text{ such that}, \quad \forall y \in C, \\
\langle A u_n, y - u_n\rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \\
C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n\}, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.62}$$

where $J : E \to E^*$ *is the normalized duality mapping,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$*, and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$*,* $\xi_n = \sup_{u\in G}(k_n - 1)\phi(u, x_n)$*. If* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy conditions (i)-(ii) in Theorem 3.1, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap F(T)\cap \mathrm{VI}(C,A,\psi)}x_0$*, where* $\mathrm{VI}(C,A,\psi)$ *is the set of solutions for the mixed variational inequality* (1.5).

Proof. Putting $\Theta = 0$ in Theorem 3.1, the conclusion of Theorem 3.5 can be obtained from Theorem 3.1. □

Theorem 3.6. *Let* E *be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and* C *a nonempty closed convex subset of* E*. Let* $\Theta : C \times C \to \mathbb{R}$ *be a bifunction satisfying conditions* (A_1)*–*(A_4)*. Let* $S, T : C \to C$ *be two closed and uniformly quasi-*ϕ*-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1,\infty)$ *and* $k_n \to 1$*. Suppose that* S *and* T *are uniformly L -Lipschitz continuous and that* $G = F(T)\bigcap F(S)\bigcap \mathrm{EP}(\Theta)$ *is a nonempty and bounded subset in* C*. Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{gathered}
x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\
y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\
u_n \in C \text{ such that}, \quad \forall y \in C,
\end{gathered}$$

$$\begin{gathered}
\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \\
C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.63}$$

where $J : E \longrightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G}(k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)\cap \mathrm{EP}(\Theta)} x_0$, where $\mathrm{EP}(\Theta)$ is the set of solutions for the equilibrium problem (1.6).

Proof. Putting $\psi = 0$ and $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.6 can be obtained from Theorem 3.1. □

Theorem 3.7. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $A : C \longrightarrow E^*$ be a continuous and monotone mapping and S, $T : C \longrightarrow C$ two closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \longrightarrow 1$. Suppose that S and T are uniformly L-Lipschitz continuous and that $G = F(T) \bigcap F(S) \bigcap \mathrm{VI}(C, A)$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered}
x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT^n x_n), \\
y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) JS^n z_n), \\
u_n \in C \text{ such that}, \quad \forall y \in C, \\
\langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \\
C_n = \{v \in C_{n-1} : \phi(v, z_n) \leq \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \leq \phi(v, x_n) + (1 + k_n)(1 - \beta_n)\xi_n\}, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.64}$$

where $J : E \longrightarrow E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in G}(k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)\cap \mathrm{VI}(C,A)} x_0$, where $\mathrm{VI}(C, A)$ is the set of solutions for the variational inequality (1.7)

Proof. Putting $\psi = 0$ and $\Theta = 0$ in Theorem 3.1, the conclusion of Theorem 3.7 can be obtained from Theorem 3.1. □

Theorem 3.8. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $A : C \longrightarrow E^*$ be a continuous and monotone* *mapping, $\psi : C \to \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A_1)–(A_4). Let $S : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Suppose that S is uniformly L-Lipschitz continuous and that $F(S) \bigcap \Omega$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{aligned}
&x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
&y_n = J^{-1}\left(\beta_n J x_n + (1 - \beta_n) J S^n x_n\right), \\
&u_n \in C \text{ such that}, \quad \forall y \in C, \\
&\Theta(u_n, y) + \langle A u_n, y - u_n \rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \\
&C_n = \left\{v \in C_{n-1} : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\right\}, \\
&Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned}
\tag{3.65}
$$

where $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\xi_n = \sup_{u \in F(S) \cap \Omega}(k_n - 1)\phi(u, x_n)$. If $\{\beta_n\}$ satisfy condition (ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap \Omega} x_0$.

Proof. Taking $T = I$ in Theorem 3.1, we have that $z_n = x_n$, $\forall n \geq 0$. Hence, the conclusion of Theorem 3.8 is obtained. □

Theorem 3.9. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ a lower semicontinuous and convex function, and $\Theta : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A_1)–(A_4). Suppose that Ω is a nonempty subset in C. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{aligned}
&x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\
&u_n \in C \text{ such that}, \quad \forall y \in C, \\
&\Theta(u_n, y) + \langle A u_n, y - u_n \rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J x_n \rangle \geq 0, \\
&C_n = \left\{v \in C_{n-1} : \phi(v, u_n) \leq \phi(v, x_n)\right\}, \\
&Q_n = \{z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\
&x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{aligned}
\tag{3.66}
$$

where $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_0$.

Proof. Taking $T = S = I$ in Theorem 3.1, the conclusion is obtained. □

Theorem 3.10. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property and C a nonempty closed convex subset of E. Let $S, T : C \to C$ be two closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Suppose that S and T are uniformly L-Lipschitz continuous and that $F(T) \bigcap F(S)$ is a nonempty and bounded subset in C. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_0 \in C, \quad C_0 = C, \quad Q_0 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\ u_n = \Pi_C y_n, \\ C_n = \{ v \in C_{n-1} : \phi(v, z_n) \le \phi(v, x_n) + \xi_n,\ \phi(v, u_n) \le \phi(v, x_n) + (1+k_n)(1-\beta_n)\xi_n \}, \\ Q_n = \{ z \in Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \ge 0 \}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{gathered} \tag{3.67}$$

where $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and $\xi_n = \sup_{u \in F(S)\cap F(T)} (k_n - 1)\phi(u, x_n)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (i)-(ii) in Theorem 3.1, then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(T)} x_0$.

Proof. Taking $A = \Theta = 0$ and $r_n = 1$, $\forall n \ge 0$ in Theorem 3.1, the conclusion of Theorem 3.10 is obtained. □

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