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Research Article

Improved Bounds for Radio k-Chromatic Number of Hypercube Q_n

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A number of graph coloring problems have their roots in a communication problem known as the channel assignment problem. The channel assignment problem is the problem of assigning channels (nonnegative integers) to the stations in an optimal way such that interference is avoidedas reported by Hale (2005). Radio k-coloring of a graph is a special type of channel assignment problem. Kchikech et al. (2005) have given a lower and an upper bound for radio k-chromatic number of hypercube Q_n, and an improvement of their lower bound was obtained by Kola and Panigrahi (2010). In this paper, we further improve Kola et al.'s lower bound as well as Kchikeck et al.'s upper bound. Also, our bounds agree for nearly antipodal number of Q_n when $n \equiv 2 \pmod 4$.

1. Introduction

Radio coloring is derived from the assignment of radio frequencies (channels) to a set of transmitters. The frequencies assigned depend on the geographical distance between the transmitters: the closer two transmitters are, the greater the potential for interference between their signals. Thus, when the distance between two transmitters is small, the difference in the frequencies assigned must be relatively large, whereas two transmitters at a large distance may be assigned relatively close frequencies.

Radio k-coloring of a graph is a variation of the channel assignment problem. For a simple connected graph G of order n and diameter q, and a positive integer k with $1 \leqslant k \leqslant q$, a *radio k-coloring* f of G is an assignment of non-negative integers to the vertices of G such that for every two distinct vertices u and v of G

$$\left|f(u) - f(v)\right| \geqslant k + 1 - d(u,v), \tag{1.1}$$

where $d(u, v)$ is the distance between u and v in G. The *span* of a radio k-coloring f, $\mathrm{span}_k(f)$, is the maximum integer assigned to a vertex of G. The *radio k-chromatic number* of G, denoted by $\mathrm{rc}_k(G)$, is defined by $\mathrm{rc}_k(G) = \min\{\mathrm{span}_k(f) : f \text{ is a radio } k\text{-coloring of } G\}$. Since $\mathrm{rc}_1(G)$ is the chromatic number $\chi(G)$, radio k-colorings are a generalization of ordinary vertex coloring of graphs. In the literature, $\mathrm{rc}_q(G)$ is termed as *radio number* of G whereas $\mathrm{rc}_{q-1}(G)$ and $\mathrm{rc}_{q-2}(G)$ are called *antipodal* and *nearly antipodal number* of G, respectively. Since the problem is to find a radio k-coloring with minimum span, we use the least color 0 in every radio k-coloring appearing in this paper.

Finding radio k-chromatic number of a graph is an interesting yet difficult combinatorial problem with potential applications to FM channel assignment. The concept of radio k-coloring was introduced by Chartrand et al. in [1]. But so far, radio k-chromatic number is known for very limited classes of graphs and for specific values of k only. Radio numbers of C_n and P_n [2], C_n^2 [3], and P_n^2 [4] are known. Recently, radio numbers of complete m-array trees have been found by Li et al. [5]. Radio number of hypercubes is determined by Khennoufa and Tongi [6] and Kola and Panigrahi [7] independently. Khennoufa et al. have also found antipodal number of Q_n. For $k = n-1$, $n-2$, $n-3$ and $n-4$ the radio k-chromatic number of path P_n has been determined in [4, 8–10], respectively. Juan and liu [11] have determined the antipodal number of cycle C_n. Kchikech et al. [12] have given an upper and a lower bound for radio k-chromatic number of cartesian product of two graphs G and G'. Also, as a particular case, they have given an upper and a lower bounds for radio k-chromatic number of hypercube Q_n, but these bounds were very loose. Recently, Kola and Panigrahi [7] have improved their lower bound for $\mathrm{rc}_k(Q_n)$ and have shown that this bound is exact for radio number of Q_n. In this paper, we obtain an improvement of Kola et al.'s lower bound as well as Kchikeck et al.'s upper bound. Further, we show that these bounds agree for $\mathrm{rc}_{n-2}(Q_n), n \equiv 2 \pmod 4$.

2. Improved Lower Bound

In this section, we give an improved lower bound of $\mathrm{rc}_k(Q_n)$ for $2 \leqslant k \leqslant n-2$. For a hypercube Q_n of dimension n, the vertex set can be taken as binary n-bit strings and two vertices being adjacent if the corresponding strings differ at exactly one bit.

Definition 2.1. For any two n-bit binary strings $a = a_0a_1 \cdots a_{n-1}$ and $b = b_0b_1 \cdots b_{n-1}$, the Hamming distance $d_H(a, b)$ between a and b is the number of bits in which they differ. In particular, if $x, y \in \{0, 1\}$, then $d_H(x, y) = 0$ or 1 according as $x = y$ or $x \neq y$.

If u and v are two vertices of Q_n with a and b as the corresponding strings, then $d_{Q_n}(u, v) = d_H(a, b)$. Two n-bit binary strings may differ in at most n positions, so diameter of Q_n is n.

The results in the following lemma may be found in [7].

Lemma 2.2. *For any three vertices x, y, and z of the hypercube Q_n, the following holds:*

(i) $d(x, y) + d(y, z) + d(x, z) \leqslant 2n$,

(ii) $d(x, y) + d(y, z) + d(x, z) = 2n$, *if* $d(x, y) = n$.

The following lower bound for $\mathrm{rc}_k(Q_n)$ was determined by Kola and Panigrahi [7].

Theorem 2.3 (see [7]). *For the hypercube Q_n of dimension $n \geqslant 2$ and for any positive integer k with $2 \leqslant k \leqslant n$,*

$$\mathrm{rc}_k(Q_n) \geqslant \begin{cases} \left(\dfrac{3(k+1)-2n}{2}\right)2^{n-1} - \dfrac{k+1}{2}, & \text{if } k \text{ is odd}, \\ \left(\dfrac{3(k+1)-2n+1}{2}\right)2^{n-1} - \dfrac{k+2}{2}, & \text{if } k \text{ is even}. \end{cases} \tag{2.1}$$

In the theorem below, we give an improvement of the above lower bound of $\mathrm{rc}_k(Q_n)$ for all values of n and $2 \leqslant k \leqslant n-2$.

Theorem 2.4. *For a hypercube Q_n of dimension $n \geqslant 2$ and for any positive integer k with $2 \leqslant k \leqslant n-2$,*

$$rc_k(Q_n) \geqslant \left\lceil \frac{3(k+1)-2n}{2} \right\rceil \left(2^{n-1}-1\right). \tag{2.2}$$

Proof. Let f be any radio k-coloring of Q_n and $x_1, x_2, \ldots, x_{2^n}$ be an ordering of the vertices of Q_n such that $f(x_{j+1}) \geqslant f(x_j)$, $1 \leqslant j \leqslant 2^n - 1$. Obviously, $f(x_1) = 0$ and $\mathrm{span}(f) = f(x_{2^n})$. Since f is a radio k-coloring of Q_n, for every i with $0 \leqslant i \leqslant 2^n - 2$, we have the following:

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geqslant k+1-d(x_{i+1}, x_i), \\ f(x_{i+2}) - f(x_{i+1}) &\geqslant k+1-d(x_{i+2}, x_{i+1}), \\ f(x_{i+2}) - f(x_i) &\geqslant k+1-d(x_{i+2}, x_i). \end{aligned} \tag{2.3}$$

Adding (2.3) we get,

$$\begin{aligned} 2\left(f(x_{i+2}) - f(x_i)\right) &\geqslant 3(k+1) - \left(d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+1}) + d(x_{i+2}, x_i)\right) \\ &\geqslant 3(k+1) - 2n \quad (\text{from (i) of Lemma 2.2}). \end{aligned} \tag{2.4}$$

From the above inequality, we have $f(x_{i+2}) - f(x_i) \geqslant \lceil (3(k+1)-2n)/2 \rceil$, for all $i = 1, 3, \ldots, 2^n - 3$ and summing up these we get, $f(x_{2^n-1}) \geqslant \lceil (3(k+1)-2n)/2 \rceil (2^{n-1}-1)$. From this inequality and the fact that $f(x_{2^n}) \geqslant f(x_{2^n-1})$, we get the result. □

3. Improved Upper Bound

In this section, we give an improved upper bound of $\mathrm{rc}_k(Q_n)$. For better presentation, we need the definition below.

Definition 3.1. For two positive integers n and l with $l < n$, a binary (n, l)-Gray code is a listing of all the n-bit binary strings such that the Hamming distance between two successive strings is exactly l. Further, a quasi (n, l)-Gray code is a listing of all the n-bit binary strings such that the Hamming distance between two successive strings is exactly l except between the two items $2^{n-1} - 1$ and 2^{n-1} for which it is $l-1$ or $l+1$.

Notation 3.2. For any positive integer n, we define $\delta(n)$ as

$$\delta(n) = \begin{cases} 0, & \text{if } n \equiv 2 \pmod{4}, \\ 1, & \text{if } n \not\equiv 2 \pmod{4}. \end{cases} \tag{3.1}$$

The following theorem gives a partition of the vertex set of Q_n that will be used to find the upper bound of $\mathrm{rc}_k(Q_n)$.

Theorem 3.3. *For a hypercube Q_n of dimension n, there exists a partition of the vertex set of Q_n with the partite sets $U_1 = \{x_i : i = 0, 1, \ldots, 2^{n-1} - 1\}$ and $U_2 = \{y_i : i = 0, 1, \ldots, 2^{n-1} - 1\}$ satisfying the followyng properties:*

(a) $d(x_i, x_{i+1}) = \lfloor n/2 \rfloor = d(y_i, y_{i+1})$, *for all $i = 0, 1, \ldots, 2^{n-1} - 2$ except $i = 2^{n-2} - 1$,*

(b) $d(x_i, x_{i+1}) = \lfloor n/2 \rfloor - \delta(n)$ *or* $\lfloor n/2 \rfloor + \delta(n)$, *for $i = 2^{n-2} - 1$,*

(c) $d(y_i, y_{i+1}) = \lfloor n/2 \rfloor - \delta(n)$ *or* $\lfloor n/2 \rfloor + \delta(n)$, *for $i = 2^{n-2} - 1$,*

(d) $d(x_i, y_i) = n$, *for all $i = 0, 1, \ldots, 2^{n-1} - 1$,*

where $\delta(n)$ is defined as in Notation 3.2.

Proof. Let Q'_{n-1} and Q''_{n-1} be two copies of Q_{n-1} induced by all the n-bit strings with starting bit 0 and 1, respectively. Let $x_0, x_1, \ldots, x_{2^{n-1}-1}$ be the ordering of the vertices of Q'_{n-1} which induce a quasi $(n-1, \lfloor n/2 \rfloor)$-Gray code if $n \not\equiv 2 \pmod{4}$ and a $(n-1, n/2)$-Gray code if $n \equiv 2 \pmod{4}$; (such code exists, see [6]). We take $y_i = \overline{x}_i$, $i = 1, 2, \ldots, 2^{n-1} - 1$, that is, $\overline{x}_i$ is obtained from x_i by changing 0*s* to 1*s* and vice versa. Now, $y_0, y_1, \ldots, y_{2^{n-1}-1}$ is an ordering of the vertices of Q''_{n-1} which induces the same Gray code as in the above. By taking $U_1 = \{x_i : i = 0, 1, \ldots, 2^{n-1} - 1\}$ and $U_2 = \{y_i : i = 0, 1, \ldots, 2^{n-1} - 1\}$, we get the result. □

The following upper bound for $\mathrm{rc}_k(Q_n)$ was determined by Kchikech et al. in [12].

Theorem 3.4 (see [12]). *For the hypercube Q_n of dimension $n \geqslant 2$ and for any $k \geqslant 2$,*

$$rc_k(Q_n) \leqslant (2^n - 1)k - 2^{n-1} + 1. \tag{3.2}$$

The theorem below gives an improvement of this upper bound of $\mathrm{rc}_k(Q_n)$.

Theorem 3.5. *For the hypercube Q_n of dimension $n \geqslant 2$ and for any positive integer k, $2 \leqslant k \leqslant n-2$,*

$$rc_k(Q_n) \leqslant \begin{cases} (2^{n-1} - 1)\left(k + 1 - \left\lfloor \dfrac{n}{2} \right\rfloor\right) + \delta(n), & \text{if } 2\left\lfloor \dfrac{n}{2} \right\rfloor - 2 \leqslant k \leqslant n - 2, \\ k2^{n-2} - \left\lceil \dfrac{k}{2} \right\rceil + \delta(n), & \text{if } \left\lfloor \dfrac{n}{2} \right\rfloor - 1 \leqslant k \leqslant 2\left\lfloor \dfrac{n}{2} \right\rfloor - 3, \\ k2^{n-2} - k, & \text{if } 2 \leqslant k \leqslant \left\lfloor \dfrac{n}{2} \right\rfloor - 2, \end{cases} \tag{3.3}$$

where $\delta(n)$ is defined as in Notation 3.2.

Table 1

$u, v \in U_1 \cup U_2$	Value of $\lvert i-j\rvert$	$d(u,v)$	$k+1-d(u,v)$	$\lvert f(u)-f(v)\rvert$
$x_i, x_j \in U_1$, or $y_i, y_j \in U_2$	$\lvert i-j\rvert = 1$	$\lfloor n/2\rfloor$	$k+1-\lfloor n/2\rfloor$	$k+1-\lfloor n/2\rfloor$
	$\lvert i-j\rvert \geqslant 2$	$\geqslant 1$	$\leqslant k$	$\geqslant k$
	$i-j=0$	n	$\leqslant 0$	0
$x_i \in U_1, y_j \in U_2$	$\lvert i-j\rvert = 1$	$\geqslant \lfloor n/2\rfloor$	$\leqslant k+1-\lfloor n/2\rfloor$	$\geqslant k+1-\lfloor n/2\rfloor$
	$\lvert i-j\rvert \geqslant 2$	$\geqslant 1$	$\leqslant k$	$\geqslant k$

Proof. Here, we consider the same partition of $V(Q_n)$ as in Theorem 3.3. We first take $2\lfloor n/2\rfloor - 2 \leqslant k \leqslant n-2$. For these values of k, we define a coloring f of $V(Q_n)$ with $f(x_0)=0$ and

$$\begin{aligned} f(x_i) &= f(x_{i-1}) + k + 1 - \left\lfloor \frac{n}{2} \right\rfloor, \quad i = 1, 2, \dots, 2^{n-1}-1, \text{ except } i = 2^{n-2}, \\ f(x_{2^{n-2}}) &= f(x_{2^{n-2}-1}) + k + 1 - \left\lfloor \frac{n}{2} \right\rfloor + \delta(n), \\ f(y_i) &= f(x_i), \quad i = 0, 1, 2, \dots, 2^{n-1}-1. \end{aligned} \tag{3.4}$$

We first check the radio condition for the vertex $x_{2^{n-2}}$ with all other vertices. Let $v \in V(Q_n) - \{x_{2^{n-2}}\}$. From the definition of f, $|f(x_{2^{n-2}}) - f(u)| \geqslant k$, except $v \in \{x_{2^{n-2}-1}, y_{2^{n-2}-1}, y_{2^{n-2}}, x_{2^{n-2}+1}, y_{2^{n-2}+1}\}$. From Theorem 3.3(b) and Lemma 2.2(ii), we have $d(x_{2^{n-2}-1}, x_{2^{n-2}})$, $d(y_{2^{n-2}-1}, x_{2^{n-2}}) \geqslant \lfloor n/2\rfloor - \delta(n)$ and therefore $f(x_{2^{n-2}}) - f(v) = k+1+\delta(n) - \lfloor n/2\rfloor \geqslant k+1-d(x_{2^{n-2}}, v)$, for $v = x_{2^{n-2}-1}$ or $y_{2^{n-2}-1}$. Similarly, for $v = x_{2^{n-2}+1}$ or $y_{2^{n-2}-1}$, we show $f(x_{2^{n-2}}) - f(u) \geqslant k+1-d(x_{2^{n-2}}, u)$. Since $f(x_{2^{n-2}}) = f(y_{2^{n-2}})$ and $d(x_{2^{n-2}}, y_{2^{n-2}}) = n$, radio condition is trivially true for the pair $(x_{2^{n-2}}, y_{2^{n-2}})$. In a similar manner one can show that $|f(y_{2^{n-2}}) - f(v)| \geqslant k+1-d(y_{2^{n-2}}, v)$, for all $v \in V(Q_n) - \{y_{2^{n-2}}\}$. For the rest of the pairs of vertices in $V(Q_n)$, checking radio condition is straightforward from the definition of f. However, in Table 1, we compute the values of $|f(u)-f(v)|$ and $k+1-d(u,v)$ for every pair of vertices $u, v \in V(Q_n) - \{x_{2^{n-2}}, y_{2^{n-2}}\}$.

Therefore, f is a radio k-coloring of Q_n and the span of f is $(2^{n-1}-1)(k+1-\lfloor n/2\rfloor)+\delta(n)$. Next, we consider the value of k as $\lfloor n/2\rfloor \leqslant k \leqslant 2\lfloor n/2\rfloor - 3$ and define a coloring f with $f(x_0)=0$, and

$$\begin{aligned} f(x_{i+1}) &= f(x_i) + \left\lfloor \frac{k}{2} \right\rfloor, \quad \text{if } i \text{ is even and } i \neq 2^{n-2}-1, \\ f(x_{i+1}) &= f(x_i) + \left\lceil \frac{k}{2} \right\rceil, \quad \text{if } i \text{ is odd}, \\ f(x_{i+1}) &= f(x_i) + \left\lfloor \frac{k}{2} \right\rfloor + \delta(n), \quad i = 2^{n-2}-1, \\ f(y_i) &= f(x_i), \quad i = 0, 1, 2, \dots, 2^{n-1}-1. \end{aligned} \tag{3.5}$$

By a similar way as in the above, one can check that f is a radio k-coloring of Q_n with the span $f(x_{2^{n-1}-1}) = 2^{n-2}\lfloor k/2\rfloor + (2^{n-2}-1)\lceil k/2\rceil + \delta(n) = k2^{n-2} - \lceil k/2\rceil + \delta(n)$.

Finally, we take $2 \leqslant k \leqslant \lfloor n/2\rfloor - 2$. From Theorem 2.4, we have $d(x_i, x_{i+1})$, $d(x_i, y_i)$, $d(x_i, y_{i+1})$, $d(x_{i+1}, y_i)$, $d(y_i, y_{i+1}) \geqslant k+1$. So, we use the same color for the vertices x_i, x_{i+1}, y_i

and y_{i+1}. In this case, we define a coloring f as $f(x_i) = (i/2)k = f(y_i)$, $f(x_{i+1}) = f(x_i)$, $f(y_{i+1}) = f(y_i)$, $i = 0, 2, \ldots, 2^{n-1} - 2$. It is easy to show that f is a radio k-coloring of Q_n with the span $k2^{n-2} - k$. □

Observe that the bounds given in Theorems 2.4 and 3.5 agree for $k = n - 2$ with $n \equiv 2 \pmod 4$. Therefore we have determined the nearly antipodal number of Q_n which is given in the theorem below.

Theorem 3.6. *For $n \equiv 2 \pmod 4$, the nearly antipodal number $rc_{n-2}(Q_n)$ of the hypercube Q_n is $(2^{n-1} - 1)(n - 2)/2$.*

4. Concluding Remarks

Improved lower and upper bounds of $\mathrm{rc}_k(Q_n)$ have been obtained in Theorems 2.4 and 3.5, respectively. It is easy to verify that the bound given in Theorem 3.5 is an improvement over the bound in Theorem 3.4. However, below, we check that lower bound given in Theorem 2.4 is really an improvement over the bound in Theorem 2.3. From Theorem 2.4, we have

$$\begin{aligned}
\mathrm{rc}_k(Q_n) &\geqslant \begin{cases} \left(\dfrac{3(k+1) - 2n}{2}\right)(2^{n-1} - 1), & \text{if } k \text{ is odd}, \\[2ex] \left(\dfrac{3(k+1) - 2n + 1}{2}\right)(2^{n-1} - 1), & \text{if } k \text{ is even}, \end{cases} \\
&= \begin{cases} \left(\dfrac{3(k+1) - 2n}{2}\right)2^{n-1} + n - \dfrac{3(k+1)}{2}, & \text{if } k \text{ is odd}, \\[2ex] \left(\dfrac{3(k+1) - 2n + 1}{2}\right)2^{n-1} + n - \dfrac{3k+4}{2}, & \text{if } k \text{ is even}. \end{cases}
\end{aligned} \tag{4.1}$$

For odd integer k,

$$\begin{aligned}
&\left[\left(\frac{3(k+1) - 2n}{2}\right)2^{n-1} + n - \frac{3(k+1)}{2}\right] - \left[\left(\frac{3(k+1) - 2n}{2}\right)2^{n-1} - \frac{k+1}{2}\right] \\
&\quad = n - k - 1 \geqslant 1, \quad \text{if } 2 \leqslant k \leqslant n - 2.
\end{aligned} \tag{4.2}$$

Again, if k is an even integer, then

$$\begin{aligned}
&\left[\left(\frac{3(k+1) - 2n + 1}{2}\right)2^{n-1} + n - \frac{3k+4}{2}\right] - \left[\left(\frac{3(k+1) - 2n + 1}{2}\right)2^{n-1} - \frac{k+2}{2}\right] \\
&\quad = n - k - 1 \geqslant 1, \quad \text{if } 2 \leqslant k \leqslant n - 2.
\end{aligned} \tag{4.3}$$

Theorem 3.6 gives the nearly antipodal number of Q_n for $n \equiv 2 \pmod 4$. From Theorems 2.4 and 3.5, one gets that the difference between the lower and upper bound of $\mathrm{rc}_{n-2}(Q_n)$, $n \equiv 0 \pmod 4$, is equal to one, that is, $(2^{n-1} - 1)(n - 2)/2 \leqslant \mathrm{rc}_{n-2}(Q_n) \leqslant (2^{n-1} - 1)(n - 2)/2 + 1$.

References

[1] G. Chartrand, D. Erwin, and P. Zhang, "A graph labeling problem suggested by FM channel restrictions," *Bulletin of the Institute of Combinatorics and Its Applications*, vol. 43, pp. 43–57, 2005.

[2] D. D.-F. Liu and M. Xie, "Radio number for square paths," *Ars Combinatoria*, vol. 90, pp. 307–319, 2009.

[3] D. D.-F. Liu and M. Xie, "Radio number for square of cycles," *Congressus Numerantium*, vol. 169, pp. 105–125, 2004.

[4] D. Liu and X. Zhu, "Multilevel distance labelings for paths and cycles," *SIAM Journal on Discrete Mathematics*, vol. 19, no. 3, pp. 610–621, 2005.

[5] X. Li, V. Mak, and S. Zhou, "Optimal radio labellings of complete m-ary trees," *Discrete Applied Mathematics*, vol. 158, no. 5, pp. 507–515, 2010.

[6] R. Khennoufa and O. Togni, "The radio antipodal and radio numbers of the hypercube," *Ars Combinatoria*. In press.

[7] S. R. Kola and P. Panigrahi, "An improved lower bound for the radio k-chromatic number of the hypercube Q_n," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 2131–2140, 2010.

[8] R. Khennoufa and O. Togni, "A note on radio antipodal colourings of paths," *Mathematica Bohemica*, vol. 130, no. 3, pp. 277–282, 2005.

[9] S. R. Kola and P. Panigrahi, "Nearly antipodal chromatic number $ac'(P_n)$ of the path P_n," *Mathematica Bohemica*, vol. 134, no. 1, pp. 77–86, 2009.

[10] S. R. Kola and P. Panigrahi, "On radio $(n-4)$-chromatic number of the path P_n," *AKCE International Journal of Graphs and Combinatorics*, vol. 6, no. 1, pp. 209–217, 2009.

[11] J. S. Juan and D. D.-F. Liu, Antipodal labeling of cycles, Manuscript, 2006, http://www.calstatela.edu/faculty/dliu/dliu.htm.

[12] M. Kchikech, R. Khennoufa, and O. Togni, "Radio k-labelings for cartesian products of graphs," *Electronic Notes in Discrete Mathematics*, vol. 22, pp. 347–352, 2005.