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Research Article

On a Third-Order Three-Point Boundary Value Problem

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We consider a third-order three-point boundary value problem. We introduce a generalized polynomial growth condition to obtain the existence of a nontrivial solution by using Leray-Schauder nonlinear alternative, then we give an example to illustrate our results.

1. Introduction

In this work, we study the existence of nontrivial solution for the following third-order three point boundary value problem (BVP):

$$u'''(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \alpha u'(0), \qquad u(1) = \beta u'(\eta), \qquad u'(1) = 0, \tag{1.2}$$

where $\eta \in (0,1), \alpha, \beta \in \mathbb{R}, f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

The parameters α and β are arbitrary in $\mathbb{R}$ such that $1 + 2\alpha + 2\beta\eta - 2\beta \neq 0$. Our aim is to give new conditions on the nonlinearity of f, then using Leray-Schauder nonlinear alternative, we establish the existence of nontrivial solution. We only assume that $f(t,0) \neq 0$ and a generalized polynomial growth condition, that is, there exist two nonnegative functions $k, h \in L^1[0,1]$ such that

$$\left| f(t,x) \right| \leq k(t)|x|^p + h(t), \quad \forall (t,x) \in [0,1] \times \mathbb{R}, \tag{1.3}$$

where $p \in \mathbb{R}_+$, so our conditions are new and more general than [1].

Such problems arise in the study of the equilibrium states of a heated bar. Very recently, there have been several papers on third-order boundary value problems. Graef and Yang [2, 3], Guo et al. [4], Hopkins and Kosmatov [5], and Sun [6] have all considered third-order problems. Anderson [7] considered the three-point boundary value problem for (1.1) in the case $t_1 < t < t_2$ and the three-point conditions $u(t_1) = u'(t_2) = 0, \gamma u(t_3) + \delta u''(t_3) = 0$; using the Krasnoselskii and Leggett-Williams fixed-point theorems, the existence of solutions to the nonlinear problem is proved. Excellent surveys of theoretical results can be found in Agarwal [8], Agarwal et al. [9], and Ma [10]. More results can be found in [11–13].

This paper is organized as follows. First, we list some preliminary materials to be used later. Then in Section 3, we present and prove our main results which consist in existence theorems. We end our work with some illustrating examples.

2. Preliminary Lemmas

Let $E = (C[0,1], \mathbb{R})$, with supremum norm $\|y\| = \sup_{t\in[0,1]}|y(t)|$, for all $y \in E$. Now, we state two preliminary results.

Lemma 2.1. *Let $y \in E$. If $\zeta = 1 + 2\alpha + 2\beta\eta - 2\beta \neq 0$, then the three-point BVP*

$$\begin{gathered} u'''(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = \alpha u'(0), \qquad u(1) = \beta u'(\eta), \qquad u'(1) = 0 \end{gathered} \tag{2.1}$$

has a unique solution

$$\begin{aligned} u(t) = & -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{\beta}{\zeta}\left(t^2 - 2t - 2\alpha\right)\int_0^\eta (\eta - s)y(s)ds \\ & + \frac{1}{\zeta}\int_0^1 (1-s)\left(\frac{t^2}{2}(1 + s + 2\alpha - 2\beta) + (t+\alpha)(2\beta\eta - s)\right)y(s)ds. \end{aligned} \tag{2.2}$$

Proof. Integrating $u'''(t) = -y(t)$ over the interval $[0, t]$, we see that

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + \frac{1}{2}At^2 + Bt + C. \tag{2.3}$$

The constants A, B, and C are given by the three-point boundary conditions (1.2). □

We define the integral operator $T : E \to E$ by

$$\begin{aligned} Tu(t) = & -\frac{1}{2}\int_0^t (t-s)^2 f(s,u(s))ds + \frac{\beta}{\zeta}\left(t^2 - 2t - 2\alpha\right)\int_0^\eta (\eta - s) f(s,u(s))ds \\ & \cdot \frac{1}{\zeta}\int_0^1 (1-s)\left(\frac{t^2}{2}(1 + s + 2\alpha - 2\beta) + (t+\alpha)(2\beta\eta - s)\right)f(s,u(s))ds. \end{aligned} \tag{2.4}$$

By Lemma 2.1, the BVP (1.1)-(1.2) has a solution if and only if the operator T has a fixed point in E. By Ascoli-Arzela theorem, we prove that T is a completely continuous operator. Now we cite the Leray-Schauder as nonlinear alternative.

Lemma 2.2 (see [14]). *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. Let $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Main Results

In this section, we present and prove our main results.

Theorem 3.1. *It is assumed that $f(t,0) \neq 0$, $\zeta \neq 0$, $p \in \mathbb{R}_+$, and there exist two nonnegative functions $k, h \in L^1[0,1]$ such that*

$$\left|f(t,x)\right| \leq k(t)|x|^p + h(t), \quad \forall (t,x) \in [0,1] \times \mathbb{R}, \tag{3.1}$$

$$\left(1 + \left|\beta\frac{(1+2\alpha)}{\zeta}\right| + \left|\frac{1}{\zeta}\right|\left(2|1+\alpha|\left(\left|\beta\right|\eta + 1\right) + \left|\beta\right|\right)\right)\int_0^1 (1-s)k(s)ds < \frac{1}{2}, \tag{3.2}$$

$$\left(1 + \left|\beta\frac{(1+2\alpha)}{\zeta}\right| + \left|\frac{1}{\zeta}\right|\left(2|1+\alpha|\left(\left|\beta\right|\eta + 1\right) + \left|\beta\right|\right)\right)\int_0^1 (1-s)h(s)ds < \frac{1}{2}. \tag{3.3}$$

Then the BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C([0,1],\mathbb{R})$.

Proof. Setting

$$\begin{aligned}
M &= \left(1 + \left|\beta\frac{(1+2\alpha)}{\zeta}\right| + \left|\frac{1}{\zeta}\right|\left(2|1+\alpha|\left(\left|\beta\right|\eta + 1\right) + \left|\beta\right|\right)\right)\int_0^1 (1-s)k(s)ds, \\
N &= \left(1 + \left|\beta\frac{(1+2\alpha)}{\zeta}\right| + \left|\frac{1}{\zeta}\right|\left(2|1+\alpha|\left(\left|\beta\right|\eta + 1\right) + \left|\beta\right|\right)\right)\int_0^1 (1-s)h(s)ds,
\end{aligned} \tag{3.4}$$

by hypothesis (3.2), we have $M < 1/2$. Since $f(t,0) \neq 0$, then there exists an interval $[\sigma,\tau] \subset [0,1]$ such that $\min_{\sigma \leq t \leq r}|f(t,0)| > 0$, then $N > 0$. Let $m = M/N$, $\Omega = \{u \in C[0,1] : ||u|| < 1\}$. Assume that $u \in \partial\Omega, \lambda > 1$ such $Tu = \lambda u$, then

$$\lambda\|u\| = \|Tu\| = \max_{0 \leq t \leq 1}|Tu(t)| \leq M\|u\|^p + N, \tag{3.5}$$

as $||u|| = 1$, then

$$\lambda \leq M + N. \tag{3.6}$$

First, if $m \leq 1$, then $\lambda \leq 2N < 1$; hence, $\lambda < 1$, and this contradicts the fact that $\lambda > 1$. By Lemma 2.2, we deduce that T has a fixed point $u^* \in \overline{\Omega}$, and then the BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C([0,1],\mathbb{R})$.

Second, if $m \geq 1$, then $\lambda \leq 2M < 1$. By arguing as above, we complete the proof. □

Let us define the following notation:

$$a = \left(1 + \left|\beta \frac{(1+2\alpha)}{\zeta}\right| + \left|\frac{1}{\zeta}\right| \left(2|1+\alpha|\left(|\beta|\eta + 1\right) + |\beta|\right)\right). \tag{3.7}$$

Theorem 3.2. *Under the conditions of Theorem 3.1 ($p \in \mathbb{R}_+$) and if one of the following conditions is satisfied*

(1) *there exist $n > 1$ and $r > 1$ such that*

$$\left(\int_0^1 k^n(s)ds\right)^{1/n} < \frac{(1+q)^{1/q}}{2a} \left(\frac{1}{n} + \frac{1}{q} = 1\right), \tag{3.8}$$

$$\left(\int_0^1 h^r(s)ds\right)^{1/r} < \frac{(1+l)^{1/l}}{2a} \left(\frac{1}{r} + \frac{1}{l} = 1\right), \tag{3.9}$$

(2) *there exist two constants $\mu, \tau > -1$ such that*

$$k(s) < \frac{(\mu+1)(\mu+2)}{2a} s^{\mu}, \tag{3.10}$$

$$h(s) < \frac{(\tau+1)(\tau+2)}{2a} s^{\tau}, \tag{3.11}$$

$$\begin{gathered} \operatorname{meas}\left\{ s \in [0,1], k(s) < \frac{(\mu+1)(\mu+2)}{2a} s^{\mu} \right\} > 0, \\ \operatorname{meas}\left\{ s \in [0,1], h(s) < \frac{(\tau+1)(\tau+2)}{2a} s^{\tau} \right\} > 0, \end{gathered} \tag{3.12}$$

(3) *the functions $k(s)$ and $h(s)$ satisfy*

$$k(s) < \frac{1}{a}, \tag{3.13}$$

$$h(s) < \frac{1}{a}, \tag{3.14}$$

$$\begin{gathered} \operatorname{meas}\left\{ s \in [0,1], k(s) < \frac{1}{a} \right\} > 0, \\ \operatorname{meas}\left\{ s \in [0,1], h(s) < \frac{1}{a} \right\} > 0, \end{gathered} \tag{3.15}$$

(4) *the function* $f(t,x)$ *satisfies*

$$\omega_1 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} \frac{|f(t,x)|}{|x|^p} < \frac{1}{2a}, \tag{3.16}$$

$$\omega_2 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)| < \frac{1}{2a}. \tag{3.17}$$

Then the BVP (1.1)-(1.2) *has at least one nontrivial solution* $u^* \in C([0,1],\mathbb{R})$.

Proof. Let M and N be defined as in the proof of Theorem 3.1. To prove Theorem 3.2, we only need to prove that $M < 1/2$ and $N < 1/2$.

(1) By using Hölder inequality, we get

$$M \le a\left(\int_0^1 k^n(s)ds\right)^{1/n}\left(\int_0^1 (1-s)^q ds\right)^{1/q}. \tag{3.18}$$

Integrating, using (3.8), and remarking that $a > 1$, we arrive at $M < 1/2$.

Using Hölder inequality a second time, we get

$$N \le a\left(\int_0^1 h^r(s)ds\right)^{1/r}\left(\int_0^1 (1-s)^l ds\right)^{1/l}. \tag{3.19}$$

Integrating and then using (3.9), we arrive at $N < 1/2$.

(2) Taking into account (3.10), it yields

$$M < \frac{(\mu+1)(\mu+2)}{2a}\left(a\int_0^1 (1-s)s^\mu ds\right) = \frac{1}{2}. \tag{3.20}$$

On the other hand, using (3.11), we obtain

$$N < \frac{(\tau+1)(\tau+2)}{2a}\left(a\int_0^1 (1-s)s^\mu ds\right) = \frac{1}{2}. \tag{3.21}$$

(3) Using the same reasoning as in the proof of the second statement, we prove the third statement.

(4) From the condition $\omega_1 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)|/|x|^p$, we deduce that there exists $c_1 > 0$ such that for $|x| > c_1$, we get $|f(t,x)| \le (\omega_1 + \varepsilon)|x|^p$, for all $\varepsilon > 0$. Choosing $\varepsilon = \omega_1$, then $|f(t,x)| \le 2\omega_1|x|^p$. Now from the condition $\omega_2 = \lim_{|x|\to\infty} \sup \max_{t\in[0,1]} |f(t,x)|$, we deduce that there exists $c_2 > 0$ such that for $|x| > c_2$, we have

$$|f(t,x)| \le \omega_2 + \varepsilon, \quad \forall x \in \mathbb{R} \setminus\,]-c_2, c_2[, \tag{3.22}$$

choosing $\varepsilon = \omega_2$, then $|f(t,x)| \leq 2\omega_2$, for all $x \in \mathbb{R} \setminus]-c_2, c_2[$; consequently,

$$\left|f(t,x)\right| \leq 2\omega_1 |x|^p + 2\omega_2, \quad \forall x \in \mathbb{R} \setminus]-c, c[, \tag{3.23}$$

where $c = \max(c_1, c_2)$, setting $R = \max\{|f(t,x)| : (t,x) \in [0,1] \times (-c,c)\}$, then for all $(t,x) \in [0,1] \times \mathbb{R}$, we get $|f(t,x)| \leq k(t)|x|^p + h(t)$, where $k(t) = 2\omega_1$ and $h(t) = 2\omega_2$. Using (3.16), we obtain

$$k(t) = 2\omega_1 < \frac{1}{a}, \tag{3.24}$$

then from (3.13), we get $M < 1/2$. Using (3.14), we arrive at

$$h(t) = 2\omega_2 < \frac{1}{a}, \tag{3.25}$$

then $N < 1/2$. Now applying the third statement, we achieve the proof of Theorem 3.2. □

Example 3.3. Consider the three-point BVP,

$$\begin{gathered} u''' + \frac{1}{7}u^5\left(\frac{\sqrt[2]{t^3}}{2} + \cos t\right) + \frac{\arcsin t}{5} = 0, \quad 0 < t < 1, \\ u(0) = -\frac{1}{2}u'(0), \qquad u(1) = -\frac{1}{2}u'\left(\frac{1}{3}\right), \qquad u'(1) = 0. \end{gathered} \tag{3.26}$$

We have

$$f(t,x) = \frac{x^5}{7}\left(\frac{\sqrt[2]{t^3}}{2} + \cos t\right) + \frac{\arcsin t}{5}, \quad \alpha = \beta = -\frac{1}{2}, \eta = \frac{1}{3}, \zeta = \frac{2}{3}. \tag{3.27}$$

So,

$$\left|f(t,x)\right| \leq \frac{1}{7}\left(\frac{\sqrt[2]{t^3}}{2} + \cos t\right)|x|^5 + \frac{\arcsin t}{5} = k(t)|x|^5 + h(t). \tag{3.28}$$

Applying the first statement of Theorem 3.2 for $p = 5,\ n = r = 2$, to get

$$\begin{aligned}
\left(\int_0^1 k^2(s)ds\right)^{1/2} &= \left(\frac{1}{49}\int_0^1\left(\frac{\sqrt[2]{s^3}}{2}+\cos s\right)^2 ds\right)^{1/2} = 0.1488 \\
&< \frac{(1+q)^{1/q}}{2a} = 0.16496, \\
\left(\int_0^1 h^2(s)ds\right)^{1/2} &= \left(\frac{1}{25}\int_0^1 (\arcsin s)^2 ds\right)^{1/2} = 0.13673 \\
&< \frac{(1+l)^{1/l}}{2a} = 0.16496.
\end{aligned} \tag{3.29}$$

Then the BVP (3.26) has at least one nontrivial solution u^* in $C[0,1]$.

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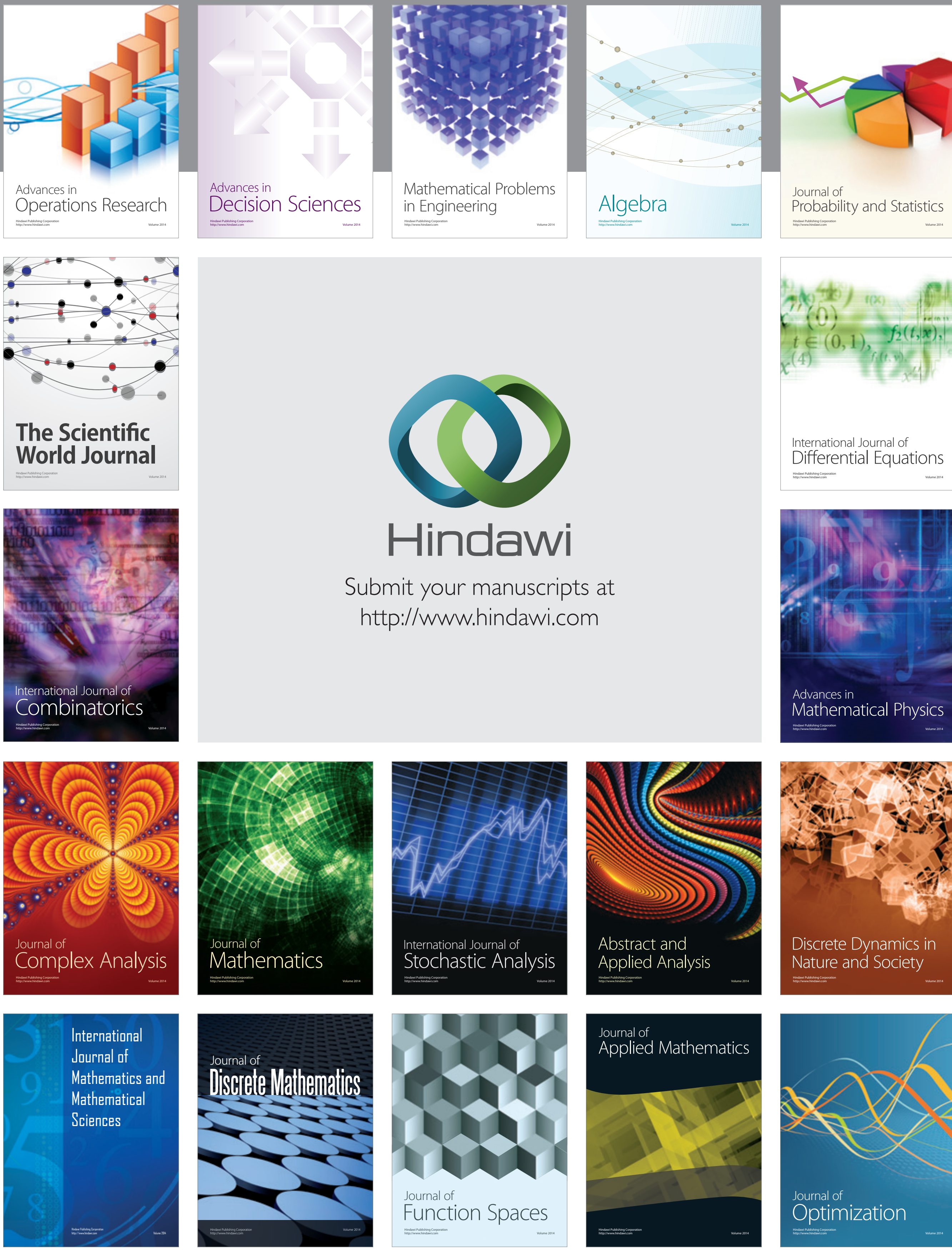

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