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Research Article

Convex Combinations of Minimal Graphs

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Given a collection of minimal graphs, $M_1, M_2, \ldots, M_n$, with isothermal parametrizations in terms of the Gauss map and height differential, we give sufficient conditions on $M_1, M_2, \ldots, M_n$ so that a convex combination of them will be a minimal graph. We will then provide two examples, taking a convex combination of Scherk's doubly periodic surface with the catenoid and Enneper's surface, respectively.

1. Introduction

Consider a surface M in $\mathbb{R}^3$.

Definition 1.1. The *normal curvature* at a point $p \in M$ in the $\mathbf{w}$ direction is

$$k(\mathbf{w}) = \alpha'' \cdot \mathbf{n}, \tag{1.1}$$

where $\mathbf{n}$ is the unit normal at p, $\mathbf{w}$ is a tangent vector of M at p, and α is an arclength parametrization of the curve created by the intersection of M with the the plane containing $\mathbf{w}$ and $\mathbf{n}$.

Definition 1.2. A *minimal surface* is a surface M with mean curvature

$$H = \frac{k_1 + k_2}{2} = 0, \tag{1.2}$$

at all points $p \in M$, where k_1 and k_2 are the maximum and minimum normal curvature values at p.

The standard tool for representing minimal surfaces is the Weierstrass representation as the following theorem demonstrates.

Theorem 1.3 (Weierstrass representation). *Every regular minimal surface in $\mathbb{R}^3$ has a local isothermal parametric representation of the following form:*

$$X(z) = \left(\operatorname{Re}\int \varphi_1(z)dz, \operatorname{Re}\int \varphi_2(z)dz, \operatorname{Re}\int \varphi_3(z)dz\right), \tag{1.3}$$

where each φ_k is analytic, $\phi^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$, and $|\phi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \neq 0$, and is finite.

A common way to use the Weierstrass representation is in terms of the Gauss map, G, and height differential, dh. These are analytic functions that provide information about the geometry of the surface (see [1, 2]). When represented in these terms, the Weierstrass representation becomes

$$X(z) = \operatorname{Re}\int \left(\frac{1}{2}\left(\frac{1}{G} - G\right), \frac{i}{2}\left(\frac{1}{G} + G\right), 1\right) dh. \tag{1.4}$$

Another way to represent minimal surfaces is in terms of planar harmonic mappings. Planar harmonic mappings have been studied independently of minimal surfaces and results about them can be used to establish results about minimal surfaces (see [3]). The following definitions and theorems will be useful in this discussion.

Definition 1.4. A continuous function $f(x,y) = u(x,y) + iv(x,y)$ defined in a domain $D \subset \mathbb{C}$ is a *planar harmonic mapping* or *harmonic function* in D if u and v are real harmonic functions in D.

In this paper, we will only consider harmonic functions defined on the unit disk, $\mathbb{D} = \{z : |z| < 1\}$.

Theorem 1.5 (see [4]). *If $f = u + iv$ is harmonic in $\mathbb{D}$, then f can be written as $f = h + \overline{g}$, where h and g are analytic.*

Definition 1.6. The *dilatation* of $f = h + \overline{g}$ is $\omega(z) = g'(z)/h'(z)$.

Theorem 1.7 (see [5]). *The harmonic function $f = h + \overline{g}$ is locally univalent and orientation preserving in $\mathbb{D}$ if and only if $|w(z)| < 1$, for all $z \in \mathbb{D}$.*

Notice that the first and second coordinates of the Weierstrass representation (1.3) are the real part of analytic functions and are thus harmonic. The projection of a minimal surface onto the x_1x_2-plane can then be viewed as the image of a planar harmonic mapping in the complex plane. This gives rise to another Weierstrass representation in terms of the planar harmonic mapping $f = h + \overline{g}$. One advantage of this representation is that the univalence of the harmonic mapping f guarantees that the corresponding minimal surface will be a graph over the image of f and will thus be embedded. These ideas are summarized in the following theorem.

Theorem 1.8 (Weierstrass representation (h, g), see [6]). *Let $f = h + \overline{g}$ be an orientation-preserving harmonic univalent mapping of a domain $\mathbb{D}$ onto some domain Ω with dilatation, ω, that is, the square of an analytic function in $\mathbb{D}$. Then*

$$X(z) = \left(\operatorname{Re}\{h(z) + g(z)\}, \operatorname{Im}\{h(z) - g(z)\}, 2\operatorname{Im}\left\{\int_0^z \sqrt{g'(\zeta)h'(\zeta)}d\zeta\right\}\right) \tag{1.5}$$

gives an isothermal parametrization of a minimal graph whose projection onto the complex plane is $f(\mathbb{D})$. Conversely, if a minimal graph is parameterized by orientation-preserving isothermal parameters $z = x + iy \in \mathbb{D}$, then the projection onto its base plane defines a harmonic univalent mapping $f(z) = \operatorname{Re}\{h(z) + g(z)\} + i\operatorname{Im}\{h(z) - g(z)\}$ whose dilatation is the square of an analytic function.

It can be derived from (1.4) and (1.5) that the Gauss map and height differential are related to the harmonic mapping $f = h + \overline{g}$ by

$$G = -i\sqrt{\frac{h'}{g'}} = \frac{-i}{\sqrt{\omega}}, \qquad dh = -2i\sqrt{g'h'}dz. \tag{1.6}$$

2. Harmonic Univalent Functions

We wish to establish conditions on a collection of minimal graphs to guarantee that a convex combination of them will be a minimal graph. To do this, we will make use of Theorem 1.8 and some established results concerning the univalence of planar harmonic mappings. We will first need some background information.

Definition 2.1. A domain Ω is *convex in the direction $e^{i\alpha}$* if for every $a \in \mathbb{C}$ the set

$$\Omega \cap \left\{a + te^{i\alpha} : t \in \mathbb{R}\right\} \tag{2.1}$$

is either connected or empty. In particular, a domain is *convex in the imaginary direction* (CID) if every line parallel to the imaginary axis has a connected intersection with Ω.

In general, it is difficult to establish the univalence of a planar harmonic mapping. The shearing technique of Clunie and Sheil-Small however provides one way to do this.

Theorem 2.2 (see [4]). *A harmonic function $f = h+\overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the $e^{i\alpha}$ direction if and only if $\varphi = h - e^{2i\alpha}g$ is a analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the $e^{i\alpha}$ direction.*

We will also need the following from Hengartner and Schober [7].

Condition 1. Let φ be a nonconstant analytic function in $\mathbb{D}$, and there exist sequences z_n', z_n'' converging to $z = 1$, $z = -1$, respectively, such that

$$\begin{aligned} \lim_{n\to\infty} \operatorname{Re}\{\varphi(z_n')\} &= \sup_{|z|<1} \operatorname{Re}\{\varphi(z)\}, \\ \lim_{n\to\infty} \operatorname{Re}\{\varphi(z_n'')\} &= \inf_{|z|<1} \operatorname{Re}\{\varphi(z)\}. \end{aligned} \tag{2.2}$$

Theorem 2.3 (see [7]). *Suppose that ψ is analytic and nonconstant in $\mathbb{D}$. Then*

$$\operatorname{Re}\left\{\left(1-z^2\right)\psi'(z)\right\} \geq 0, \quad z \in \mathbb{D} \tag{2.3}$$

if and only if ψ is univalent in $\mathbb{D}$, $\psi(\mathbb{D})$ is convex in the imaginary direction, and Condition 1 holds.

Note that the normalization in (2.2) can be thought of in some sense as if $\psi(1)$ and $\psi(-1)$ are the right and left extremes in the image domain in the extended complex plane.

3. Convex Combinations of Minimal Graphs

We are now ready to prove our main result.

Theorem 3.1. *Let $M_1, \ldots, M_n : \mathbb{D} \to \mathbb{R}^3$ be minimal graphs with isothermal parametrizations $\phi_k = \operatorname{Re}(\phi_k^1, \phi_k^2, \phi_k^3) = \operatorname{Re}\int((1/2)(1/G_k - G_k), (i/2)(1/G_k + G_k), 1)dh_k$, where G_k is the Gauss map and dh_k is the height differential ($k = 1, \ldots, n$). Let*

(1) *$G_k = G_1$ for each k,*

(2) *the projection of M_k on to the x_1x_2-plane, Ω_k, be* CID,

(3) *Condition 1 holds for each ϕ_k^1, for $k = 1, \ldots, n$.*

If $(t_1\phi_1^1 + \cdots + t_n\phi_n^1)' \neq 0$, then $M = t_1M_1 + \cdots + t_nM_n$ is a minimal graph for all $0 \leq t_k \leq 1$, where $t_1 + \cdots + t_n = 1$ with $G = G_1$ and $dh = t_1dh_1 + \cdots + t_ndh_n$.

Remark 3.2. This definition of the convex combinations of minimal graphs is very close to the definition of the sum of two complete minimal surfaces with finite total curvature given by Rosenberg and Toubiana in [8].

Proof. By Theorem 1.8, the projection of each minimal graph, M_k, onto the x_1x_2-plane defines a univalent harmonic mapping $f_k = h_k + \overline{g_k}$ with dilatation $\omega_k = g_k'/h_k'$. Let

$$f = h + \overline{g} = (t_1h_1 + \cdots + t_nh_n) + \overline{(t_1g_1 + \cdots + t_ng_n)}. \tag{3.1}$$

We will show that f is a univalent harmonic mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction. Since $G_1 = G_k$, we see from (1.6) that $\omega_1 = \omega_k$ for all $k = 2, \ldots, n$. Also, $\omega = g'/h'$ equals ω_1 because

$$\omega = \frac{t_1g_1' + \cdots + t_ng_n'}{t_1h_1' + \cdots + t_nh_n'} = \frac{t_1h_1'\omega_1 + \cdots + t_nh_n'\omega_n}{t_1h_1' + \cdots + t_nh_n'} = \omega_1. \tag{3.2}$$

Hence, f is locally univalent since $|\omega(z)| = |\omega_1(z)| < 1$ for every $z \in \mathbb{D}$. We now will show that $h + g$ is a univalent analytic mapping of $\mathbb{D}$ onto a domain convex in the imaginary direction, so we can apply the shearing theorem. By Theorem 2.2, we know that each $h_k + g_k$ is univalent and CID. Also, $h_k + g_k$ satisfies Condition 1 since $\operatorname{Re}\{h_k + g_k\} = \operatorname{Re}\{\phi_k^1\}$. Applying Theorem 2.3, we have

$$\operatorname{Re}\left\{\left(1-z^2\right)\left(h_k'(z) + g_k'(z)\right)\right\} \geq 0 \tag{3.3}$$

for every $k \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned} &\operatorname{Re}\left\{\left(1-z^2\right)\left(h'(z)+g'(z)\right)\right\} \\ &\quad= \operatorname{Re}\left\{\left(1-z^2\right)\left[t_1\left(h_1'(z)+g_1'(z)\right)+\cdots+t_n\left(h_n'(z)+g_n'(z)\right)\right]\right\} \\ &\quad= t_1 \operatorname{Re}\left\{\left(1-z^2\right)\left(h_1'(z)+g_1'(z)\right)\right\}+\cdots+t_n \operatorname{Re}\left\{\left(1-z^2\right)\left(h_n'(z)+g_n'(z)\right)\right\} \geq 0. \end{aligned} \tag{3.4}$$

Since $h' + g' = (t_1\phi_1^1 + \cdots + t_n\phi_n^1)' \neq 0$, by applying Theorem 2.3 in the other direction, we have that $h + g$ is a conformal univalent mapping of $\mathbb{D}$ onto a CID domain. Thus, by Theorem 2.2, f is a harmonic univalent mapping with $f(\mathbb{D})$ being convex in the imaginary direction. We can now apply the Weierstrass representation from Theorem 1.8 to lift $f = h + \overline{g}$ to a minimal graph $\widetilde{M} = (u, v, F(u, v))$. Notice that

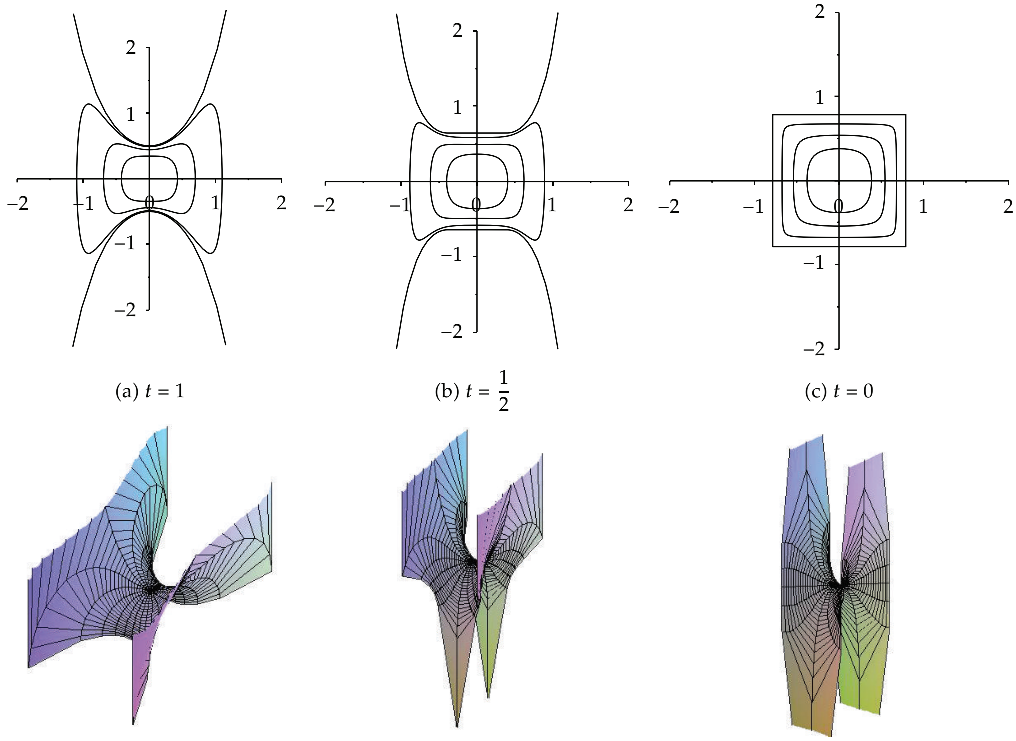
$$\begin{aligned} u &= \operatorname{Re}\{h+g\} \\ &= \operatorname{Re}\{(t_1h_1+t_1g_1)+\cdots+(t_nh_n+t_ng_n)\} \\ &= t_1 \operatorname{Re}\left\{\phi_1^1\right\}+\cdots+t_n \operatorname{Re}\left\{\phi_n^1\right\}. \end{aligned} \tag{3.5}$$

Similarly, $v = \operatorname{Im}\{h - g\} = t_1 \operatorname{Re}\{\phi_1^2\} + \cdots + t_n \operatorname{Re}\{\phi_n^2\}$. Finally,

$$\begin{aligned} F(u,v) &= 2\operatorname{Im}\left\{\int_0^z \sqrt{\left(t_1g_1'(\zeta)+\cdots+t_ng_n'(\zeta)\right)\left(t_1h_1'(\zeta)+\cdots+t_nh_n'(\zeta)\right)}\,d\zeta\right\} \\ &= 2\operatorname{Im}\left\{\int_0^z \sqrt{\left(t_1\omega_1(\zeta)h_1'(\zeta)+\cdots+t_n\omega_n(\zeta)h_n'(\zeta)\right)\left(t_1h_1'(\zeta)+\cdots+t_nh_n'(\zeta)\right)}\,d\zeta\right\} \\ &= 2\operatorname{Im}\left\{\int_0^z \sqrt{\omega_1(\zeta)}\left(t_1h_1'(\zeta)+\cdots+t_nh_n'(\zeta)\right)d\zeta\right\} \\ &= 2\operatorname{Im}\left\{\int_0^z \left(t_1\sqrt{g_1'(\zeta)h_1'(\zeta)}+\cdots+t_n\sqrt{g_n'(\zeta)h_n'(\zeta)}\right)d\zeta\right\} \\ &= t_1 \operatorname{Re}\left\{\phi_1^3\right\}+\cdots+t_n \operatorname{Re}\left\{\phi_n^3\right\}. \end{aligned} \tag{3.6}$$

Thus, $\widetilde{M} = t_1M_1 + \cdots + t_nM_n = M$. □

Using this theorem we can take a convex combination of several classical minimal surfaces to produce new minimal graphs.



(a) $t = 1$ (b) $t = \frac{1}{2}$ (c) $t = 0$

Figure 1: Images of concentric circles under f and corresponding minimal surfaces for various values of t in Example 3.3.

Example 3.3. Consider the Weierstrass data for the catenoid $G_1 = -1/z$ and $dh_1 = z/(1-z^2)^2 dz$, where $z \in \mathbb{D}$. Using (1.6), we get

$$h_1 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z^2},$$
$$g_1 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{1}{2}\frac{z}{1-z^2}. \tag{3.7}$$

Notice that $\phi_1^1 = h_1 + g_1 = (1/2)\log((1+z)/(1-z))$ and $\operatorname{Re}\{(1-z^2)((\phi_1^1)')\} = 1 \geq 0$. So by Theorem 2.3, $\phi_1^1(\mathbb{D})$ is convex in the imaginary direction and ϕ_1^1 satisfies Condition 1. Since $\omega_1 = -z^2$, the harmonic map $f_1 = h_1 + \overline{g_1}$ lifts to a minimal graph by Theorem 1.8.

Similarly, the Weierstrass data $G_2 = -1/z$ and $dh_2 = (z/(z^4-1))dz$, where $z \in \mathbb{D}$, results in a graph of Scherk's doubly periodic surface with $\omega_2 = -z^2$, and

$$h_2 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right),$$
$$g_2 = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right). \tag{3.8}$$

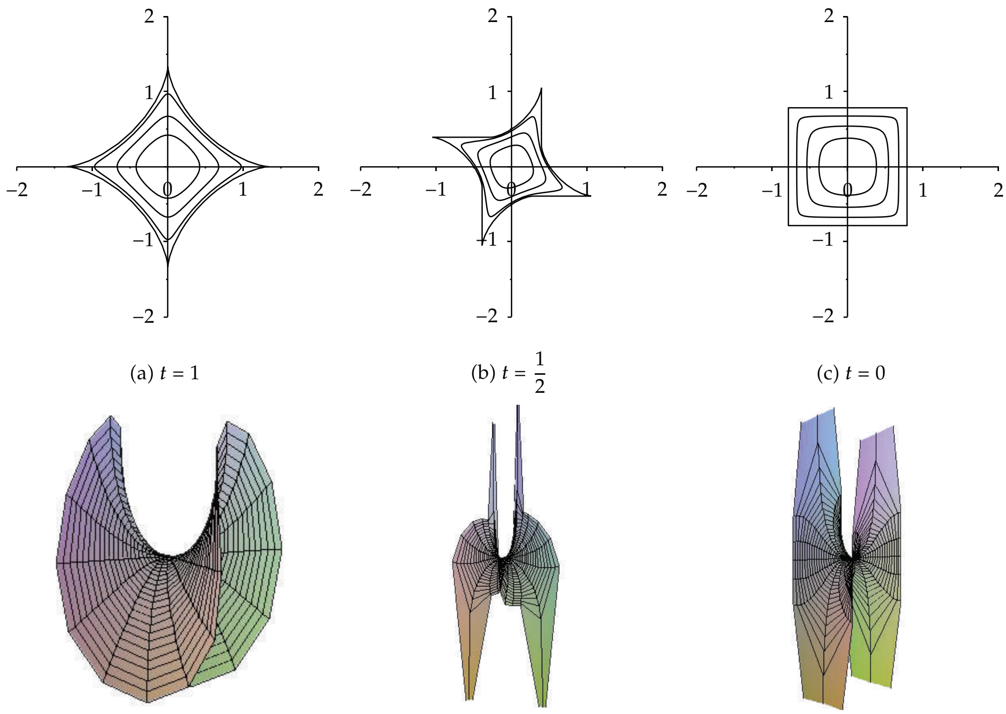


Figure 2: Images of concentric circles under f and corresponding minimal surfaces for various values of t in Example 3.4.

Now $\phi_2^1 = h_2 + g_2 = (i/2)\log((i+z)/(i-z))$ and

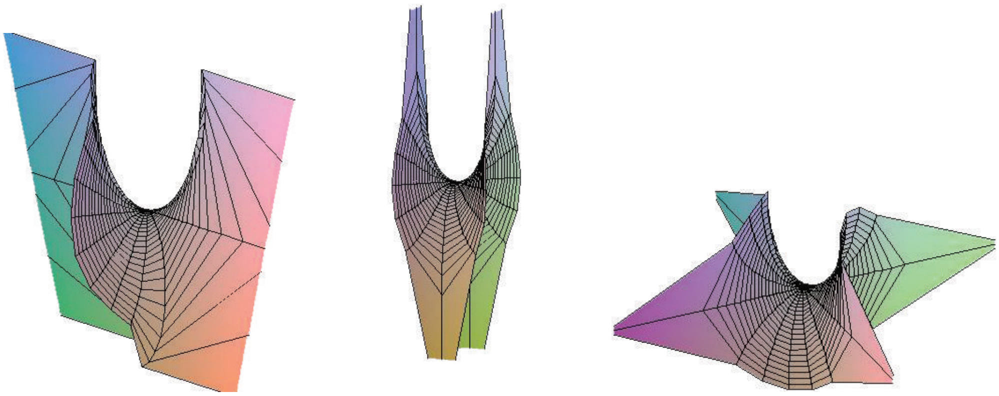
$$\operatorname{Re}\left\{\left(1-z^2\right)\left(\left(\phi_2^1\right)'\right)\right\} = \operatorname{Re}\left\{\frac{1-z^2}{1+z^2}\right\} \geq 0. \tag{3.9}$$

Again, by Theorem 2.3, $\phi_2^1(\mathbb{D})$ is convex in the imaginary direction and ϕ_2^1 satisfies Condition 1.

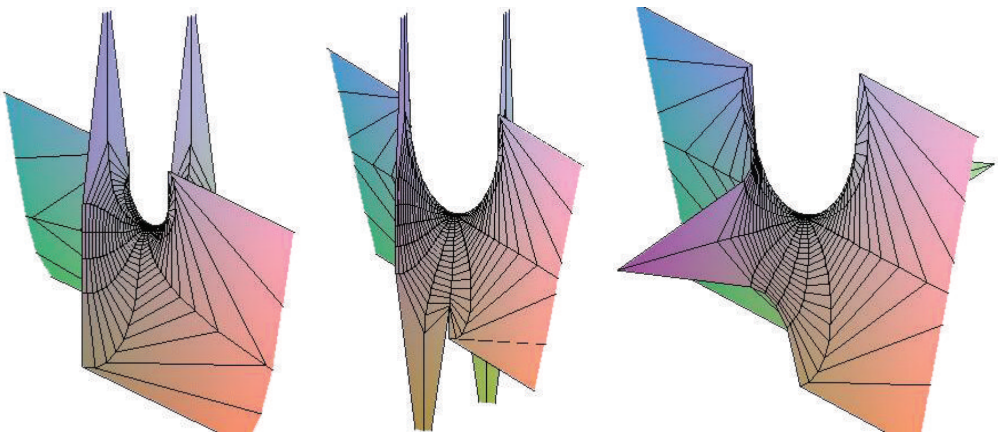
Since both parametrizations satisfy the hypotheses of Theorem 3.1, the harmonic map $f = t(h_1 + \overline{g_1}) + (1-t)(h_2 + \overline{g_2})$ will lift to a minimal graph over $\mathbb{D}$ with Weierstrass data $G = -1/z$ and $dh = (t)dh_1 + (1-t)dh_2$ for all $0 \leq t \leq 1$ (see Figure 1).

Example 3.4. The Weierstrass data $G_1 = -i/z$ and $dh_1 = -iz$ gives a parametrization of Enneper's surface. The functions $G_2 = -i/z$ and $dh_2 = z/(z^4 - 1)$ give a different paramaterization of Scherk's doubly periodic surface than in Example 3.3. Notice that $\phi_1^1 = z + (1/3)z^3$ and $\phi_2^1 = -(i/2)\log((1+z)/(1-z))$. By Theorem 2.3, ϕ_1^1 and ϕ_2^1 satisfy Condition 1. Thus, both surfaces satisfy the hypotheses of Theorem 3.1, and the function $f = t(h_1 + \overline{g_1}) + (1-t)(h_2 + \overline{g_2})$ will lift to a minimal graph for all $0 \leq t \leq 1$ (see Figure 2).

Using this method, we were able to show that the combinations shown in Figure 3 are also minimal graphs.



(a) Enneper-Helicoid (b) Enneper-Scherk's doubly periodic (c) Enneper-Scherk's singly periodic



(d) Helicoid-Scherk's doubly periodic (e) Helicoid-Scherk's doubly periodic 2 (f) Helicoid-Scherk's singly periodic

Figure 3: Various combinations of minimal graphs.

Remark 3.5. In the examples of this paper we have only taken convex combinations using two minimal graphs. It is possible, however, to take a convex combination of any finite number of minimal graphs using Theorem 3.1.

Area for Further Investigation

The condition that two minimal graphs share the same Gauss map does not seem to be necessary. It would be interesting to find an example of two minimal graphs with different Gauss maps such that a convex combination of them is a minimal graph.

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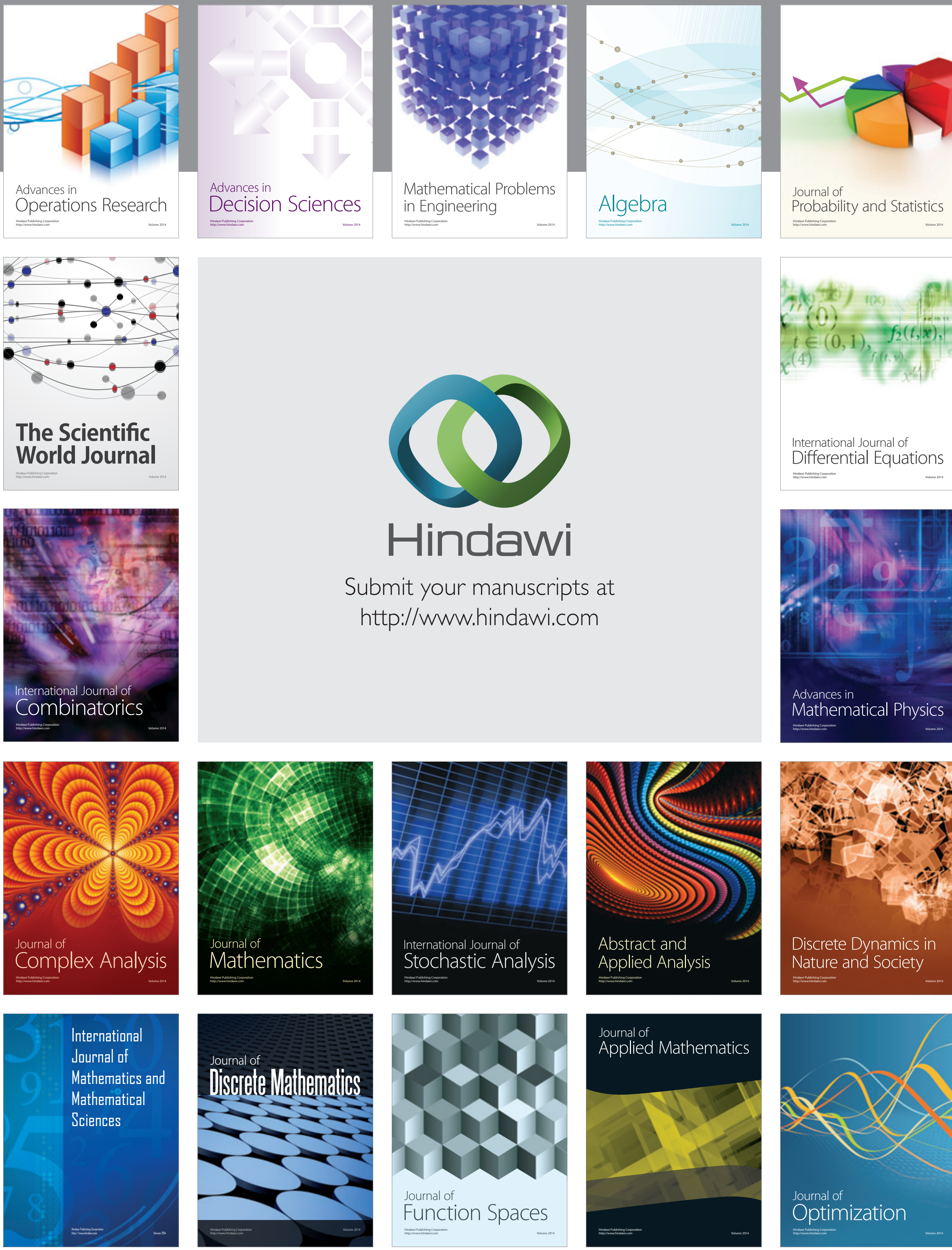

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