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Research Article

The Monotonicity Results for the Ratio of Certain Mixed Means and Their Applications

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We continue to adopt notations and methods used in the papers illustrated by Yang (2009, 2010) to investigate the monotonicity properties of the ratio of mixed two-parameter homogeneous means. As consequences of our results, the monotonicity properties of four ratios of mixed Stolarsky means are presented, which generalize certain known results, and some known and new inequalities of ratios of means are established.

1. Introduction

Since the Ky Fan [1] inequality was presented, inequalities of ratio of means have attracted attentions of many scholars. Some known results can be found in [2–14]. Research for the properties of ratio of bivariate means was also a hotspot at one time.

In this paper, we continue to adopt notations and methods used in the paper [13, 14] to investigate the monotonicity properties of the functions Q_{if} $(i = 1, 2, 3, 4)$ defined by

$$\begin{gathered}
Q_{1f}(p) := \frac{g_{1f}(p;a,b)}{g_{1f}(p;c,d)}, \\
Q_{2f}(p) := \frac{g_{2f}(p;a,b)}{g_{2f}(p;c,d)}, \\
Q_{3f}(p) := \frac{g_{3f}(p;a,b)}{g_{3f}(p;c,d)}, \\
Q_{4f}(p) := \frac{g_{4f}(p;a,b)}{g_{4f}(p;c,d)},
\end{gathered} \tag{1.1}$$

where

$$g_{1f}(p) = g_{1f}(p;a,b) := \sqrt{\mathcal{H}_f(p,q)\mathcal{H}_f(2k-p,q)}, \tag{1.2}$$

$$g_{2f}(p) = g_{2f}(p;a,b) := \sqrt{\mathcal{H}_f(p,p+m)\mathcal{H}_f(2k-p,2k-p+m)}, \tag{1.3}$$

$$g_{3f}(p) = g_{3f}(p;a,b) := \sqrt{\mathcal{H}_f(p,2m-p)\mathcal{H}_f(2k-p,2m-2k+p)}, \tag{1.4}$$

$$g_{4f}(p) = g_{4f}(p;a,b) := \sqrt{\mathcal{H}_f(pr,ps)\mathcal{H}_f((2k-p)r,(2k-p)s)}, \tag{1.5}$$

the $q,r,s,k,m \in \mathbb{R}$, $a,b,c,d \in \mathbb{R}_+$ with $b/a > d/c \geq 1$, $\mathcal{H}_f(p,q)$ is the so-called two-parameter homogeneous functions defined by [15, 16]. For conveniences, we record it as follows.

Definition 1.1. Let $f\colon \mathbb{R}_+^2 \setminus \{(x,x), x \in \mathbb{R}_+\} \to \mathbb{R}_+$ be a first-order homogeneous continuous function which has first partial derivatives. Then, $\mathcal{H}_f : \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a homogeneous function generated by f with parameters p and q if $\mathcal{H}_f$ is defined by for $a \neq b$

$$\begin{aligned} \mathcal{H}_f(p,q;a,b) &= \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{1(p-q)}, \quad \text{if } pq(p-q) \neq 0, \\ \mathcal{H}_f(p,p;a,b) &= \exp\left(\frac{a^p f_x(a^p,b^p)\ln a + b^p f_y(a^p,b^p)\ln b}{f(a^p,b^p)}\right), \quad \text{if } p = q \neq 0, \end{aligned} \tag{1.6}$$

where $f_x(x,y)$ and $f_y(x,y)$ denote first-order partial derivatives with respect to first and second component of $f(x,y)$, respectively.

If $\lim_{y\to x} f(x,y)$ exits and is positive for all $x \in \mathbb{R}_+$, then further define

$$\begin{aligned} \mathcal{H}_f(p,0;a,b) &= \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{1/p}, \quad \text{if } p \neq 0,\ q = 0, \\ \mathcal{H}_f(0,q;a,b) &= \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{1/q}, \quad \text{if } p = 0,\ q \neq 0, \\ \mathcal{H}_f(0,0;a,b) &= a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)}, \quad \text{if } p = q = 0, \end{aligned} \tag{1.7}$$

and $\mathcal{H}_f(p,q;a,a) = a$.

Remark 1.2. Witkowski [17] proved that if the function $(x,y) \to f(x,y)$ is a symmetric and first-order homogeneous function, then for all p,q $\mathcal{H}_f(p,q;a,b)$ is a mean of positive numbers a and b if and only if f is increasing in both variables on $\mathbb{R}_+$. In fact, it is easy to see that the condition "$f(x,y)$ is symmetric" can be removed.

If $\mathcal{H}_f(p,q;a,b)$ is a mean of positive numbers a and b, then it is called two-parameter homogeneous mean generated by f.

For simpleness, $\mathcal{H}_f(p,q;a,b)$ is also denoted by $\mathcal{H}_f(p,q)$ or $\mathcal{H}_f(a,b)$.

The two-parameter homogeneous function $\mathcal{H}_f(p,q;a,b)$ generated by f is very important because it can generates many well-known means. For example, substituting

$L = L(x,y) = (x-y)/(\ln x - \ln y)$ if $x, y > 0$ with $x \neq y$ and $L(x,x) = x$ for f yields Stolarsky means $\mathscr{H}_L(p,q;a,b) = S_{p,q}(a,b)$ defined by

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q}{p}\dfrac{a^p - b^p}{a^q - b^q}\right)^{1/(p-q)}, & \text{if } pq(p-q) \neq 0, \\ L^{1/p}(a^p, b^p), & \text{if } p \neq 0,\ q = 0, \\ L^{1/q}(a^q, b^q), & \text{if } q \neq 0, p = 0, \\ I^{1/p}(a^p, b^p), & \text{if } p = q \neq 0, \\ \sqrt{ab}, & \text{if } p = q = 0, \end{cases} \tag{1.8}$$

where $I(x,y) = e^{-1}(x^x/y^y)^{1/(x-y)}$ if $x, y > 0$, with $x \neq y$, and $I(x,x) = x$ is the identric (exponential) mean (see [18]). Substituting $A = A(x,y) = (x+y)/2$ for f yields Gini means $\mathscr{H}_A(p,q;a,b) = G_{p,q}(a,b)$ defined by

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{1/(p-q)}, & \text{if } p \neq q, \\ Z^{1/p}(a^p, b^p), & \text{if } p = q, \end{cases} \tag{1.9}$$

where $Z(a,b) = a^{a/(a+b)} b^{b/(a+b)}$ (see [19]).

As consequences of our results, the monotonicity properties of four ratios of mixed Stolarsky means are presented, which generalize certain known results, and some known and new inequalities of ratios of means are established.

2. Main Results and Proofs

In [15, 16, 20], two decision functions play an important role, that are,

$$\begin{gathered} \mathcal{I} = \mathcal{I}(x,y) = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}, \\ \mathcal{J} = \mathcal{J}(x,y) = (x-y)\frac{\partial(x\mathcal{I})}{\partial x} = (x-y)(x\mathcal{I})_x. \end{gathered} \tag{2.1}$$

In [14], it is important to another key decision function defined by

$$\mathcal{T}_3(x,y) := -xy(x\mathcal{I})_x \ln^3\left(\frac{x}{y}\right), \quad \text{where } \mathcal{I} = (\ln f)_{xy},\ x = a^t,\ y = b^t. \tag{2.2}$$

Note that the function T defined by

$$T(t) := \ln f(a^t, b^t), \quad t \neq 0 \tag{2.3}$$

has well properties (see [15, 16]). And it has shown in [14, (3.4)], [16, Lemma 4] the relation among $T'''(t)$, $\mathcal{J}(x,y)$ and $\mathcal{T}_3(x,y)$:

$$T'''(t) = t^{-3}\mathcal{T}_3(x,y), \quad \text{where } x = a^t,\ y = b^t, \tag{2.4}$$

$$T'''(t) = -Ct^{-3}\mathcal{J}(x,y), \quad \text{where } C = xy(x-y)^{-1}(\ln x - \ln y)^3 > 0. \tag{2.5}$$

Moreover, it has revealed in [14, (3.5)] that

$$\mathcal{T}_3(x,y) = \mathcal{T}_3\left(\frac{x}{y},1\right) = \mathcal{T}_3\left(1,\frac{y}{x}\right). \tag{2.6}$$

Now, we observe the monotonicities of ratio of certain mixed means defined by (1.1).

Theorem 2.1. *Suppose that $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, first-order homogenous, and three-time differentiable function, and $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$. Then, for any $a,b,c,d > 0$ with $b/a > d/c \geq 1$ and fixed $q \geq 0$, $k \geq 0$, but q,k are not equal to zero at the same time, Q_{1f} is strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$.*

The monotonicity of Q_{1f} is converse if $q \leq 0$, $k \leq 0$, but q,k are not equal to zero at the same time.

Proof. Since $f(x,y) > 0$ for $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, so $T'(t)$ is continuous on $[p,q]$ or $[q,p]$ for $p,q \in \mathbb{R}$, then (2.13) in [13] holds. Thus we have

$$\ln g_{1f}(p) = \frac{1}{2}\ln \mathcal{H}_f(p,q) + \frac{1}{2}\ln \mathcal{H}_f(2k-p,q) = \frac{1}{2}\int_0^1 T'(t_{11})dt + \frac{1}{2}\int_0^1 T'(t_{12})dt, \tag{2.7}$$

where

$$t_{12} = tp + (1-t)q, \qquad t_{11} = t(2k-p) + (1-t)q. \tag{2.8}$$

Partial derivative leads to

$$\begin{aligned}
(\ln g_{1f}(p))' &= \frac{1}{2}\int_0^1 tT''(t_{12})dt - \frac{1}{2}\int_0^1 tT''(t_{11})dt \\
&= \frac{1}{2}\int_0^1 tT''(|t_{12}|)dt - \frac{1}{2}\int_0^1 tT''(|t_{11}|)dt \quad (\text{by}[13], (2.7)) \\
&= \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v)dv\,dt,
\end{aligned} \tag{2.9}$$

and then

$$\begin{aligned}
(\ln Q_{1f}(p))' &= (\ln g_{1f}(p;a,b))' - (\ln g_{1f}(p;c,d))' \\
&= \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v)dv\,dt - \frac{1}{2}\int_0^1 t\int_{|t_{11}|}^{|t_{12}|} T'''(v;c,d)dv\,dt \\
&= \int_0^1 t(|t_{12}|-|t_{11}|)\frac{\int_{|t_{11}|}^{|t_{12}|}(T'''(v;a,b)-T'''(v;c,d))dv}{|t_{12}|-|t_{11}|}dt \\
&:= \int_0^1 t(|t_{12}|-|t_{11}|)h(|t_{11}|,|t_{12}|)dt,
\end{aligned} \tag{2.10}$$

where

$$h(x,y) := \begin{cases} \dfrac{\int_x^y (T'''(v;a,b)-T'''(v;c,d))dv}{y-x}, & \text{if } x \neq y, \\ T'''(x;a,b)-T'''(x;c,d), & \text{if } x = y. \end{cases} \tag{2.11}$$

Since $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$, (2.4) and (2.6) together with $b/a > d/c \geq 1$ yield

$$\begin{aligned}
T'''(v;a,b) - T'''(v;c,d) &= v^{-3}(\mathcal{T}_3(a^v,b^v) - \mathcal{T}_3(c^v,d^v)) \\
&= v^{-3}\left(\mathcal{T}_3\left(1,\left(\frac{b}{a}\right)^v\right) - \mathcal{T}_3\left(1,\left(\frac{d}{c}\right)^v\right)\right) > (<)0, \quad \text{for } v > 0,
\end{aligned} \tag{2.12}$$

and therefore $h(x,y) > (<)0$ for $x,y > 0$. Thus, in order to prove desired result, it suffices to determine the sign of $(|t_{12}|-|t_{11}|)$. In fact, if $q \geq 0$, $k \geq 0$, then for $t \in [0,1]$

$$|t_{12}|-|t_{11}| = \frac{t_{12}^2 - t_{11}^2}{|t_{12}|+|t_{11}|} = 4t\frac{q(1-t)+kt}{t_{12}+t_{11}}(p-k) = \begin{cases} > 0, & \text{if } p > k, \\ < 0, & \text{if } p < k. \end{cases} \tag{2.13}$$

It follows that

$$(\ln Q_{1f}(p))' = \begin{cases} > (<)0, & \text{if } p > k, \\ < (>)0, & \text{if } p < k. \end{cases} \tag{2.14}$$

Clearly, the monotonicity of Q_{1f} is converse if $q \leq 0$, $k \leq 0$.

This completes the proof. □

Theorem 2.2. *The conditions are the same as those of Theorem 2.1. Then, for any $a,b,c,d > 0$ with $b/a > d/c \geq 1$ and fixed m,k with $k \geq 0$, $k+m \geq 0$, but m,k are not equal to zero at the same time, Q_{2f} is strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$.*

The monotonicity of Q_{2f} is converse if $k \le 0$ and $k+m \le 0$, but m, k are not equal to zero at the same time.

Proof. By (2.13) in [13] we have

$$\begin{aligned}\ln g_{2f}(p) &= \frac{1}{2}\ln \mathscr{H}_f(p, p+m) + \frac{1}{2}\ln \mathscr{H}_f(2k-p, 2k-p+m)\\ &= \frac{1}{2}\int_0^1 T'(t_{22})dt + \frac{1}{2}\int_0^1 T'(t_{21})dt,\end{aligned} \tag{2.15}$$

where

$$t_{22} = tp + (1-t)(p+m), \qquad t_{21} = t(2k-p) + (1-t)(2k-p+m). \tag{2.16}$$

Direct calculation leads to

$$(\ln g_{2f}(p))' = \frac{1}{2}\int_0^1 T''(t_{22})dt - \frac{1}{2}\int_0^1 T''(t_{21})dt = \frac{1}{2}\int_0^1\int_{|t_{21}|}^{|t_{22}|} T'''(v)dv\,dt, \tag{2.17}$$

and then

$$\begin{aligned}(\ln Q_{2f}(p))' &= (\ln g_{2f}(p; a, b))' - (\ln g_{2f}(p; c, d))'\\ &= \frac{1}{2}\int_0^1\int_{|t_{21}|}^{|t_{22}|} T'''(v; a, b)dv\,dt - \frac{1}{2}\int_0^1\int_{|t_{21}|}^{|t_{22}|} T'''(v; c, d)dv\,dt\\ &= \frac{1}{2}\int_0^1 (|t_{22}| - |t_{21}|)h(|t_{21}|, |t_{22}|)dt,\end{aligned} \tag{2.18}$$

where $h(x, y)$ is defined by (2.11). As shown previously, $h(x, y) > (<)0$ for $x, y > 0$ if $\mathcal{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$; it remains to determine the sign of $(|t_{22}| - |t_{21}|)$. It is easy to verify that if $k \ge 0$ and $k + m \ge 0$, then

$$|t_{22}| - |t_{21}| = \frac{t_{22}^2 - t_{21}^2}{|t_{22}| + |t_{21}|} = 4\frac{k + m(1-t)}{|t_{22}| + |t_{21}|}(p-k) = \begin{cases} > 0, & \text{if } p > k,\\ < 0, & \text{if } p < k.\end{cases} \tag{2.19}$$

Thus, we have

$$(\ln Q_{2f}(p))' = \begin{cases} > (<)0, & \text{if } p > k,\\ < (>)0, & \text{if } p < k.\end{cases} \tag{2.20}$$

Clearly, the monotonicity of Q_{2f} is converse if $k \le 0$ and $k + m \le 0$.

The proof ends. □

Theorem 2.3. *The conditions are the same as those of Theorem 2.1. Then, for any $a, b, c, d > 0$ with $b/a > d/c \geq 1$ and fixed $m > 0$, $0 \leq k \leq 2m$, Q_{3f} is strictly increasing (decreasing) in p on (k, ∞) and decreasing (increasing) on $(-\infty, k)$.*

The monotonicity of Q_{2f} is converse if $m < 0$, $2m \leq k \leq 0$.

Proof. From (2.13) in [13], it is derived that

$$\ln g_{3f}(p) = \frac{1}{2}\ln \mathscr{H}_f(p, 2m-p) + \frac{1}{2}\ln \mathscr{H}_f(2k-p, 2m-2k+p)$$
$$= \frac{1}{2}\int_0^1 T'(t_{32})dt + \frac{1}{2}\int_0^1 T'(t_{31})dt, \tag{2.21}$$

where

$$t_{32} = \bigl(tp + (1-t)(2m-p)\bigr), \qquad t_{31} = \bigl(t(2k-p) + (1-t)(2m-2k+p)\bigr). \tag{2.22}$$

Simple calculation yields

$$(\ln g_{3f}(p))' = \frac{1}{2}\int_0^1 (2t-1)(T''(t_{32}) - T''(t_{31}))dt = \frac{1}{2}\int_0^1 (2t-1)\int_{|t_{31}|}^{|t_{32}|} T'''(v; a, b)dv\, dt. \tag{2.23}$$

Hence,

$$(\ln Q_{3f}(p))' = (\ln g_{3f}(p; a, b))' - (\ln g_{3f}(p; c, d))'$$
$$= \frac{1}{2}\int_0^1 (2t-1)\int_{|t_{31}|}^{|t_{32}|} (T'''(v; a, b) - T'''(v; c, d))dv\, dt \tag{2.24}$$
$$= \frac{1}{2}\int_0^1 (2t-1)(|t_{32}| - |t_{31}|)h(|t_{31}|, |t_{32}|)dt,$$

where $h(x, y)$ is defined by (2.11). It has shown that $h(x, y) > (<)0$ for $x, y > 0$ if $\mathfrak{T}_3(1, u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$, and we have also to check the sign of $(2t-1)(|t_{32}| - |t_{31}|)$. Easy calculation reveals that if $m > 0$, $0 \leq k \leq 2m$, then

$$(2t-1)(|t_{32}| - |t_{31}|) = (2t-1)\frac{(t_{32}^2 - t_{31}^2)}{|t_{32}| + |t_{31}|}$$
$$= 4(2t-1)^2 \frac{tk + (1-t)(2m-k)}{|t_{32}| + |t_{31}|}(p-k) \tag{2.25}$$
$$= \begin{cases} > 0, & \text{if } p > k, \\ < 0, & \text{if } p < k, \end{cases}$$

which yields

$$(\ln Q_{3f}(p))' = \begin{cases} > (<)0, & \text{if } p > k, \\ < (>)0, & \text{if } p < k. \end{cases} \tag{2.26}$$

It is evident that the monotonicity of Q_{3f} is converse if $m < 0, 2m \leq k \leq 0$.

Thus the proof is complete. □

Theorem 2.4. *The conditions are the same as those of Theorem 2.1. Then, for any $a, b, c, d > 0$ with $b/a > d/c \geq 1$ and fixed $k, r, s \in \mathbb{R}$ with $r + s \neq 0$, Q_{4f} is strictly increasing (decreasing) in p on (k, ∞) and decreasing (increasing) on $(-\infty, k)$ if $k(r + s) > 0$.*

The monotonicity of Q_{4f} is converse if $k(r + s) < 0$.

Proof. By (2.13) in [13], $\ln \mathscr{H}_f(pr, ps)$ can be expressed in integral form

$$\ln \mathscr{H}_f(pr, ps) = \begin{cases} \dfrac{1}{r - s} \int_s^r T'(pt)dt, & \text{if } r \neq s, \\ T'(pr), & \text{if } r = s. \end{cases} \tag{2.27}$$

The case $r = s \neq 0$ has no interest since it can come down to the case of $m = 0$ in Theorem 2.2. Therefore, we may assume that $r \neq s$. We have

$$\begin{aligned} \ln g_{4f}(p) &= \ln \sqrt{\mathscr{H}_f(pr, ps)\mathscr{H}_f((2k - p)r, (2k - p)s)} \\ &= \frac{1}{2}\frac{1}{r - s}\int_s^r T'(pt)dt + \frac{1}{2}\frac{1}{r - s}\int_s^r T'((2k - p)t)dt, \end{aligned} \tag{2.28}$$

and then

$$\begin{aligned} (\ln g_{4f}(p))' &= \frac{1}{2}\frac{1}{r - s}\int_s^r tT''(pt)dt - \frac{1}{2}\frac{1}{r - s}\int_s^r tT''((2k - p)t)dt \\ &= \frac{1}{2}\frac{1}{r - s}\int_s^r t(T''(pt) - T''((2k - p)t)). \end{aligned} \tag{2.29}$$

Note that $T''(t)$ is even (see [13, (2.7)]) and so $t(T''(pt) - T''((2k - p)t))$ is odd, then make use of Lemma 3.3 in [13], $(\ln g_{4f}(p))'$ can be expressed as

$$\begin{aligned} (\ln g_{4f}(p))' &= \frac{1}{2}\frac{r + s}{|r| - |s|}\int_{|s|}^{|r|} t(T''(|pt|) - T''(|(2k - p)t|))dt \\ &= \frac{1}{2}\frac{r + s}{|r| - |s|}\int_{|s|}^{|r|} t\int_{|t_{41}|}^{|t_{42}|} T'''(v)dv\, dt, \end{aligned} \tag{2.30}$$

where

$$t_{42} = pt, \qquad t_{41} = (2k - p)t. \tag{2.31}$$

Hence,

$$\begin{aligned}(\ln Q_{4f}(p))' &= (\ln g_{4f}(p;a,b))' - (\ln g_{4f}(p;c,d))' \\ &= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t\int_{|t_{41}|}^{|t_{42}|}(T'''(v;a,b) - T'''(v;c,d))\,dv\,dt \\ &= \frac{1}{2}\frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t(|t_{42}| - |t_{41}|)h(|t_{41}|,|t_{42}|)dt,\end{aligned} \tag{2.32}$$

where $h(x,y)$ is defined by (2.11). We have shown that $h(x,y) > (<)0$ for $x,y > 0$ if $\mathcal{T}_3(1,u)$ strictly increase (decrease) with $u > 1$ and decrease (increase) with $0 < u < 1$, and we also have

$$\operatorname{sgn}(|t_{42}| - |t_{41}|) = \operatorname{sgn}\left(t_{42}^2 - t_{41}^2\right) = \operatorname{sgn}(k)\operatorname{sgn}(p-k). \tag{2.33}$$

It follows that

$$\begin{aligned}\operatorname{sgn} Q'_{4f}(p) &= \operatorname{sgn}(r+s)\operatorname{sgn}(k)\operatorname{sgn}(p-k)\operatorname{sgn} h(|t_{41}|,|t_{42}|) \\ &= \begin{cases} > (<)0, & \text{if } k(r+s) > 0,\ p > k, \\ < (>)0, & \text{if } k(r+s) > 0,\ p < k, \\ < (>)0, & \text{if } k(r+s) < 0,\ p > k, \\ > (<)0, & \text{if } k(r+s) < 0,\ p < k. \end{cases}\end{aligned} \tag{2.34}$$

This proof is accomplished. □

3. Applications

As shown previously, $S_{p,q}(a,b) = \mathcal{H}_L(p,q;a,b)$, where $L = L(x,y)$ is the logarithmic mean. Also, it has been proven in [14] that $\mathcal{T}'_3(1,u) < 0$ if $u > 1$ and $\mathcal{T}'_3(1,u) > 0$ if $0 < u < 1$. From the applications of Theorems 2.1–2.4, we have the following.

Corollary 3.1. *Let $a,b,c,d > 0$ with $b/a > d/c \geq 1$. Then, the following four functions are all strictly decreasing (increasing) on (k,∞) and increasing (decreasing) on $(-\infty,k)$:*

(i) *Q_{1L} is defined by*

$$Q_{1L}(p) = \frac{\sqrt{S_{p,q}(a,b)S_{2k-p,q}(a,b)}}{\sqrt{S_{p,q}(c,d)S_{2k-p,q}(c,d)}}, \tag{3.1}$$

for fixed $q \geq (\leq)0$, $k \geq (\leq)0$, but q,k are not equal to zero at the same time,

(ii) Q_{2L} *is defined by*

$$Q_{2L}(p) = \frac{\sqrt{S_{p,p+m}(a,b)S_{2k-p,2k-p+m}(a,b)}}{\sqrt{S_{p,p+m}(c,d)S_{2k-p,2k-p+m}(c,d)}}, \tag{3.2}$$

for fixed m, k *with* $k \geq (\leq)0$ *and* $k + m \geq (\leq)0$, *but* m, k *are not equal to zero at the same time,*

(iii) Q_{3L} *is defined by*

$$Q_{3L}(p) = \frac{\sqrt{S_{p,2m-p}(a,b)S_{2k-p,2m-2k+p}(a,b)}}{\sqrt{S_{p,2m-p}(c,d)S_{2k-p,2m-2k+p}(c,d)}}, \tag{3.3}$$

for fixed $m > (<)0$, $k \in [0, 2m]$ $([2m, 0])$.

(iv) Q_{4L} *is defined by*

$$Q_{4L}(p) = \frac{\sqrt{S_{pr,ps}(a,b)S_{(2k-p)r,(2k-p)s}(a,b)}}{\sqrt{S_{pr,ps}(c,d)S_{(2k-p)r,(2k-p)s}(c,d)}}, \tag{3.4}$$

for fixed $k, r, s \in \mathbb{R}$ *with* $k(r+s) > (<)0$.

Remark 3.2. Letting in the first result of Corollary 3.1, $q = k$ yields Theorem 3.4 in [13] since $\sqrt{S_{p,k}S_{2k-p,k}} = S_{p,2k-p}$. Letting $q = 1$, $k = 0$ yields

$$\frac{G(a,b)}{G(c,d)} = Q_{1L}(\infty) < \frac{\sqrt{S_{p,1}(a,b)S_{-p,1}(a,b)}}{\sqrt{S_{p,1}(c,d)S_{-p,1}(c,d)}} < Q_{1L}(0) = \frac{L(a,b)}{L(c,d)}. \tag{3.5}$$

Inequalities (3.5) in the case of $d = c$ were proved by Alzer in [21]. By letting $q = 1$, $k = 1/2$ from $Q_{1L}(1/2) > Q_{1L}(1) > Q_{1L}(2)$, we have

$$\frac{A(a,b)+G(a,b)}{A(c,d)+G(c,d)} > \frac{\sqrt{L(a,b)I(a,b)}}{\sqrt{L(c,d)I(c,d)}} > \frac{\sqrt{A(a,b)G(a,b)}}{\sqrt{A(c,d)G(c,d)}}. \tag{3.6}$$

Inequalities (3.6) in the case of $d = c$ are due to Alzer [22].

Remark 3.3. Letting in the second result of Corollary 3.1, $m = 1$, $k = 0$ yields Cheung and Qi's result (see [23, Theorem 2]). And we have

$$\frac{G(a,b)}{G(c,d)} = Q_{2L}(\infty) < \frac{\sqrt{S_{p,p+1}(a,b)S_{-p,-p+1}(a,b)}}{\sqrt{S_{p,p+1}(c,d)S_{-p,-p+1}(c,d)}} < Q_{2L}(0) = \frac{L(a,b)}{L(c,d)}. \tag{3.7}$$

When $d = c$, inequalities (3.7) are changed as Alzer's ones given in [24].

Remark 3.4. In the third result of Corollary 3.1, letting $k = m$ also leads to Theorem 3.4 in [13]. Put $m = 1/2$, $k = 1/4$. Then from $Q_{3L}(1/4) > Q_{3L}(1/2)$, we obtain a new inequality

$$\frac{He_{1/2}(a,b)}{He_{1/2}(c,d)} > \frac{\sqrt{L(a,b)I_{1/2}(a,b)}}{\sqrt{L(c,d)I_{1/2}(c,d)}}. \tag{3.8}$$

Putting $m = 1/2$, $k = 1/3$ leads to another new inequality

$$\frac{A_{1/3}(a,b)}{A_{1/3}(c,d)} > \frac{\sqrt{S_{1/6,5/6}(a,b)I_{1/2}(a,b)}}{\sqrt{S_{1/6,5/6}(c,d)I_{1/2}(c,d)}}. \tag{3.9}$$

Remark 3.5. Letting in the third result of Corollary 3.1, $k = 1/2$ and $(r,s) = (1,0), (1,1), (2,1)$, and we deduce that all the following three functions

$$p \longrightarrow \frac{\sqrt{L_p(a,b)L_{1-p}(a,b)}}{\sqrt{L_p(c,d)L_{1-p}(c,d)}}, \qquad p \longrightarrow \frac{\sqrt{I_p(a,b)I_{1-p}(a,b)}}{\sqrt{I_p(c,d)I_{1-p}(c,d)}}, \qquad p \longrightarrow \frac{\sqrt{A_p(a,b)A_{1-p}(a,b)}}{\sqrt{A_p(c,d)A_{1-p}(c,d)}}, \tag{3.10}$$

are strictly decreasing on $(1/2,\infty)$ and increasing on $(-\infty,1/2)$, where $L_p = L^{1/p}(a^p,b^p)$, $I_p = I^{1/p}(a^p,b^p)$, and $A_p = A^{1/p}(a^p,b^p)$ are the p-order logarithmic, identric (exponential), and power mean, respectively, particularly, so are the functions $\sqrt{L_pL_{1-p}}$, $\sqrt{I_pI_{1-p}}$, $\sqrt{A_pA_{1-p}}$.

4. Other Results

Let $d = c$ in Theorems 2.1–2.4. Then, $\mathcal{H}_f(p,q;c,d) = c$ and $T'''(t;c,c) = 0$. From the their proofs, it is seen that the condition "$\mathcal{T}_3(1,u)$ strictly increases (decreases) with $u > 1$ and decreases (increases) with $0 < u < 1$" can be reduce to "$T'''(v) > (<)0$ for $v > 0$", which is equivalent with $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$, by (2.4). Thus, we obtain critical theorems for the monotonicities of g_{if}, $i = 1-4$, defined as (1.2)–(1.5).

Theorem 4.1. *Suppose that $f: \mathbb{R}_+\times\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a symmetric, first-order homogenous, and three-time differentiable function and $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$. Then, for $a,b > 0$ with $a \neq b$, the following four functions are strictly increasing (decreasing) in p on (k,∞) and decreasing (increasing) on $(-\infty,k)$:*

(i) g_{1f} *is defined by* (1.2), *for fixed $q,k \geq 0$, but q,k are not equal to zero at the same time;*

(ii) g_{2f} *is defined by* (1.3), *for fixed m,k with $k \geq 0$ and $k+m \geq 0$, but m,k are not equal to zero at the same time;*

(iii) g_{3f} *is defined by* (1.4), *for fixed $m > 0$ and $0 \leq k \leq 2m$;*

(iv) g_{4f} *is defined by* (1.5), *for fixed $k,r,s \in \mathbb{R}$ with $k(r+s) > 0$.*

If f is defined on $\mathbb{R}_+^2 \setminus \{(x,x), x \in \mathbb{R}_+\}$, then $T'(t)$ may be not continuous at $t = 0$, and (2.13) in [13] may not hold for $p,q \in \mathbb{R}$ but must be hold for $p,q \in \mathbb{R}_+$. And then, we easily derive the following from the proofs of Theorems 2.1–2.4.

Theorem 4.2. *Suppose that $f: \mathbb{R}_+^2 \setminus \{(x,x), x \in \mathbb{R}_+\} \to \mathbb{R}_+$ is a symmetric, first-order homogenous and three-time differentiable function and $\mathcal{Q} = (x-y)(x\mathcal{I})_x < (>)0$, where $\mathcal{I} = (\ln f)_{xy}$. Then for $a, b > 0$ with $a \neq b$ the following four functions are strictly increasing (decreasing) in p on $(k, 2k)$ and decreasing (increasing) on $(0, k)$:*

(i) *g_{1f} is defined by (1.2), for fixed $q, k > 0$;*

(ii) *g_{2f} is defined by (1.3), for fixed m, k with $k > 0$ and $k + m > 0$;*

(iii) *g_{3f} is defined by (1.4), for fixed $m > 0$ and $0 \leq k \leq 2m$;*

(iv) *g_{4f} is defined by (1.5), for fixed $k, r, s > 0$.*

If we substitute L, A, and I for f, where L, A, and I denote the logarithmic, arithmetic, and identric (exponential) mean, respectively, then from Theorem 4.1, we will deduce some known and new inequalities for means. Similarly, letting in Theorem 4.2 $f(x,y) = D(x,y) = |x-y|$, $K(x,y) = (x+y)|\ln(x/y)|$, where $x, y > 0$ with $x \neq y$, we will obtain certain companion ones of those known and new ones. Here no longer list them.

Disclosure

This paper is in final form and no version of it will be submitted for publication elsewhere.

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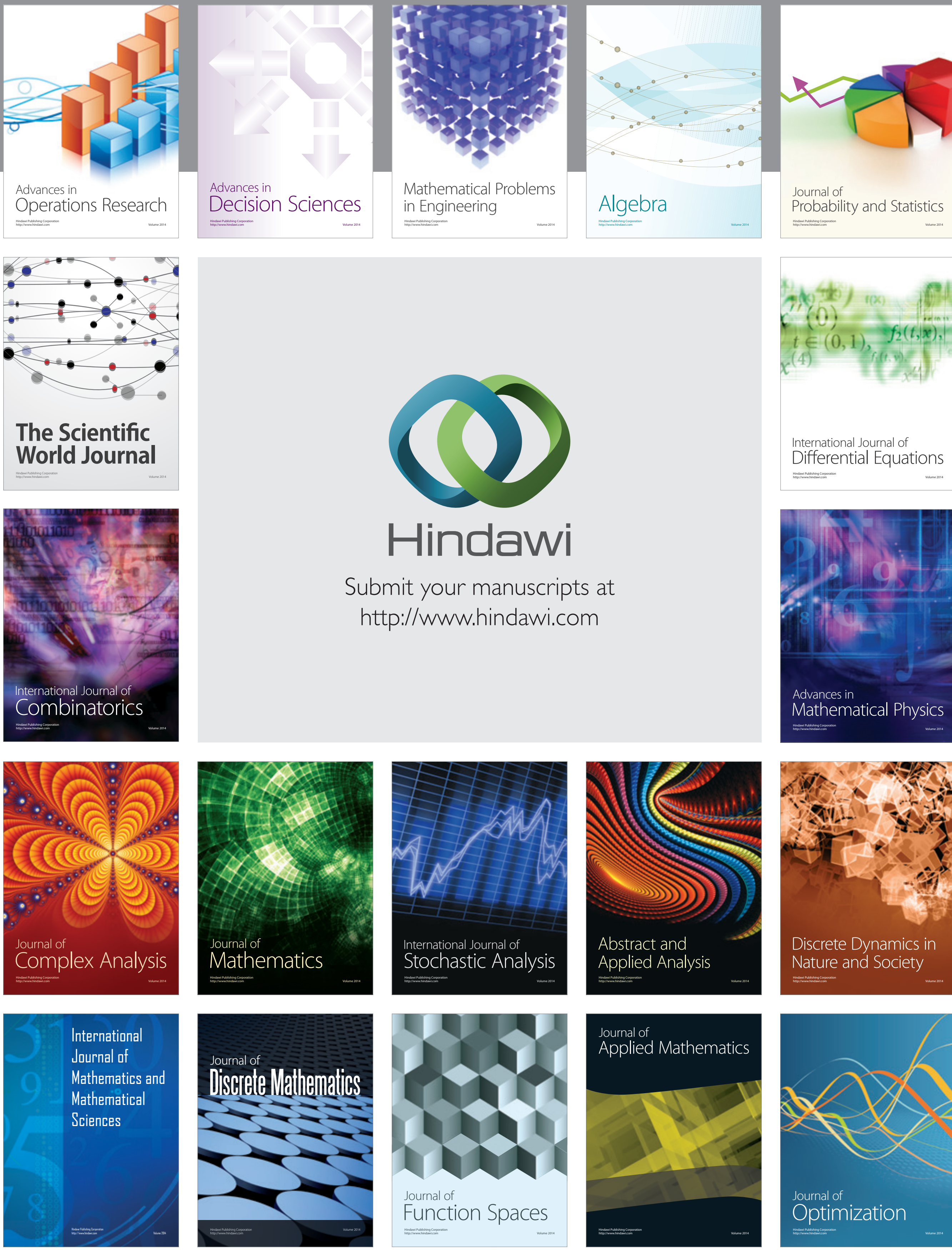
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