Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 129132, 15 pages
doi:10.1155/2012/129132

Research Article

On Numerical Radius of a Matrix and Estimation of Bounds for Zeros of a Polynomial

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Received 31 March 2012; Accepted 5 June 2012

Academic Editor: Teodor Bulboaca

We obtain inequalities involving numerical radius of a matrix $A \in M_n(\mathbb{C})$. Using this result, we find upper bounds for zeros of a given polynomial. We also give a method to estimate the spectral radius of a given matrix $A \in M_n(\mathbb{C})$ up to the desired degree of accuracy.

1. Introduction

Suppose $A \in M_n(\mathbb{C})$. Let $W(A)$, $\sigma(A)$ denote respectively the numerical range, spectrum of A and $w(A)$, $r_\sigma(A)$ denote respectively the numerical radius, spectral radius of A, that is,

$$\begin{aligned} W(A) &= \{\langle Ax, x\rangle : \|x\| = 1\}, \\ w(A) &= \sup\{|\lambda| : \lambda \in W(A)\}, \\ \sigma(A) &= \{\lambda : \lambda \text{ is an eigenvalue of } A\}, \\ r_\sigma(A) &= \sup\{|\lambda| : \lambda \in \sigma(A)\}. \end{aligned} \tag{1.1}$$

It is well known that

(i) $\|A\|/2 \le w(A) \le \|A\|$.

Kittaneh [1] improved on the second inequality to prove that.

(ii) $w(A) \le 1/2\|A\| + 1/2\|A^2\|^{1/2}$.

Clearly, $(1/2)\|A\| + (1/2)\|A^2\|^{1/2} \le \|A\|$ so that inequality (ii) is sharper than the second inequality of (i).

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ be a monic polynomial where $a_0, a_1, \ldots, a_{n-1}$ are complex numbers and let

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdot & \cdot & -a_1 & -a_0 \\ 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix} \tag{1.2}$$

be the Frobenius companion matrix of the polynomial $p(z)$. Then, it is well known that zeros of p are exactly the eigenvalues of $C(p)$. Considering $C(p)$ as an element of $M_n(\mathbb{C})$, we see that if z is root of the polynomial equation $p(z) = 0$, then

$$|z| \le w(C(p)), \qquad |z| \le r_\sigma(C(p)). \tag{1.3}$$

Based on inequality (ii), Kittaneh [1] obtained an estimation for $w(C(p))$ which gives an upper bound for zeros of the polynomial $p(z)$.

In Section 1 we find numerical radius of some special class of matrices and use the results obtained to give a better estimation of bounds for zeros of a polynomial.

2. On Numerical Radius of a Matrix

We first obtain bounds for numerical radius of a matrix in $M_n(\mathbb{C})$ and use it to obtain numerical radius for some special class of matrices.

Theorem 2.1. *Suppose $T \in M_n(\mathbb{C})$ and*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where $A \in M_r(\mathbb{C})$, $B \in M_{r,n-r}(\mathbb{C})$, $C \in M_{n-r,r}(\mathbb{C})$ and $D \in M_{n-r}(\mathbb{C})$. Then,

(i) $w(T) \le (1/2)[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2}]$ *and*

(ii) $\|T\|^2 \le (1/2)(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2) + (1/2)\sqrt{(\|A\|^2 + \|C\|^2 - \|B\|^2 - \|D\|^2)^2 + 4(\|A\|\|B\| + \|C\|\|D\|)^2}$.

Proof. (i) Let $Z \in \mathbb{C}^n$ and

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{2.2}$$

where $X \in \mathbb{C}^r$ and $Y \in \mathbb{C}^{n-r}$ with $\|Z\| = 1$.

Then,

$$\langle TZ, Z\rangle = \left\langle \begin{pmatrix} AX + BY \\ CX + DY \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \right\rangle = \langle AX, X\rangle + \langle BY, X\rangle + \langle CX, Y\rangle + \langle DY, Y\rangle \tag{2.3}$$

and so

$$|\langle TZ, Z\rangle| \leq |\langle AX, X\rangle| + |\langle DY, Y\rangle| + \|B\|\|X\|\|Y\| + \|C\|\|X\|\|Y\|. \tag{2.4}$$

Therefore, we have

$$\begin{aligned} w(T) &\leq \sup_{\|X\|^2+\|Y\|^2=1} \left[w(A)\|X\|^2 + w(D)\|Y\|^2 + (\|B\| + \|C\|)\|X\|\|Y\| \right], \\ &= \sup_{\theta\in[0,2\pi]} \left[w(A)\cos^2\theta + w(D)\sin^2\theta + (\|B\| + \|C\|)\cos\theta\sin\theta \right], \\ &\leq \frac{1}{2}\left[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2} \right]. \end{aligned} \tag{2.5}$$

This completes the first part of the proof.

(ii) Proceeding as in (i) we can prove the second part. This completes the proof of the theorem. □

Remark 2.2. As an application of (i) in Theorem 2.1, $\|T\|$ has another estimation by $\|T\|^2 = \|T^*T\| = w(T^*T)$ as follows:

$$\begin{aligned} \|T\|^2 \leq \frac{1}{2}\Big[& w(A^*A + C^*C) + w(B^*B + D^*D) \\ & + \sqrt{(w(A^*A + C^*C) - w(B^*B + D^*D))^2 + 4\|A^*B + C^*D\|^2} \Big]. \end{aligned} \tag{2.6}$$

Furuta [2] obtained numerical radius for a bounded linear operator T of the above form with $A = aI_r$, $B = bA$, $C = cA^*$, $D = dI_{n-r}$, and $a, b, c, d \in \mathbb{R}^+$. If we consider $A = aI_r$, $D = dI_{n-r}$, $C = 0_{n-r,r}$ where $a, d \in \mathbb{R}$, then we can exactly calculate $w(T)$ and $\|T\|$ as proved in the next theorem.

Theorem 2.3. *Suppose $B \in M_{r,n-r}(\mathbb{C})$ and*

$$T = \begin{pmatrix} aI_r & B \\ O_{n-r,r} & dI_{n-r} \end{pmatrix}. \tag{2.7}$$

Then

(i) $w(T) = (1/2)(|a + d| + \sqrt{(a - d)^2 + \|B\|^2})$ *and*

(ii) $\|T\| = (1/\sqrt{2})\sqrt{(a^2 + d^2 + \|B\|^2) + \sqrt{(a^2 - d^2 - \|B\|^2)^2 + 4a^2\|B\|^2}}$.

Proof. (i) Following the method employed in the previous theorem, we can show that

$$w(T) \leq \frac{1}{2}\left(|a+d| + \sqrt{(a-d)^2 + \|B\|^2}\right). \tag{2.8}$$

We only need to show that there exists z_0, $\|z_0\| = 1$ such that $|\langle Tz_0, z_0\rangle|$ equals the quantity in the RHS.

Suppose B attains its norm at y with $\|y\| = 1$.

Let $z = (By\, ky)^t$ where k is a scalar. Then, $\|z\|^2 = \|B\|^2 + |k|^2$. Now

$$\langle Tz, z\rangle = \left\langle \begin{pmatrix} aI_r & B \\ O_{n-r,r} & dI_{n-r}\end{pmatrix}\begin{pmatrix} B_y \\ k_y\end{pmatrix}, \begin{pmatrix} B_y \\ k_y\end{pmatrix}\right\rangle \tag{2.9}$$

so that

$$\left|\langle Tz, z\rangle \cdot \frac{1}{\|z\|^2}\right| = \frac{\left|(a+k)\|B\|^2 + dk^2\right|}{\|B\|^2 + k^2}. \tag{2.10}$$

Thus for all scalar k, we get

$$w(T) \geq \frac{\left|(a+k)\|B\|^2 + dk^2\right|}{\|B\|^2 + k^2}. \tag{2.11}$$

Case 1 ($d + a \geq 0$). Define a function $\phi : \mathbb{R}^+ \to \mathbb{R}$ by

$$\phi(k) = \frac{(a+k)\|B\|^2 + dk^2}{\|B\|^2 + k^2}. \tag{2.12}$$

Then using elementary calculus, we can show that $\phi(k)$ attains its maximum at $k_0 = (d-a) + \sqrt{(d-a)^2 + \|B\|^2}$ so that for $z_0 = (1/\sqrt{\|By\|^2 + k_0\|y\|^2})(By\, k_0 y)^t$ we get

$$|\langle Tz_0, z_0\rangle| = \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + \|B\|^2}\right). \tag{2.13}$$

Thus, we get

$$w(T) = \frac{1}{2}\left(|a+d| + \sqrt{(a-d)^2 + \|B\|^2}\right). \tag{2.14}$$

Case 2 ($d + a \le 0$). As before we can show that there exists $k_0 = (d-a) - \sqrt{(d-a)^2 + \|B\|^2}$ so that for $z_0 = (1/\sqrt{\|By\|^2 + k_0\|y\|^2})(By\, k_0 y)^t$ we get

$$|\langle T z_0, z_0\rangle| = \frac{1}{2}\left(|a+d| + \sqrt{(a-d)^2 + \|B\|^2}\right). \tag{2.15}$$

Thus in all cases, we get

$$w(T) = \frac{1}{2}\left(|a+d| + \sqrt{(a-d)^2 + \|B\|^2}\right). \tag{2.16}$$

This completes the proof of (i).

(ii) The proof is similar to the earlier one.

This completes the proof of the theorem. □

Using Theorem 2.3, we can find numerical radius of an idempotent matrix A, that is, a matrix for which $A^2 = A$ and also for a matrix for which $A^2 = I$.

Corollary 2.4. *Suppose $A \in M_n(\mathbb{C})$ with $A^2 = A$. Then*

$$w(A) = \frac{\|A\|}{2} + \frac{1}{2}. \tag{2.17}$$

Proof. By Schur's theorem, A is unitarily equivalent to an upper triangular matrix. So without loss of generality, we can assume that

$$A = \begin{pmatrix} I_r & B_{r,n-r} \\ O_{n-r,r} & 0_{n-r} \end{pmatrix}, \tag{2.18}$$

where I_r is the identity matrix, $B_{r,n-r}$ is any matrix. Using the last theorem, we get

$$w(A) = \frac{\|A\|}{2} + \frac{1}{2}. \tag{2.19}$$

□

Corollary 2.5. *Suppose $A \in M_n(\mathbb{C})$ and $A^2 = I$. Then*

$$w(A) = \frac{1}{2}\left(\|A\| + \frac{1}{\|A\|}\right). \tag{2.20}$$

Proof. A can be expressed as

$$A = \begin{pmatrix} I_r & B_{r,n-r} \\ O_{n-r,r} & -I_{n-r} \end{pmatrix}, \tag{2.21}$$

where I_r is the identity matrix, $B_{r,n-r}$ is any matrix. By Theorem 2.3, we have

$$\begin{gathered} \|A\| = \sqrt{1 + \frac{1}{2}\|B\|^2 + \frac{1}{2}\|B\|\sqrt{4 + \|B\|^2}}, \\ w(A) = \frac{1}{2}\sqrt{\left(4 + \|B\|^2\right)}. \end{gathered} \tag{2.22}$$

Therefore

$$\begin{aligned} \|A\|^2 &= 1 + \frac{1}{2}\|B\|^2 + \frac{1}{2}\|B\|\sqrt{4 + \|B\|^2}, \\ \frac{1}{\|A\|^2} &= \frac{1}{1 + (1/2)\|B\|^2 + (1/2)\|B\|\sqrt{4 + \|B\|^2}} \\ &= 1 + \frac{1}{2}\|B\|^2 - \frac{1}{2}\|B\|\sqrt{4 + \|B\|^2}. \end{aligned} \tag{2.23}$$

By adding, we get

$$\begin{aligned} \|A\|^2 + \frac{1}{\|A\|^2} &= 2 + \|B\|^2 \\ &\Longrightarrow \left(\|A\| + \frac{1}{\|A\|}\right)^2 = 4 + \|B\|^2 \\ &\Longrightarrow w(A) = \frac{1}{2}\left(\|A\| + \frac{1}{\|A\|}\right). \end{aligned} \tag{2.24}$$

□

Corollary 2.6. *Suppose* $A \in M_n(\mathbb{C})$ *with* $A^{2n} = I$. *Then*

$$w(A) \geq \left(\frac{1}{2}\left(\|A^n\| + \frac{1}{\|A^n\|}\right)\right)^{1/n}. \tag{2.25}$$

Proof. It follows from the fact that $(A^n)^2 = I$ and $w(A)^n \geq w(A^n)$. □

3. Bounds for Zeros of Polynomials

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ be a monic polynomial where $a_0, a_1, \ldots, a_{n-1}$ are complex numbers and let

$$C(p) = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdot & \cdot & -a_1 & -a_0 \\ 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix} \tag{3.1}$$

be the Frobenius companion matrix of the polynomial $p(z)$. Then, it is well known that zeros of p are exactly the eigenvalues of $C(p)$. Considering $C(p)$ as a linear operator on C^n, we see that if z is root of the polynomial equation $p(z) = 0$ then

$$|z| \leq w(C(p)) \quad \text{as } \sigma(C(p)) \subset W(C(p)), \tag{3.2}$$

where $\sigma(C(p))$ is the spectrum of operator $C(p)$. Estimation of the roots of zeros of the polynomial $p(z)$ has been done by many mathematicians over the years, some of them are mentioned below. Let λ be a root of the polynomial equation $p(z) = 0$.

(i) Carmichael and Mason [3] proved that

$$|\lambda| \leq \left[1 + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2\right]^{1/2}. \tag{3.3}$$

(ii) Montel [4, 5] proved that

$$\begin{aligned} |\lambda| &\leq |a_0| + |a_0 - a_1| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} + 1|, \\ |\lambda| &\leq (n-1) + |a_0| + |a_1| + \cdots + |a_{n-1}|. \end{aligned} \tag{3.4}$$

(iii) Cauchy [3] proved that

$$|\lambda| \leq 1 + \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}. \tag{3.5}$$

(iv) Fujii and Kubo [6, 7] proved that

$$|\lambda| \leq \cos\frac{\pi}{n+1} + \frac{1}{2}\left[\left(\sum_{i=0}^{n-2}|a_i|^2\right)^{1/2} + |a_{n-1}|\right]. \tag{3.6}$$

(v) Alpin et al. [8] proved that

$$|\lambda| \leq \max_{1\leq k\leq n}\left[(1 + |a_{n-1}|)(1 + |a_{n-2}|)\cdots(1 + |a_{n-k}|)\right]^{1/k}. \tag{3.7}$$

(vi) Kittaneh [1] proved that

$$|\lambda| \leq \frac{1}{2}\left[\|C(p)\| + \left\|C(p)^2\right\|^{1/2}\right]. \tag{3.8}$$

We develop an inequality involving numerical radius with the help of which we estimate the zeros of the polynomial p. We show with examples that our estimation is better than the estimations mentioned above.

Theorem 3.1. *If λ is a zero of the polynomial $p(z)$, then*

$$|\lambda| \leq \left|\frac{a_{n-1}}{n}\right| + \frac{1}{2}\left[\cos\frac{\pi}{n} + \sqrt{\cos^2\frac{\pi}{n} + \left(1 + \sqrt{\sum_{r=0}^{n-2}|\alpha_r|^2}\right)^2}\right], \tag{3.9}$$

where $\alpha_r = \sum_{k=0}^{n-r} {}_nC_k(-a_{n-1}/n)^k a_{n-k}$, $r = 0,1,2,\ldots,n-2$.

Proof. Putting $z = \xi + h$ in the polynomial equation $p(z) = z^n + a_n z^{n-1} + \cdots a_2 z + a_1 = 0$, we get

$$(\xi + h)^n + a_{n-1}(\xi + h)^{n-1} + \cdots + a_1(\xi + h) + a_0 = 0. \tag{3.10}$$

Substituting $h = -a_{n-1}/n$, we get

$$\xi^n + \alpha_{n-2}\xi^{n-2} + \alpha_{n-3}\xi^{n-3} + \cdots + \alpha_1\xi + \alpha_0 = 0, \tag{3.11}$$

where $\alpha_r = \sum_{k=0}^{n-r} {}_nC_K(-a_{n-1}/n)^k a_{n-k}$, $r = 0,1,2,\ldots,n-2$.

Let $C(\xi)$ be the Frobenius companion matrix of the polynomial $q(\xi) = \xi^n + \alpha_{n-2}\xi^{n-2} + \alpha_{n-3}\xi^{n-3} + \cdots + \alpha_1\xi + \alpha_0$.

Then $C(\xi) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A = (0)_1$, $B = (-\alpha_{n-2}, -\alpha_{n-3}, \ldots, -\alpha_0)_{1,n-1}$

$$C = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{n-1,1}, \qquad D = \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 1 & 0 \end{pmatrix}_{n-1,n-1}. \tag{3.12}$$

Using Theorem 2.1, we get

$$\begin{aligned} w(C(\xi)) &\leq \frac{1}{2}\left[w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|C\|)^2}\right] \\ \Longrightarrow w(C(\xi)) &\leq \frac{1}{2}\left[\cos\frac{\pi}{n} + \sqrt{\cos^2\frac{\pi}{n} + \left(1 + \sqrt{\sum_{r=0}^{n-2}|\alpha_r|^2}\right)^2}\right]. \end{aligned} \tag{3.13}$$

This shows that if ξ_0 is a zero of the polynomial $q(\xi)$, then

$$|\xi_0| \leq \frac{1}{2}\left[\cos\frac{\pi}{n} + \sqrt{\cos^2\frac{\pi}{n} + \left(1 + \sqrt{\sum_{r=0}^{n-2}|\alpha_r|^2}\right)^2}\right]. \tag{3.14}$$

Thus if λ is a zero of the polynomial $p(z)$, then

$$|\lambda| \leq \left|\frac{a_{n-1}}{n}\right| + \frac{1}{2}\left[\cos\frac{\pi}{n} + \sqrt{\cos^2\frac{\pi}{n} + \left(1 + \sqrt{\sum_{r=0}^{n-2}|\alpha_r|^2}\right)^2}\right]. \tag{3.15}$$

This completes the proof of the theorem. □

Example 3.2. Consider the polynomial equation $p(z) = z^3 - 3z^2 + 2z = 0$. Then the bounds estimated by different mathematicians are as shown in Table 1.

But our estimation shows that if λ is a zero of the polynomial then $|\lambda| \leq 2.280776406$ which is much better than all the estimations mentioned above.

The companion matrix of the polynomial after removing the second term can be written as

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A_{2,2} & B_{2,1} \\ C_{1,1} & D_{1,1} \end{pmatrix}. \tag{3.16}$$

Then using the above theorems, it is easy to show that $|\lambda| \leq 2.207$ which is even better estimation.

Example 3.3. Consider the polynomial equation $p(z) = z^5 - 8z^4 + 25z^3 - 38z^2 + 28z - 8 = 0$. Then, the bounds estimated by different mathematicians are as shown in Table 2.

But our estimation shows that if λ is a zero of the polynomial then $|\lambda| \leq 2.703669110$ which is much better than all the estimations mentioned above.

Theorem 3.4. *Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ having α_i $(i = 1, 2, \ldots, n)$ as zeros and for each $m \in N$, $p_m(z) = z^n + a_{n-1}^{(m)} z^{n-1} + \cdots a_2^{(m)} z + a_1^{(m)} z + a_0^{(m)}$ is a polynomial having $\alpha_i^{2^m}$ $(i = 1, 2, \ldots, n)$ as zeros. If λ is a zero of the polynomial $p(z)$, then for all m*

$$|\lambda| \leq \left(\frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(\left|a_{n-1}^{(m)}\right| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \left(1 + \sqrt{\sum_{k=2}^{n}\left|a_{n-k}^{(m)}\right|^2}\right)^2}\right]\right)^{1/2^m}. \tag{3.17}$$

Table 1

Carmichael and Mason	3.741657387
Montel	7
Cauchy	4
Fujii and Kubo	4.0098824
Yuri, Chien and Yeh	3.464101615
Kittaneh	3.44572894

Table 2

Carmichael and Mason	54.60769176
Montel	215
Cauchy	39
Fujii and Kubo	32.16529279
Yuri, Chien and Yeh	9

Proof. We first prove the lemma which shows that the coefficients of $p_m(z)$ can be expressed in terms of coefficients of $p(z)$. □

Lemma 3.5. *Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ is a monic polynomial, where $a_0, a_1, \ldots, a_{n-1}$ are complex numbers and α_i, $(i = 1, 2, \ldots, n)$ are the zeros of this polynomial. If $p_1(z) = z^n + a_{n-1}^{(1)} z^{n-1} + \cdots a_1^{(1)} z + a_0^{(1)}$ is the polynomial having α_i^2 $(i = 1, 2, \ldots, n)$ as zeros, then for $r = 1, 2, \ldots, n$:*

$$a_r^{(1)} = (-1)^{2n-r}\left(a_r^2 + 2\sum_{k=1}^{n-r}(-1)^k a_{r+k} a_{r-k}\right), \quad \text{where } a_n = 1,\ a_{n+k} = a_{n-k} = 0. \tag{3.18}$$

Proof. We have

$$\begin{aligned}
\det\left(z^2 I - C(p)^2\right) &= \det(zI - C(p)) \det(zI + C(p)) \\
&\Longrightarrow p_1\left(z^2\right) = p(z)p(-z) \\
&\Longrightarrow z^{2n} + a_{n-1}^{(1)} z^{2(n-1)} + \cdots a_1^{(1)} z^2 + a_0^{(1)} \\
&= (-1)^n \left(z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0\right) \\
&\quad \times \left(z^n - a_{n-1} z^{n-1} + \cdots + (-1)^{n-1} a_1 z + (-1)^n a_0\right).
\end{aligned} \tag{3.19}$$

Comparing the coefficient of z^{2r}, we get for $r = 1, 2, \ldots n$:

$$a_r^{(1)} = (-1)^{2n-r}\left(a_r^2 + 2\sum_{k=1}^{n-r}(-1)^k a_{r+k} a_{r-k}\right), \quad \text{where } a_n = 1,\ a_{n+k} = a_{n-k} = 0. \tag{3.20}$$

This completes the proof of lemma. □

The companion matrix of the monic polynomial $p_m(z)$ is

$$C(p_m) = \begin{pmatrix} -a_{n-1}^{(m)} & -a_{n-2}^{(m)} & \cdot & \cdot & -a_1^{(m)} & -a_0^{(m)} \\ 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 \end{pmatrix}. \tag{3.21}$$

We have

$$w(C(p_m)) \geq r_\sigma(C(p_m)) = r_\sigma(C(p))^{2^m}. \tag{3.22}$$

So

$$r_\sigma(C(p)) \leq (w(C(p_m)))^{1/2^m}. \tag{3.23}$$

Using Theorem 2.1, we get

$$w(C(p_m)) \leq \frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(\left|a_{n-1}^{(m)}\right| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \left(1 + \sqrt{\sum_{k=2}^{n}\left|a_{n-k}^{(m)}\right|^2}\right)^2}\right]. \tag{3.24}$$

Thus if λ is a zero of the polynomial $p(z)$, then

$$|\lambda| \leq \left(\frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(\left|a_{n-1}^{(m)}\right| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \left(1 + \sqrt{\sum_{k=2}^{n}\left|a_{n-k}^{(m)}\right|^2}\right)^2}\right]\right)^{1/2^m}. \tag{3.25}$$

This completes the proof of the theorem.

We next prove the theorem.

Theorem 3.6. *Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots a_1 z + a_0$ is a monic polynomial and α_i are the roots of this equation $i = 1, 2, \ldots, n$, where $a_0, a_1, \ldots, a_{n-1}$ are complex numbers with $|\alpha_1| > 1 > |\alpha_2| > \cdots > |\alpha_n|$. If the equation having roots $\alpha_i^{2^m}$ for $i = 1, 2, \ldots, n$ is $p_m(z) = z^n + a_{n-1}^{(m)}z^{n-1} + \cdots a_1^{(m)}z + a_0^{(m)}$, then there exists $m_0 \in \mathbb{N}$ such that*

(1) *$|a_{n-k}^{(m)}| \leq |a_{n-1}^{(m)}|$ whenever $m \geq m_0$ and for $k = 2, 3, \ldots, n$;*

(2) *$[w(C_m(p))]^{1/2^m}$ converges to $r_\sigma[C(p)]$.*

Proof. (1) We prove this for $k = 2$ and the rest are similar.

First observe that

$$\left|a_{n-1}^{(m)}\right| = \left|\sum_{k=1}^{n}\alpha_k^{2^m}\right| \geq |\alpha_1|^{2^m} - \sum_{k=2}^{n}|\alpha_k|^{2^m},$$
$$\left|a_{n-2}^{(m)}\right| = \left|\sum_{j\neq k=1}^{n}\alpha_j^{2^m}\alpha_k^{2^m}\right| \leq |\alpha_1|^{2^m}\left(\sum_{k=2}^{n}|\alpha_k|^{2^m}\right) + \sum_{j\neq k=2}^{n}\left|\alpha_j\right|^{2^m}|\alpha_k|^{2^m}. \tag{3.26}$$

Now in order to have

$$\left|a_{n-2}^{(m)}\right| \leq \left|a_{n-1}^{(m)}\right|. \tag{3.27}$$

We get

$$|\alpha_1|^{2^m} - \sum_{k=2}^{n}|\alpha_k|^{2^m} \geq |\alpha_1|^{2^m}\left(\sum_{k=2}^{n}|\alpha_k|^{2^m}\right) + \sum_{j\neq k=2}^{n}\left|\alpha_j\right|^{2^m}|\alpha_k|^{2^m}, \tag{3.28}$$

that is,

$$|\alpha_1|^{2^m} \geq \left(1+|\alpha_1|^{2^m}\right)\left[\sum_{k=2}^{n}|\alpha_k|^{2^m}\right] + \sum_{j\neq k=2}^{n}\left|\alpha_j\right|^{2^m}|\alpha_k|^{2^m}, \tag{3.29}$$

that is,

$$\frac{|\alpha_1|^{2^m}}{1+|\alpha_1|^{2^m}} \geq \frac{\sum_{k=2}^{n}|\alpha_k|^{2^m}}{1+|\alpha_1|^{2^m}} + \frac{\sum_{j\neq k=2}^{n}\left|\alpha_j\right|^{2^m}|\alpha_k|^{2^m}}{1+|\alpha_1|^{2^m}}. \tag{3.30}$$

Clearly, this inequality holds good as the left-hand side converges to 1, but the right-hand side converges to 0.

(2) We have

$$w(C_m(p)) \leq \frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + \cos\left(\frac{\pi}{n}\right) + \sqrt{\left(\left|a_{n-1}^{(m)}\right| - \cos\left(\frac{\pi}{n}\right)\right)^2 + \left(1+\sqrt{\sum_{k=2}^{n}\left|a_{n-k}^{(m)}\right|^2}\right)^2}\right], \tag{3.31}$$

that is,

$$w(C_m(p)) \leq \frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + 1 + \sqrt{\left(\left|a_{n-1}^{(m)}\right|^2 + (1+\sqrt{n-2}\left|a_{n-1}^{(m)}\right|\right)^2}\right], \tag{3.32}$$

that is,

$$w(C_m(p)) \leq \frac{1}{2}\left[\left|a_{n-1}^{(m)}\right| + \left|a_{n-1}^{(m)}\right| + \sqrt{\left(\left|a_{n-1}^{(m)}\right|^2 + (\left|a_{n-1}^{(m)}\right| + \sqrt{n-2}\left|a_{n-1}^{(m)}\right|\right)^2}\right]. \tag{3.33}$$

So we get

$$w(C_m(p)) \leq K\left|a_{n-1}^{(m)}\right| \leq K\left|\sum_{i=1}^{n}\alpha_i^{2^m}\right| \leq K|\alpha_1|^{2^m}\left(1 + \sum_{i=2}^{n}\left|\frac{\alpha_i}{\alpha_1}\right|^{2^m}\right). \tag{3.34}$$

Now

$$(r_\sigma(C(p)))^{2^m} = r_\sigma(C_m(p)) \leq w(C_m(p)) \leq K|\alpha_1|^{2^m}\left(1 + \sum_{i=2}^{n}\left|\frac{\alpha_i}{\alpha_1}\right|^{2^m}\right). \tag{3.35}$$

Therefore,

$$r_\sigma(C(p)) \leq [w(C_m(p))]^{1/2^m} \leq |\alpha_1|\left(K\left(1 + \sum_{i=2}^{n}\left|\frac{\alpha_i}{\alpha_1}\right|^{2^m}\right)\right)^{1/2^m}. \tag{3.36}$$

As the terms inside the bracket on the RHS converges to 1, we get the desired result.

This completes the proof of the theorem. □

Application. As an application we can exactly find the spectral radius of a given matrix. Consider a given matrix A of order n.

Step 1. We first find the characteristic polynomial $q(z) = z^n + b_{n-1}z^{n-1} + \cdots b_1 z + b_0$. Suppose $\alpha_i,\ i = 1, 2, \ldots, r$ are the distinct roots of $q(z) = 0$ with $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_r|$.

Step 2. Find $p(z) = q(z)/\gcd[q(z), q'(z)] = z^r + a_{r-1}z^{r-1} + \cdots a_1 z + a_0$. Then, roots of $q(z) = 0$ are $\alpha_i,\ i = 1, 2, \ldots, r$ without multiplicity. Let $p_m(z) = z^n + a_{n-1}^{(m)}z^{n-1} + \cdots a_1^{(m)}z + a_0^{(m)}$ be the polynomial having $\alpha_i^{2^m}$ for $i = 1, 2, \ldots, r$ as its zeros.

Step 3. Since $|\alpha_2| < |\alpha_1|$, taking $\epsilon < (1/4)(|\alpha_1| - |\alpha_2|)$, we can see that $|\alpha_2| < |\alpha_1 - 2\epsilon$. Again using a result of [9], we get $|a_{n-1}^{(m)}|^{1/2^m}$ converging to $|\alpha_1|$. So for this ϵ there exists an $m_0 \in \mathbb{N}$ such that $|\alpha_1| - \epsilon < |a_{n-1}^{(m)}|^{1/2^m} < |\alpha_1| + \epsilon$ for all $m \geq m_0$. Therefore,

$$|\alpha_2| + \epsilon < |\alpha_1| - \epsilon < \left|a_{n-1}^{(m)}\right|^{1/2^m} < |\alpha_1| + \epsilon \quad \forall m \geq m_0. \tag{3.37}$$

Table 3

No. of iterations	$a_4^{(m)}$	$a_3^{(m)}$	$a_2^{(m)}$	$a_1^{(m)}$	$a_0^{(m)}$	$r_\sigma C((q(z)))$
0	−1	0	0	0	-2^{-5}	2.85
1	−1	0	-2^{-4}	0	-2^{-10}	2.40
2	−1	-2^{-3}	-2^{-9}	-2^{-13}	-2^{-20}	2.20
3	−1.25	31×2^{-12}	13×2^{-19}	3×2^{-28}	-2^{-40}	2.11
4	1.547363281	0.000119	0	0	0	2.07
5	2.394094544	0	0	0	0	2.05

Step 4. Let $t = |a_{n-1}^{(m_0)}|^{1/2^{m_0}} - \epsilon$.

Find $s(z) = z^r + \sum_{k=1}^{r}(a_{r-k}/t^k)z^{r-k} = z^r + c_{r-1}z^{r-1} + \cdots c_1 z + c_0$. If the roots of $s(z) = 0$ are β_i, then $\beta_i = \alpha_i/t,\ i = 1, 2, \ldots, r$ and

$$|\beta_1| = \left|\frac{\alpha_1}{t}\right| > 1 > \left|\frac{\alpha_2}{t}\right| = |\beta_2| > |\beta_3| > \cdots > |\beta_r|. \tag{3.38}$$

Then, $s(z)$ satisfies all the criterion of Theorem 3.6.

Step 5. The required sequence is $x_m = t[w(C_m(s))]^{1/2^m}$, which converges to the spectral radius of matrix A.

Example 3.7. Consider the 5th-degree polynomial $q(z) = z^5 + 2z^4 + 1$.

By Rouche's theorem, it is easy to see that all the roots except one are enclosed by the simple closed curve $|z| = 2$.

Consider $s(z) = z^5 + z^4 + (1/2^5)$ and then iterating the coefficints of $Q(z)$ we get the following.

The highest absolute value of the zeros of the polynomial is 2.055 and by 5th iteration we get 2.05. Continuing the above process, we can find the highest absolute value of the zeros of the polynomial up to the desired degree of accuracy. The previous best result for this is known to be 2.414 given by Alpin [8]. The iterations are shown in Table 3.

Acknowledgments

The authors would like to thank the referees for their suggestions, one of the referees suggested the inclusion of Remark 2.2 following Theorem 2.1. The research of the first author is partially supported by PURSE-DST, Govt. of India and the research of the second author is supported by CSIR, India.

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