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Research Article

Iteration and Iterative Roots of Fractional Polynomial Function

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Iteration is involved in the fields of dynamical systems and numerical computation and so forth. The computation of iteration is difficult for general functions (even for some simple functions such as linear fractional functions). In this paper, we discuss fractional polynomial function and use the method of conjugate similitude to obtain its expression of general iterate of order n under two different conditions. Furthermore, we also give iterative roots of order n for the function under two different conditions.

1. Introduction

Iteration is a repetition of the same operation. Given a nonempty set X and a self-mapping $f : X \to X$, define $f^0(x) = x$, $f^1(x) = f(x)$, $f^n(x) = f \circ f^{n-1}(x)$, where $n \in \mathbb{Z}^+$ and $\circ$ denotes the composition of mappings. f^n is called the nth iterate of f, and n is the iterate index of f^n concerning f. Iteration is often observed in mathematics, science, engineering, and daily life, but the computation of iteration of some elementary functions is very complicated and sometimes rather difficult (see [1–7]), such as linear fractional functions $f(x) = (ax+b)/(x+c)$, where $a, b, c \in \mathbb{R}$, $ac - b \neq 0$. Using the numerical computation method, we only make some partitions on the defined interval of x to obtain pointwise data and approximately curves of f^n. Although computer algebra system such as Maple provided the symbol computational tool, we still need to calculate the nth iterate of f for a given n, and the expression of iteration is complicated even for $n = 12$ (see [8]). However, using the method of conjugate similar, we can effectively calculate its iteration of order n (see the following $(*)$). This example shows that computer is not universal, and we need to find good mathematical method.

Given mapping f and g, if there exists invertible mapping h such that $f = h^{-1} \circ g \circ h$, then f is conjugating to g. Obviously, if $f = h^{-1} \circ g \circ h$, then $f^n = h^{-1} \circ g^n \circ h$. We usually use this method to turn iteration of complicated function into iteration of simple function which is easy to get general iteration. We call it as the method of conjugation. For example, using the method of conjugation in reference [9], fractional linear function is conjugated to a linear function by conjugation function $h(x) = 1/(x-s)$, where s is a root of the equation $s^2 - (a-c)s - b = 0$. Thus, the nth iterate of the fractional linear function f is

$$f^n(x) = \begin{cases} s + \dfrac{(a-s)^n(x-s)}{((a-s)^n - (s+c)^n)x_0(x-s) + (s+c)^n}, & \left(\dfrac{a-c}{2}\right)^2 + b \neq 0, \\ s + \dfrac{(a-s)(x-s)}{nx + a - (n+1)s}, & \left(\dfrac{a-c}{2}\right)^2 + b = 0, \end{cases} \tag{$*$}$$

where $x_0 = 1/(a-c-2s)$. By the same method, in reference [10], Jin et al. discuss that the nth iterate of polynomial function

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m, \tag{1.1}$$

where $m \in \mathbb{Z},\ m \geq 2,\ a_i \in \mathbb{C},\ i = 0,1,\ldots,m$, under the conditions that $a_i = a_0C_m^i(a_1/ma_0)^i$, $(i = 2,3,\ldots,m-1)$, $a_m = (a_1/ma_0)[a_0(a_1/ma_0)^{m-1} - 1]$, and $a_0 \neq 0$, is

$$f^n(x) = a_0^{(m^n-1)/(m-1)}\left(x + \frac{a_1}{ma_0}\right)^{m^n} - \frac{a_1}{ma_0}, \tag{1.2}$$

where C_m^i denotes the number of combination, that is, $C_m^i = m!/i!(m-i)!$.

Given a nonempty set I and an integer $n > 0$, an iterative root of order n of a given self-mapping $f : I \to I$ is a self-mapping $\varphi : I \to I$ such that

$$\varphi^n(x) = f(x), \quad \forall\ x \in I,\ n \in \mathbb{Z}^+, \tag{1.3}$$

where φ^n denotes the nth iterate of φ, that is, $\varphi^n = \varphi \circ \varphi^{n-1}$. The problem of iterative roots of mapping is an important problem in the iteration theory (see [9, 11–13]). It was studied early from the 19th century, but great advances have been made since 1950s, most of which were given for monotone self-mappings on compact interval. For nonmonotonic cases, there are also some progress in references (see [14–19]).

In this paper, we study iteration and iterative roots of the fractional polynomial function

$$f(x) = \frac{a_kx^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1x + a_0}{b_kx^k + b_{k-1}x^{k-1} + b_{k-2}x^{k-2} + \cdots + b_1x + b_0}, \tag{1.4}$$

where $k \in \mathbb{Z}, k \geq 1, a_i, b_i \in \mathbb{R}, i = 0,1,\ldots,k$. It can be treated as a nonmonotonic mapping on $\mathbb{R}$. Using the method of conjugation, we get the expression of f^n and iterative roots of order n of f under some conditions.

2. Iteration of Fractional Polynomial Functions

Theorem 2.1. *Let either $a_0 \neq 0$ when k is odd or $a_0 > 0$ when k is even. Suppose that the fractional polynomial function f defined by* (1.4) *satisfies $a_i = C_k^i(\sqrt[k]{a_0})^{k-i}$, $(i = 0, 1, \ldots, k), b_j = -a_j/\sqrt[k]{a_0}$, $(j = 0, 1, \ldots, k-1)$, then*

$$f^n(x) = \frac{1}{-1/\sqrt[k]{a_0} - \left[b(-\sqrt[k]{a_0})^{-k}\right]^{(1-(-k)^n)/(1+k)}(-1/x - 1/\sqrt[k]{a_0})^{(-k)^n}}, \tag{2.1}$$

where $b = -1/\sqrt[k]{a_0} - b_k$, when $k = 1$, $\sqrt[k]{a_0} := a_0$; $n \in \mathbb{Z}^+$.

Proof. On the basis of what f satisfies, the fractional polynomial function f defined by (1.4) transforms into

$$\begin{aligned} &f(x) \\ &= \frac{x^k + C_k^{k-1}(\sqrt[k]{a_0})x^{k-1} + C_k^{k-2}(\sqrt[k]{a_0})^2 x^{k-2} + \cdots + C_k^1(\sqrt[k]{a_0})^{k-1}x + C_k^0(\sqrt[k]{a_0})^k}{-\mathcal{A}x^k - \mathcal{A}C_k^{k-1}(\sqrt[k]{a_0})x^{k-1} - \mathcal{A}C_k^{k-2}(\sqrt[k]{a_0})^2x^{k-2} - \cdots - \mathcal{A}C_k^1(\sqrt[k]{a_0})^{k-1}x - \mathcal{A}C_k^0(\sqrt[k]{a_0})^k - bx^k} \\ &= \frac{(x + \sqrt[k]{a_0})^k}{-\mathcal{A}(x + \sqrt[k]{a_0})^k - bx^k}, \end{aligned} \tag{2.2}$$

where $\mathcal{A}$ denotes $(1/\sqrt[k]{a_0})$. Set $h_1(x) = 1 - 1/x$, then the inverse of h_1 is $h_1^{-1}(x) = 1/(1-x)$, it follows that

$$\begin{aligned} h_1 \circ f \circ h_1^{-1}(x) &= h_1\left(f\left(h_1^{-1}(x)\right)\right) = h_1\left(f\left(\frac{1}{1-x}\right)\right) \\ &= h_1\left(\frac{(1 + \sqrt[k]{a_0} - \sqrt[k]{a_0}x)^k}{(-\sqrt[k]{a_0})^{-1}(1 + \sqrt[k]{a_0} - \sqrt[k]{a_0}x)^k - b}\right) \\ &= \frac{\sqrt[k]{a_0} + 1}{\sqrt[k]{a_0}} + \frac{b}{(-\sqrt[k]{a_0})^k}\left(x - \frac{1 + \sqrt[k]{a_0}}{\sqrt[k]{a_0}}\right)^{-k} := g(x). \end{aligned} \tag{2.3}$$

Set $h_2(x) = x - (1 + \sqrt[k]{a_0})/\sqrt[k]{a_0}$, then the inverse of h_2 is $h_2^{-1}(x) = x + (1 + \sqrt[k]{a_0})/\sqrt[k]{a_0}$, it follows that

$$\begin{aligned} h_2 \circ g \circ h_2^{-1}(x) &= h_2\left(g\left(h_2^{-1}(x)\right)\right) = h_2\left(g\left(x + \frac{1 + \sqrt[k]{a_0}}{\sqrt[k]{a_0}}\right)\right) \\ &= h_2\left(\frac{b}{(-\sqrt[k]{a_0})^k}x^{-k} + \frac{\sqrt[k]{a_0} + 1}{\sqrt[k]{a_0}}\right) = \frac{b}{(-\sqrt[k]{a_0})^k}x^{-k} := G(x). \end{aligned} \tag{2.4}$$

By induction, we obtain easily

$$G^n(x) = \left[\frac{b}{(-\sqrt[k]{a_0})^k}\right]^{(1-(-k)^n)/(1+k)} x^{(-k)^n}. \tag{2.5}$$

By (2.4) and (2.5), we have

$$g^n(x) = h_2^{-1} \circ G^n \circ h_2(x) = \left[\frac{b}{(-\sqrt[k]{a_0})^k}\right]^{(1-(-k)^n)/(1+k)} \left(x - \frac{1+\sqrt[k]{a_0}}{\sqrt[k]{a_0}}\right)^{(-k)^n} + \frac{1+\sqrt[k]{a_0}}{\sqrt[k]{a_0}}. \tag{2.6}$$

By (2.3) and (2.6), we get

$$f^n(x) = h_1^{-1} \circ g^n \circ h_1(x) = \frac{1}{-1/\sqrt[k]{a_0} - \left[b(-\sqrt[k]{a_0})^{-k}\right]^{(1-(-k)^n)/(1+k)} (-1/x - 1/\sqrt[k]{a_0})^{(-k)^n}}. \tag{2.7}$$

This completes the proof. □

Theorem 2.2. *Let either $b_0 \neq 0$ when k is odd or $b_0 > 0$ when k is even. Suppose that the fractional polynomial function f defined by* (1.4) *satisfies $b_i = C_k^i(\sqrt[k]{b_0})^{k-i}$ $(i = 0,1,2,\ldots,k)$, $a_j = (-\sqrt[k]{b_0})b_j$ $(j = 1,2,\ldots,k)$, then*

$$f^n(x) = d^{(1-(-k)^n)/(1+k)} \left(x + \sqrt[k]{b_0}\right)^{(-k)^n} - \sqrt[k]{b_0}, \tag{2.8}$$

where $d = a_0 + b_0\sqrt[k]{b_0}$; when $k = 1$, $\sqrt[k]{b_0} := b_0$, $n \in \mathbb{Z}^+$.

Proof. On the basis of what f satisfies, the fractional polynomial function f defined by (1.4) transforms into

$$\begin{aligned} f(x) &= \frac{-\sqrt[k]{b_0}x^k - \sqrt[k]{b_0}C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} - \sqrt[k]{b_0}C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} - \cdots - \sqrt[k]{b_0}C_k^1\left(\sqrt[k]{b_0}\right)^{k-1}x + d - b_0\sqrt[k]{b_0}}{x^k + C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} + C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} + \cdots + C_k^1\left(\sqrt[k]{b_0}\right)^{k-1}x + C_k^0\left(\sqrt[k]{b_0}\right)^k} \\ &= \frac{-\sqrt[k]{b_0}\left[x^k + C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} + C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} + \cdots + C_k^1\left(\sqrt[k]{b_0}\right)^{k-1}x + b_0\right] + d}{\left(x + \sqrt[k]{b_0}\right)^k} \\ &= \frac{-\sqrt[k]{b_0}\left(x + \sqrt[k]{b_0}\right)^k + d}{\left(x + \sqrt[k]{b_0}\right)^k}. \end{aligned} \tag{2.9}$$

Set $h(x) = x + \sqrt[k]{b_0}$, then the inverse of h is $h^{-1}(x) = x - \sqrt[k]{b_0}$, it follows that

$$\begin{aligned} h \circ f \circ h^{-1}(x) &= h\left(f\left(h^{-1}(x)\right)\right) = h\left(f\left(x - \sqrt[k]{b_0}\right)\right) = h\left(\frac{-\sqrt[k]{b_0}x^k + d}{x^k}\right) \\ &= h\left(-\sqrt[k]{b_0} + dx^{-k}\right) = dx^{-k} := g(x). \end{aligned} \tag{2.10}$$

By induction, we obtain easily

$$g^n(x) = d^{(1-(-k)^n)/(1+k)} x^{(-k)^n}. \tag{2.11}$$

By (2.10) and (2.11), we get

$$f^n(x) = h^{-1} \circ g^n \circ h(x) = d^{(1-(-k)^n)/(1+k)} \left(x + \sqrt[k]{b_0}\right)^{(-k)^n} - \sqrt[k]{b_0}. \tag{2.12}$$

This completes the proof. □

3. Iterative Roots of Fractional Polynomial Functions

We first give some useful lemmas.

Lemma 3.1. *If $f(x) = l(x+c)^k - c$, where $l, k, c \in \mathbb{R}$, $lk \neq 0$, then*

$$f^n(x) = \begin{cases} l^n x + l^n c - c, & k = 1, \\ l^{(1-k^n)/(1-k)} (x+c)^{k^n} - c, & k \neq 1, \end{cases} \tag{3.1}$$

where $n \in \mathbb{Z}^+$.

Proof. Set $h(x) = x + c$, then the inverse of h is $h^{-1}(x) = x - c$, it follows that

$$h \circ f \circ h^{-1}(x) = h\left(f\left(h^{-1}(x)\right)\right) = h(f(x-c)) = h\left(lx^k - c\right) = lx^k := g(x). \tag{3.2}$$

By induction, we obtain that the nth iterate of g is

$$g^n(x) = \begin{cases} l^n x, & k = 1, \\ l^{(1-k^n)/(1-k)} x^{k^n}, & k \neq 1. \end{cases} \tag{3.3}$$

By (3.2) and (3.3), we get

$$f^n(x) = h^{-1} \circ g^n \circ h(x) = \begin{cases} l^n x + l^n c - c, & k = 1, \\ l^{(1-k^n)/(1-k)} (x+c)^{k^n} - c, & k \neq 1. \end{cases} \tag{3.4}$$

This completes the proof. □

Lemma 3.2. *If $f(x) = x^k/(ax^k - b(x-c)^k)$, where $a, b, c, k \in \mathbb{R}$, $abck \neq 0, k \neq 1$, satisfies $ac = 1$, then*

$$f^n(x) = \frac{1}{1/c - \left(bc^k\right)^{(1-k^n)/(1-k)} (-1/x + 1/c)^{k^n}}, \tag{3.5}$$

where $n \in \mathbb{Z}^+$.

Proof. Set $h(x) = 1 - 1/x$, then the inverse of h is $h^{-1}(x) = 1/(1-x)$, it follows that

$$\begin{aligned} h \circ f \circ h^{-1}(x) &= h\left(f\left(h^{-1}(x)\right)\right) = h\left(f\left(\frac{1}{1-x}\right)\right) = h\left(\frac{1}{a - b(1-c+cx)^k}\right) \\ &= 1 - a + b(1-c+cx)^k = bc^k\left(x + \frac{1-c}{c}\right)^k + 1 - a := g(x). \end{aligned} \tag{3.6}$$

By Lemma 3.1, when $(1-c)/c = a-1$, that is, when $ac = 1$, we get that the nth iterate of g is

$$g^n(x) = \left(bc^k\right)^{(1-k^n)/(1-k)}\left(x + \frac{1-c}{c}\right)^{k^n} - \frac{1-c}{c}. \tag{3.7}$$

By (3.6) and (3.7), we get

$$f^n(x) = h^{-1} \circ g^n \circ h(x) = \frac{1}{1/c - \left(bc^k\right)^{(1-k^n)/(1-k)}(-1/x + 1/c)^{k^n}}. \tag{3.8}$$

This completes the proof. □

In Theorems 2.1 and 2.2, we get the expression of f^n of the fractional polynomial function (1.4) under different conditions, they can be treated as a mapping which involves parameter n, then n is extended from $\mathbb{Z}$ to $\mathbb{Q}$, we can obtain the iterative roots. For example, we can get the iterative roots of the fractional polynomial function (1.4) by extending the results of Theorems 2.1 and 2.2.

Theorem 3.3. *Suppose that the fractional polynomial function f defined by* (1.4) *satisfies conditions in Theorem 2.1, then f has iterative roots of any odd order n:*

$$\varphi(x) = \frac{1}{-1/\sqrt[k]{a_0} - \left[b(-\sqrt[k]{a_0})^{-k}\right]^{(1-(-k)^{1/n})/(1+k)}(-1/x - 1/\sqrt[k]{a_0})^{(-k)^{1/n}}}, \tag{3.9}$$

where $n \in \mathbb{Z}^+$.

Proof. In what follows, we only need to prove that $\varphi^n(x) = f(x)$ holds under the case that f satisfies the conditions in Theorem 2.1 and n is positive odd.

In fact, suppose that the fractional polynomial functions f defined by (1.4) satisfy the conditions in Theorem 2.1, then we have

$$\begin{aligned} &f(x) \\ &= \frac{x^k + C_k^{k-1}(\sqrt[k]{a_0})x^{k-1} + C_k^{k-2}(\sqrt[k]{a_0})^2x^{k-2} + \cdots + C_k^1(\sqrt[k]{a_0})^{k-1}x + C_k^0(\sqrt[k]{a_0})^k}{-\mathcal{A}x^k - \mathcal{A}C_k^{k-1}(\sqrt[k]{a_0})x^{k-1} - \mathcal{A}C_k^{k-2}(\sqrt[k]{a_0})^2x^{k-2} - \cdots - \mathcal{A}C_k^1(\sqrt[k]{a_0})^{k-1}x - \mathcal{A}C_k^0(\sqrt[k]{a_0})^k - bx^k} \\ &= \frac{(x + \sqrt[k]{a_0})^k}{-\mathcal{A}(x + \sqrt[k]{a_0})^k - bx^k}, \end{aligned} \tag{3.10}$$

where $\mathcal{A}$ denotes $(1/\sqrt[k]{a_0})$.

Because

$$\begin{aligned}\varphi(x) &= \frac{1}{\left(-1/\sqrt[k]{a_0}\right) - \left[b\left(-\sqrt[k]{a_0}\right)^{-k}\right]^{\left(1-(-k)^{1/n}\right)/(1+k)}\left(-1/x - 1/\sqrt[k]{a_0}\right)^{(-k)^{1/n}}} \\ &= \frac{x^{(-k)^{1/n}}}{-1/\sqrt[k]{a_0}x^{(-k)^{1/n}} - \left[b\left(-\sqrt[k]{a_0}\right)^{-k}\right]^{\left(1-(-k)^{1/n}\right)/(1+k)}\left(-\sqrt[k]{a_0}\right)^{-(-k)^{1/n}}\left(x+\sqrt[k]{a_0}\right)^{(-k)^{1/n}}},\end{aligned} \tag{3.11}$$

when n is positive odd, by Lemma 3.2, we have

$$\begin{aligned}&\varphi^n(x) \\ &= \frac{1}{-1/\sqrt[k]{a_0} - \left[\left(b\left(-\sqrt[k]{a_0}\right)^{-k}\right)^{(1+k^{1/n})/(1+k)}\left(-\sqrt[k]{a_0}\right)^{k^{1/n}}\left(-\sqrt[k]{a_0}\right)^{-k^{1/n}}\right]^{\left(1-(-k^{1/n})^n\right)/(1+k^{1/n})}\left(-1/x - 1/\sqrt[k]{a_0}\right)^{(-k^{1/n})^n}} \\ &= \frac{1}{-1/\sqrt[k]{a_0} - \left(b/\left(-\sqrt[k]{a_0}\right)^k\right)\left(-1/x - 1/\sqrt[k]{a_0}\right)^{-k}} = \frac{\left(x+\sqrt[k]{a_0}\right)^k}{-\left(1/\sqrt[k]{a_0}\right)\left(x+\sqrt[k]{a_0}\right)^k - bx^k} = f(x).\end{aligned} \tag{3.12}$$

Thus, f has iterative roots of any odd order n:

$$\varphi(x) = \frac{1}{-1/\sqrt[k]{a_0} - \left[b\left(-\sqrt[k]{a_0}\right)^{-k}\right]^{\left(1-(-k)^{1/n}\right)/(1+k)}\left(-1/x - 1/\sqrt[k]{a_0}\right)^{(-k)^{1/n}}}, \tag{3.13}$$

This completes the proof. □

Remark 3.4. When n is positive even, the above $\varphi(x)$ is not well defined, thus, under the condition in Theorem 3.3, f has no iterative roots of any order n in the form of $\varphi(x)$.

By Theorems 2.1 and 3.3, we are easy to get the following corollary.

Corollary 3.5. *Let $f(x) = (x + a_0)/(b_1 x - 1)$ and $a_0 \neq 0$, then*

(1) *the expression of f^n is*

$$f^n(x) = \frac{1}{-1/a_0 - \left[1/a_0^2 + b_1/a_0\right]^{\left(1-(-1)^n\right)/2}\left(-1/x - 1/a_0\right)^{(-1)^n}}, \quad \left(n \in \mathbb{Z}^+\right) \tag{3.14}$$

(2) *f has iterative roots f of any odd order n, that is, $f^n = f$, where $n = 2m - 1$, $m \in \mathbb{Z}^+$.*

Theorem 3.6. *Suppose that the fractional polynomial function f defined by* (1.4) *satisfies conditions in Theorem 2.2, then f has iterative roots of any odd order n:*

$$F(x) = d^{\left(1-(-k)^{1/n}\right)/(1+k)}\left(x + \sqrt[k]{b_0}\right)^{(-k)^{1/n}} - \sqrt[k]{b_0}, \tag{3.15}$$

where $n \in \mathbb{Z}^+$.

Proof. In what follows, we only need to prove that $F^n(x) = f(x)$ holds under the case that f satisfies the conditions in Theorem 2.2 and n is positive odd.

In fact, suppose that the fractional polynomial functions f defined by (1.4) satisfy the conditions in Theorem 2.2, then we have

$$\begin{aligned} f(x) &= \frac{-\sqrt[k]{b_0}x^k - \sqrt[k]{b_0}C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} - \sqrt[k]{b_0}C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} - \cdots - \sqrt[k]{b_0}C_k^1\left(\sqrt[k]{b_0}\right)^{k-1} x + d - b_0\sqrt[k]{b_0}}{x^k + C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} + C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} + \cdots + C_k^1\left(\sqrt[k]{b_0}\right)^{k-1} x + C_k^0\left(\sqrt[k]{b_0}\right)^k} \\ &= \frac{-\sqrt[k]{b_0}\left[x^k + C_k^{k-1}\left(\sqrt[k]{b_0}\right)x^{k-1} + C_k^{k-2}\left(\sqrt[k]{b_0}\right)^2 x^{k-2} + \cdots + C_k^1\left(\sqrt[k]{b_0}\right)^{k-1} x + b_0\right] + d}{\left(x + \sqrt[k]{b_0}\right)^k} \\ &= \frac{-\sqrt[k]{b_0}\left(x + \sqrt[k]{b_0}\right)^k + d}{\left(x + \sqrt[k]{b_0}\right)^k}. \end{aligned} \tag{3.16}$$

When n is positive odd, by Lemma 3.1, we have

$$\begin{aligned} F^n(x) &= \left[d^{(1-(-k)^{1/n})/(1+k)}\right]^{(1-[(-k)^{1/n}]^n)/(1-(-k)^{1/n})}\left(x + \sqrt[k]{b_0}\right)^{[(-k)^{1/n}]^n} - \sqrt[k]{b_0} \\ &= d\left(x + \sqrt[k]{b_0}\right)^{-k} - \sqrt[k]{b_0} = \frac{-\sqrt[k]{b_0}\left(x + \sqrt[k]{b_0}\right)^k + d}{\left(x + \sqrt[k]{b_0}\right)^k} = f(x). \end{aligned} \tag{3.17}$$

Thus, f has iterative roots of any odd order n:

$$F(x) = d^{(1-(-k)^{1/n})/(1+k)}\left(x + \sqrt[k]{b_0}\right)^{(-k)^{1/n}} - \sqrt[k]{b_0}. \tag{3.18}$$

This completes the proof. □

Remark 3.7. When n is positive even, the above $F(x)$ is not well defined, thus, under the conditions in Theorem 3.6, f has no iterative roots of any order n in the form of $F(x)$.

By Theorems 2.2 and 3.6, we are easy to get the following corollary.

Corollary 3.8. *Let $f(x) = (a_1x + a_0)/(x - a_1)$ and $a_0 + a_1^2 \neq 0$, then*

(1) *the expression of f^n is*

$$f^n(x) = \left(a_0 + a_1^2\right)^{(1-(-1)^n)/2}(x - a_1)^{(-1)^n} + a_1, \quad (n \in \mathbb{Z}^+) \tag{3.19}$$

(2) *f has iterative roots f of any odd order n, that is, $f^n = f$, where $n = 2m - 1$, $m \in \mathbb{Z}^+$.*

4. Examples

We demonstrate our theorems with the following examples.

Example 4.1. Consider the fractional polynomial function $f(x) = (x^3+6x^2+12x+8)/((9/2)x^3-3x^2-6x-4)$, by Theorem 2.1, we know $k=3$, $a_0=8$, $b=-5$, thus iteration of order n of the fractional polynomial functions f:

$$f^n(x) = \frac{1}{-1/2-(5/8)^{(1-(-3)^n)/4}(-1/2-1/x)^{(-3)^n}}. \tag{4.1}$$

By Theorem 3.3, we know f has iterative roots of any odd order n:

$$\varphi(x) = \frac{1}{-1/2-(5/8)^{(1-(-3)^{1/n})/4}(-1/2-1/x)^{(-3)^{1/n}}}, \tag{4.2}$$

where $n \in \mathbb{Z}^+$.

Example 4.2. Consider the fractional polynomial function $f(x) = (x^2+2x+1)/(-2x-1)$, by Theorem 2.1, we know $k=2$, $a_0=1$, $b=-1$, thus iteration of order n of the fractional polynomial functions f:

$$f^n(x) = \frac{1}{-1-(-1)^{(1-(-2)^n)/3}(-1-1/x)^{(-2)^n}}. \tag{4.3}$$

By Theorem 3.3, we know f has iterative roots of any odd order n:

$$\varphi(x) = \frac{1}{-1-(-1)^{(1-(-2)^{1/n})/3}(-1-1/x)^{(-2)^{1/n}}}, \tag{4.4}$$

where $n \in \mathbb{Z}^+$.

Example 4.3. Consider the fractional polynomial function $f(x) = (2x^3-12x^2+24x-13)/(x^3-6x^2+12x-8)$, by Theorem 2.2, we know $k=3$, $a_0=-13$, $a_3=2$, $d=3$, thus iteration of order n of the fractional polynomial functions f:

$$f^n(x) = 3^{(1-(-3)^n)/4}(x-2)^{(-3)^n}+2. \tag{4.5}$$

By Theorem 3.6, we know f has iterative roots of any odd order n:

$$F(x) = 3^{(1-(-3)^{1/n})/4}(x-2)^{(-3)^{1/n}}+2, \tag{4.6}$$

where $n \in \mathbb{Z}^+$.

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