Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 871845, 29 pages
doi:10.1155/2012/871845

Research Article

Recent Developments of Hilbert-Type Discrete and Integral Inequalities with Applications

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Received 7 March 2012; Revised 12 June 2012; Accepted 27 June 2012

Academic Editor: Tibor K. Pogány

This paper deals with recent developments of Hilbert-type discrete and integral inequalities by introducing kernels, weight functions, and multiparameters. Included are numerous generalizations, extensions, and refinements of Hilbert-type inequalities involving many special functions such as beta, gamma, logarithm, trigonometric, hyper-bolic, Bernoulli's functions and Bernoulli's numbers, Euler's constant, zeta function, and hypergeometric functions with many applications. Special attention is given to many equivalent inequalities and to conditions under which the constant factors involved in inequalities are the best possible. Many particular cases of Hilbert-type inequalities are presented with numerous applications. A large number of major books and recent research papers published during 2009–2012 are included to stimulate new interest in future study and research.

"As long as a branch of knowledge offers an abundance of problems, it is full of vitality"
David Hilbert

1. Introduction

Historically, mathematical analysis has been the major and significant branch of mathematics for the last three centuries. Indeed, inequalities became the heart of mathematical analysis. Many great mathematicians have made significant contributions to many new developments of the subject, which led to the discovery of many new inequalities with proofs and useful applications in many fields of mathematical physics, pure and applied mathematics. Indeed, mathematical inequalities became an important branch of modern mathematics in the twentieth century through the pioneering work entitled *Inequalities* by G. H. Hardy, J. E. Littlewood, and G. Pòlya, which was first published treatise in 1934. This unique publication represents a paradigm of precise logic, full of elegant inequalities with rigorous proofs and useful applications in mathematics.

During the twentieth century, discrete and integral inequalities played a fundamental role in mathematics and have a wide variety of applications in many areas of pure and applied mathematics. In particular, David Hilbert (1862–1943) first proved Hilbert's double series inequality without exact determination of the constant in his lectures on integral equations. If $\{a_m\}$ and $\{b_n\}$ are two real sequences such that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the *Hilbert's double series inequality* is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}. \tag{1.1}$$

In 1908, Weyl [1] published a proof of Hilbert's inequality (1.1), and in 1911, Schur [2] proved that π in (1.1) is the best possible constant and also discovered the integral analogue of (1.1), which became known as the *Hilbert's integral inequality* in the form

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy < \pi\left\{\int_0^{\infty}f^2(x)\mathrm{d}x\int_0^{\infty}g^2(y)dy\right\}^{1/2}, \tag{1.2}$$

where f and g are measurable functions such that $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(y)dy < \infty$, and π in (1.2) is still the best possible constant factor. A large number of generalizations, extensions, and refinements of both (1.1) and (1.2) are available in the literature in Hardy et al. [3], Mitrinović et al. [4], Kuang [5], and Hu [6].

Considerable attention has been given to the well-known classical Hardy-Littlewood-Sobolev (HLS) inequality (see Hardy et al. [3]) in the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{\lambda}}dx\,dy \le C_{n,\lambda,p}\|f\|_p\|g\|_q, \tag{1.3}$$

for every $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where $0 < \lambda < n$, $1 < p, q < \infty$ such that $(1/p) + (1/q) + (\lambda/n) = 2$, and $\|f\|_p$ is the $L^p(\mathbb{R}^n)$ norm of the function f. For arbitrary p and q, an estimate of the upper bound of the constant $C_{n,\lambda,p}$ was given by Hardy, Littlewood, and Sobolev, but no sharp value is known up till now. However, for the special case, $p = q = 2n/(2n-\lambda)$, the sharp value of the constant was found as

$$C_{p,\lambda,n} = C_{n,\lambda} = \pi^{\lambda/2}\,\frac{\Gamma((n-\lambda)/2)}{\Gamma(n-(\lambda/2))}\left[\frac{\Gamma(n/2)}{\Gamma(n)}\right]^{(\lambda/n)-1}, \tag{1.4}$$

and the equality in (1.3) holds if and only if $g(x) = c_1 f(x)$ and $f(x) = c_2 h((x/\mu^2) - a)$, where $h(x) = (1+|x|^2)^{-d}$, $2d = (\lambda - 2n)$, $a \in \mathbb{R}^n$, $c_1, c_2, \mu \in \mathbb{R}/\{0\}$.

In 1958, Stein and Weiss [7] generalized the double-weighted inequality of Hardy and Littlewood in the form with the same notation as in (1.3):

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(x)}{|x|^{\alpha}|x-y|^{\lambda}|y|^{\beta}}dx\,dy\right| \leq C_{\alpha,\beta,n,\lambda,p}\|f\|_p\|g\|_q, \tag{1.5}$$

where $\alpha+\beta \geq 0$, and the powers of α and β of the weights satisfy the following conditions $1-(1/p)-(\lambda/n)<\alpha/n<1-(1/p)$, and $(1/p)+(1/q)+(1/n)(\lambda+\alpha+\beta)=2$. Inequality (1.5) and its proof given by Stein and Weiss [7] represent some major contribution to the subject.

On the other hand, Chen et al. [8] used weighted Hardy-Littlewood-Sobolev inequalities (1.3) and (1.5) to solve systems of integral equations. In 2011, Khotyakov [9] suggested two proofs of the sharp version of the HLS inequality (1.3). The first proof is based on the invariance property of the inequality (1.3), and the second proof uses some properties of the fast diffusion equation with the conditions $\lambda = n-2$, $n \geq 3$ on the sharp HLS inequality (1.3).

The main purpose of this paper is to describe recent developments of Hilbert's discrete and integral inequalities in different directions with many applications. Included are many generalizations, extensions, and refinements of Hilbert-type inequalities involving many special functions such as beta, gamma, logarithm, trigonometric, hyperbolic, Bernoulli's functions and Bernoulli's numbers, Euler's constant, zeta function, and hypergeometric functions. Special attention is given to many equivalent inequalities and to conditions under which constant factors involved inequalities are the best possible. Many particular cases of Hilbert-type inequalities are presented with applications. A large number of major books and recent research papers published during 2009–2012 are included in references to stimulate new interest in future study and research.

2. Operator Formulation of Hilbert's Inequality

Suppose that $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+^m = \overbrace{(0,\infty)\times\cdots\times(0,\infty)}^{m}$, for $p>1$, $l^p := \{a = \{a_n\}_{n=1}^{\infty} | \|a\|_p = \{\sum_{n=1}^{\infty}|a_n|^p\}^{1/p} < \infty\}$ and $L^p(\mathbb{R}_+) := \{f | \|f\|_p = \{\int_0^{\infty}|f(x)|^p dx\}^{1/p} < \infty\}$ are real normal spaces with the norms $\|a\|_p$ and $\|f\|_p$. We express inequality (1.1) using the form of operator as follows: $T : l^2 \to l^2$ is a linear operator, for any $a = \{a_m\}_{m=1}^{\infty} \in l^2$, there exists a sequence $c = \{c_n\}_{n=1}^{\infty} \in l^2$, satisfying

$$c_n = (Ta)(n) = \sum_{m=1}^{\infty}\frac{a_m}{m+n}, \quad n \in \mathbb{N}, \tag{2.1}$$

where $\mathbb{N}$ is the set of positive integers. Hence, for any sequence $b = \{b_n\}_{n=1}^{\infty} \in l^2$, we define the inner product of Ta and b as follows:

$$(Ta,b) = (c,b) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)b_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}. \tag{2.2}$$

Using (2.2), inequality (1.1) can be rewritten in the operator form

$$(Ta,b) < \pi\|a\|_2\|b\|_2, \tag{2.3}$$

where $\|a\|_2$ and $\|b\|_2 > 0$. It follows from Wilhelm [10] that T is a bounded operator and the norm $\|\mathrm{T}\| = \pi$ and T is called *Hilbert's operator* with the kernel $1/(m+n)$. For $\|a\|_2 > 0$, the equivalent form of (2.3) is given as $\|Ta\|_2 < \pi\|a\|_2$, that is,

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2\sum_{n=1}^{\infty}a_n^2, \tag{2.4}$$

where the constant factor π^2 is still the best possible. Obviously, inequality (2.4) and (1.1) are equivalent (see Hardy et al. [3]).

We may define Hilbert's integral operator as follows: $\widetilde{T} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, for any $f \in L^2(\mathbb{R}_+)$, there exists a function, $h = \widetilde{T}f \in L^2(\mathbb{R}_+)$, satisfying

$$\left(\widetilde{T}f\right)(y) = h(y) = \int_0^{\infty}\frac{f(x)}{x+y}dx, \quad y \in (0,\infty). \tag{2.5}$$

Hence, for any $g \in L^2(\mathbb{R}_+)$, we may still define the inner product of $\widetilde{T}f$ and g as follows:

$$\left(\widetilde{T}f,g\right) = \int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}dx\right)g(y)dy = \int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy. \tag{2.6}$$

Setting the norm of f as $\|f\|_2 = \{\int_0^{\infty} f^2(x)dx\}^{1/2}$, if $\|f\|_2$ and $\|g\|_2 > 0$, then (1.2) may be rewritten in the operator form

$$\left(\widetilde{T}f,g\right) < \pi\|f\|_2\|g\|_2. \tag{2.7}$$

It follows that $\|\widetilde{T}f\| = \pi$ (see Carleman [11]), and we have the equivalent form of (1.2) as $\|\widetilde{T}f\|_2 < \pi\|f\|_2$, (see Hardy et al. [3]), that is,

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+y}dx\right)^2 dy < \pi^2\int_0^{\infty} f^2(x)dx, \tag{2.8}$$

where the constant factor π^2 is still the best possible. It is obvious that inequality (2.8) is the integral analogue of (2.4).

3. A More Accurate Discrete Hilbert's Inequality

If we set the subscripts $m,\ n$ of the double series from zero to infinity, then, we may rewrite inequality (1.1) equivalently in the following form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+2} < \pi\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right\}^{1/2}, \tag{3.1}$$

where the constant factor π is still the best possible. Obviously, we may raise the following question: Is there a positive constant $\alpha(< 2)$, that makes inequality still valid as we replace 2 by α in the kernel $1/(m+n+2)$? The answer is positive. That is, the following is more accurate Hilbert's inequality (for short, Hilbert's inequality) (see Hardy et al. [3]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right\}^{1/2}, \tag{3.2}$$

where the constant factor π is the best possible.

Since for $a_m,\ b_n \geq 0$, $\alpha \geq 1$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+\alpha} \leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}, \tag{3.3}$$

then, by (3.2) and for $\alpha \geq 1$, we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+\alpha} < \pi\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right\}^{1/2}. \tag{3.4}$$

For $1 \leq \alpha < 2$, inequality (3.4) is a refinement of (3.1). Obviously, we have a refinement of (2.4), which is equivalent to (3.4) as follows:

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+\alpha}\right)^2 < \pi^2\sum_{n=0}^{\infty}a_n^2, \quad (1 \leq \alpha < 2). \tag{3.5}$$

For $0 < \alpha < 1$, in 1936, Ingham [12] proved that if $\alpha \geq 1/2$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m a_n}{m+n+\alpha} \leq \pi\sum_{n=0}^{\infty}a_n^2, \tag{3.6}$$

and if $0 < \alpha < 1/2$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m a_n}{m+n+\alpha} \leq \frac{\pi}{\sin(\alpha\pi)}\sum_{n=0}^{\infty}a_n^2. \tag{3.7}$$

Note 1. If we put $x = X + (\alpha/2)$, $y = Y + (\alpha/2)$, $F(X) = f(X + (\alpha/2))$, and $G(Y) = g(Y + (\alpha/2))$ ($\alpha \in \mathbb{R}$) in (1.2), then we obtain

$$\int_{-\alpha/2}^{\infty}\int_{-\alpha/2}^{\infty}\frac{F(X)G(Y)}{X+Y+\alpha}dX\,dY < \pi\left\{\int_{-\alpha/2}^{\infty}F^2(X)dX\int_{-\alpha/2}^{\infty}G^2(Y)dY\right\}^{1/2}. \tag{3.8}$$

For $\alpha \geq 1/2$, inequality (3.8) is an integral analogue of (3.6) with $G = F$. However, if $0 < \alpha < 1/2$, inequality (3.8) is not an integral analogue of (3.7), because two constant factors are different.

Using the improved version of the Euler-Maclaurin summation formula and introducing new parameters, several authors including Yang (see [13–15]) recently obtained several more accurate Hilbert-type inequalities and some new Hardy-Hilbert inequality with applications.

4. Hilbert's Inequality with One Pair of Conjugate Exponents

In 1925, by introducing one pair of conjugate exponents (p, q) with $(1/p) + (1/q) = 1$, Hardy [16] gave an extension of (1.1) as follows:

if $p > 1$, $a_m, b_n \geq 0$, such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{4.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. The equivalent discrete form of (4.1) is as follows:

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_{\mathrm{m}}}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty}a_n^p, \tag{4.2}$$

where the constant factor $[\pi/\sin(\pi/p)]^p$ is still the best possible. Similarly, inequalities (3.2) and (3.5) (for $\alpha = 1$) may be extended to the following equivalent forms (see Hardy et al. [3]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{m=0}^{\infty}a_m^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}b_n^q\right\}^{1/q}, \tag{4.3}$$

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=0}^{\infty}a_n^p, \tag{4.4}$$

where the constant factors $(\pi/\sin(\pi/p))$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. The equivalent integral analogues of (4.1) and (4.2) are given as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty g^q(y)dy\right\}^{1/q}, \tag{4.5}$$

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\int_0^\infty f^p(x)dx. \tag{4.6}$$

(4.1) and (4.3) as Hardy-Hilbert's inequality and call (4.5) as *Hardy-Hilbert's integral inequality*.

Inequality (4.3) may be expressed in the form of operator as follows: $T_p : l^p \to l^p$ is a linear operator, such that for any nonnegative sequence $a = \{a_m\}_{m=1}^\infty \in l^p$, there exists $T_p a = c = \{c_n\}_{n=1}^\infty \in l^p$, satisfying

$$c_n = (T_p a)(n) = \sum_{m=0}^\infty \frac{a_m}{m+n+1}, \quad n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{4.7}$$

And for any nonnegative sequence $b = \{b_n\}_{n=1}^\infty \in l^q$, we can define the formal inner product of $T_p a$ and b as follows:

$$(T_p a, b) = \sum_{n=0}^\infty\left(\sum_{m=0}^\infty \frac{a_m}{m+n+1}\right)b_n = \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{a_m b_n}{m+n+1}. \tag{4.8}$$

Then inequality (4.3) may be rewritten in the operator form

$$(T_p a, b) < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \tag{4.9}$$

where $\|a\|_p, \|b\|_q > 0$. The operator T_p is called *Hardy-Hilbert's operator*.

Similarly, we define the following *Hardy-Hilbert's integral operator* $\widetilde{T}_p : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ as follows: for any $f(\geq 0) \in L^p(\mathbb{R}_+)$, there exists an $h = \widetilde{T}_p f \in L^p(\mathbb{R}_+)$, defined by

$$\left(\widetilde{T}_p f\right)(y) = h(y) = \int_0^\infty \frac{f(x)}{x+y}dx, \quad y \in \mathbb{R}_+. \tag{4.10}$$

And for any $g(\geq 0) \in L^q(\mathbb{R}_+)$, we can define the formal inner product of $\widetilde{T}_p f$ and g as follows:

$$\left(\widetilde{T}_p f, g\right) = \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy. \tag{4.11}$$

Then inequality (4.5) may be rewritten as follows:

$$\left(\widetilde{T}_p f, g\right) < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q. \tag{4.12}$$

On the other hand, if (p, q) is not a pair of conjugate exponents, then we have the following results (see Hardy et al. [3]).

If $p > 1,\ q > 1,\ (1/p) + (1/q) \geq 1,\ 0 < \lambda = 2 - ((1/p) + (1/q)) \leq 1$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \leq K\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{4.13}$$

where $K = K(p, q)$ relates to p, q, only for $(1/p) + (1/q) = 1$, $\lambda = 2 - ((1/p) + (1/q)) = 1$, the constant factor K is the best possible. The integral analogue of (4.13) is given by

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dx\,dy \leq K\left\{\int_0^{\infty}f^p(x)dx\right\}^{1/p}\left\{\int_0^{\infty}g^q(y)dy\right\}^{1/q}. \tag{4.14}$$

We also find an extension of (4.14) as follows (see Mitrinović et al. [4]):

If $p > 1, q > 1, (1/p) + (1/q) > 1, 0 < \lambda = 2 - ((1/p) + (1/q)) < 1$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{|x+y|^{\lambda}}dx\,dy \leq k(p,q)\left\{\int_{-\infty}^{\infty}f^p(x)dx\right\}^{1/p}\left\{\int_{-\infty}^{\infty}g^q(x)dx\right\}^{1/q}. \tag{4.15}$$

For $f(x) = g(x) = 0$, $x \in (-\infty, 0]$, inequality (4.15) reduces to (4.14). Leven [17] also studied the expression forms of the constant factors in (4.13) and (4.14). But he did not prove their best possible property. In 1951, Bonsall [18] considered the case of (4.14) for the general kernel.

5. A Hilbert-Type Inequality with the General Homogeneous Kernel of Degree −1

If $\alpha \in \mathbb{R}$, the function $k(x, y)$ is measurable in $\mathbb{R}_+^2$, satisfying for any $x, y, u > 0$, $k(ux, uy) = u^{\alpha}k(x, y)$, then $k(x, y)$ is called the homogeneous function of degree α. In 1934, Hardy et al. [3] published the following theorem: suppose that $p > 1$, $(1/p) + (1/q) = 1$, $k_1(x, y)(\geq 0)$ is a homogeneous function of degree -1 in $\mathbb{R}_+^2$. If $f(x), g(y) \geq 0, f \in L^p(\mathbb{R}_+), g \in L^q(\mathbb{R}_+),\ k = \int_0^{\infty} k_1(u, 1)u^{-1/p}du$ is finite, then we have $k = \int_0^{\infty} k_1(1, u)u^{-1/q}du$ and the following equivalent integral inequalities:

$$\int_0^{\infty}\int_0^{\infty}k_1(x,y)f(x)g(y)dx\,dy \leq k\left\{\int_0^{\infty}f^p(x)dx\right\}^{1/p}\left\{\int_0^{\infty}g^q(y)dy\right\}^{1/q}, \tag{5.1}$$

$$\int_0^{\infty}\left(\int_0^{\infty}k_1(x,y)f(x)dx\right)^p dy \leq k^p\int_0^{\infty}f^p(x)dx, \tag{5.2}$$

where the constant factor k is the best possible. Moreover, if $a_m,\ b_n \geq 0,\ a = \{a_m\}_{m=1}^{\infty} \in l^p,\ b = \{b_n\}_{n=1}^{\infty} \in l^q$, both $k_1(u,1)u^{-1/p}$ and $k_1(1,u)u^{-1/q}$ are decreasing in $\mathbb{R}_+$, then we have the following equivalent discrete forms:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(m,n)a_m b_n \leq k\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{5.3}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}k_1(m,n)a_m\right)^p \leq k^p\sum_{n=1}^{\infty}a_n^p. \tag{5.4}$$

For $0 < p < 1$, if $k = \int_0^{\infty} k_1(u,1)u^{-1/p}du$ is finite, then we have the reverses of (5.1) and (5.2).

Note 2. We have not seen any proof of (5.1)–(5.4) and the reverse examples in [3].

We call $k_1(x,y)$ the kernel of (5.1) and (5.2). If all the integrals and series in the right-hand side of inequalities (5.1)–(5.4) are positive, then we can obtain the following particular examples (see Hardy et al. [3]):

(1) for $k_1(x,y) = 1/(x+y)$ in (5.1)–(5.4), they reduce to (4.5), (4.6), (4.1), and (4.2);

(2) if $k_1(x,y) = 1/\max\{x,y\}$ in (5.1)–(5.4), they reduce the following two pairs of equivalent forms:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < pq\left\{\int_0^{\infty}f^p(x)dx\right\}^{1/p}\left\{\int_0^{\infty}g^q(y)dy\right\}^{1/q}, \tag{5.5}$$

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{\max\{x,y\}}dx\right)^p dy < (pq)^p\int_0^{\infty}f^p(x)dx; \tag{5.6}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{m=1}^{\infty}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{5.7}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}}\right)^p < (pq)^p\sum_{n=1}^{\infty}a_n^p; \tag{5.8}$$

(3) if $k_1(x,y) = \ln(x/y)/(x-y)$ in (5.1)–(5.4), they reduce to the following two pairs of equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y) f(x) g(y)}{x-y} dx\,dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left\{\int_0^\infty f^p(x)dx\right\}^{1/p} \left\{\int_0^\infty g^q(y)dy\right\}^{1/q}, \tag{5.9}$$

$$\int_0^\infty \left(\int_0^\infty \frac{\ln(x/y) f(x)}{x-y} dx\right)^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^{2p} \int_0^\infty f^p(x)dx; \tag{5.10}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(m/n) a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left\{\sum_{m=1}^\infty a_{\mathrm{m}}^p\right\}^{1/p} \left\{\sum_{n=1}^\infty b_n^q\right\}^{1/q}, \tag{5.11}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{\ln(m/n) a_m}{m-n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^{2p} \sum_{n=1}^\infty a_n^p. \tag{5.12}$$

Note 3. The constant factors in the above inequalities are all the best possible. We call (5.7) and (5.11) *Hardy-Littlewood-Pólya's inequalities* (or *H-L-P inequalities*). We find that the kernels in the above inequalities are all decreasing functions. But this is not necessary. For example, we find the following two pairs of equivalent forms with the nondecreasing kernel (see Yang [19]):

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)| f(x) g(y)}{\max\{x,y\}} dx\,dy < \left(p^2+q^2\right) \left\{\int_0^\infty f^p(x)dx\right\}^{1/p} \left\{\int_0^\infty g^q(y)dy\right\}^{1/q}, \tag{5.13}$$

$$\int_0^\infty \left(\int_0^\infty \frac{|\ln(x/y)| f(x)}{\max\{x,y\}} dx\right)^p dy < \left(p^2+q^2\right)^p \int_0^\infty f^p(x)dx, \tag{5.14}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{|\ln(m/n)| a_m b_n}{\max\{m,n\}} < \left(p^2+q^2\right) \left\{\sum_{m=1}^\infty a_m^p\right\}^{1/p} \left\{\sum_{n=1}^\infty b_n^q\right\}^{1/q}, \tag{5.15}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{|\ln(m/n)| a_m}{\max\{m,n\}}\right)^p < \left(p^2+q^2\right)^p \sum_{n=1}^\infty a_n^p, \tag{5.16}$$

where the constant factors (p^2+q^2) and $(p^2+q^2)^p$ are the best possible. Another type inequalities with the best constant factors are as follows (see Xin and Yang [20]):

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)| f(x) g(y)}{x+y} dx\, dy < c_0(p) \left\{ \int_0^\infty f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty g^p(y) dy \right\}^{1/q}, \tag{5.17}$$

$$\int_0^\infty \left(\int_0^\infty \frac{|\ln(x/y)| f(x)}{x+y} dx \right)^p dy < c_0^p(p) \int_0^\infty f^p(x) dx; \tag{5.18}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{|\ln(m/n)| a_m b_n}{m+n} < c_0(2) \left\{ \sum_{m=1}^\infty a_m^2 \sum_{n=1}^\infty b_n^2 \right\}^{1/2}, \tag{5.19}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{|\ln(m/n)| a_m}{m+n} \right)^2 < c_0^2(2) \sum_{n=1}^\infty a_n^2, \tag{5.20}$$

where the constant factor $c_0(p) = 2\sum_{n=1}^\infty (-1)^{n-1} [1/(n-(1/p))^2 - (1/(n-(1/q))^2)]$.

6. Two Multiple Hilbert-Type Inequalities with the Homogeneous Kernels of Degree $(-n+1)$

Suppose $n \in \mathbb{N} \setminus \{1\}$, n numbers $p, q, \ldots, r$ satisfying $p, q, \ldots, r > 1$, $p^{-1} + q^{-1} + \cdots + r^{-1} = 1$, and $k(x, y, \ldots, z) \geq 0$ is a homogeneous function of degree $(-n+1)$. If

$$k = \int_0^\infty \int_0^\infty \cdots \int_0^\infty k(1, y, \ldots, z) y^{-1/q} \cdots z^{-1/r} dy \cdots dz \tag{6.1}$$

is a finite number, $f, g, \ldots, h$ are nonnegative measurable functions in $\mathbb{R}_+$, then, we have the following multiple Hilbert-type integral inequality (see Hardy et al. [3]):

$$\begin{aligned} &\int_0^\infty \int_0^\infty \cdots \int_0^\infty k(x, y, \ldots, z) f(x) g(y) \cdots h(z) dx\, dy \cdots dz \\ &\quad \leq k \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(y) dy \right)^{1/q} \cdots \left(\int_0^\infty h^r(z) dz \right)^{1/r}. \end{aligned} \tag{6.2}$$

Moreover, if $a_m, b_n, \ldots, c_s \geq 0$, $k(1, y, \ldots, z) x^0 y^{-1/q} \cdots z^{-1/r}$, $k(x, 1, \ldots, z) x^{-1/p} y^0 \cdots z^{-1/r}$, …, $k(x, y, \ldots, 1) x^{-1/p} y^{-1/q} \cdots z^0$ are all decreasing functions with respect to any single variable in $\mathbb{R}_+$, then, we have

$$\begin{aligned} &\sum_{s=1}^\infty \cdots \sum_{n=1}^\infty \sum_{m=1}^\infty k(m, n, \ldots, s) a_m b_n \cdots c_s \\ &\quad \leq k \left(\sum_{m=1}^\infty a_m^p \right)^{1/p} \left(\sum_{n=1}^\infty b_n^q \right)^{1/q} \cdots \left(\sum_{s=1}^\infty c_s^r \right)^{1/r}. \end{aligned} \tag{6.3}$$

Note 4. The authors did not write and prove that the constant factor k in the above inequalities is the best possible. For two numbers p and $q(n = 2)$, inequalities (6.2) and (6.3) reduce, respectively, to (5.1) and (5.3).

7. Modern Research for Hilbert's Integral Inequality

In 1979, based on an improvement of Hölder's inequality, Hu [21] proved a refinement of (1.2) (for $f = g$) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}dx\,dy < \pi\left\{\left(\int_0^\infty f^2(x)dx\right)^2 - \frac{1}{4}\left(\int_0^\infty f^2(x)\cos\sqrt{x}\,dx\right)^2\right\}^{1/2}. \tag{7.1}$$

Since then, Hu [6] published many interesting results similar to (7.1).

In 1998, Pachpatte [22] gave an inequality similar to (1.2) as follows: for $a, b > 0$,

$$\int_0^a \int_0^b \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\sqrt{ab}}{2}\left\{\int_0^a (a-x)f'^2(x)dx\int_0^b (b-x)g'^2(y)dy\right\}^{1/2}. \tag{7.2}$$

Some improvements and extensions have been made by Zhao and Debnath [23], Lu [24], and He and Li [25]. We can also refer to other works of Pachpatte in [26].

In 1998, by introducing parameters $\lambda \in (0,1]$ and $a, b \in \mathbb{R}_+ (a < b)$, Yang [27] gave an extension of (1.2) as follows:

$$\begin{aligned}\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy \\ < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\lambda/4}\right]\left\{\int_a^b x^{1-\lambda}f^2(x)dx\int_a^b x^{1-\lambda}g^2(y)dy\right\}^{1/2},\end{aligned} \tag{7.3}$$

where $B(u,v)$ is the beta function.

In 1999, Kuang [28] gave another extension of (1.2) as follows: for $\lambda \in (1/2, 1]$,

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda}dx\,dy < \frac{\pi}{\lambda\sin(\pi/2\lambda)}\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right\}^{1/2}. \tag{7.4}$$

We can refer to the other works of Kuang in [5, 29].

In 1999, using the methods of algebra and analysis, Gao [30] proved an improvement of (1.2) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}dxdy < \pi\sqrt{1-R}\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right\}^{1/2}, \tag{7.5}$$

where $\|f\| = \int_0^\infty f^2(x)dx^{1/2}, R = 1/\pi((u/\|g\|) - (v/\|f\|))^2$, $u = \sqrt{(2/\pi)}(g,e)$, $v = \sqrt{2\pi}(f, e^{-x})$, $e(y) = \int_0^\infty (e^x/(x+y))dx$. We also refer to works of Gao and Hsu in [31].

In 2002, using the operator theory, Zhang [32] gave an improvement of (1.2) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} dx\, dy \le \frac{\pi}{\sqrt{2}} \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx + \left(\int_0^\infty f(x)g(x)dx \right)^2 \right\}^{1/2}. \tag{7.6}$$

8. On the Way of Weight Coefficient for Giving a Strengthened Version of Hilbert's Inequality

In 1991, for making an improvement of (1.1), Hsu and Wang [33] raised the way of weight coefficient as follows: at first, using Cauchy's inequality in the left-hand side of (1.1), it follows that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m+n} \left[\left(\frac{m}{n} \right)^{1/4} a_m \right] \left[\left(\frac{n}{m} \right)^{1/4} b_n \right] \\ &\le \left\{ \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1/2} \right] a_m^2 \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1/2} \right] b_n^2 \right\}^{1/2}. \end{aligned} \tag{8.1}$$

Then, we define the weight coefficient

$$\omega(n) = \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1/2}, \quad n \in \mathbb{N}, \tag{8.2}$$

and rewrite (8.1) as follows:

$$I \le \left\{ \sum_{m=1}^{\infty} \omega(m) a_m^2 \sum_{n=1}^{\infty} \omega(n) b_n^2 \right\}^{1/2}. \tag{8.3}$$

Setting

$$\omega(n) = \pi - \frac{\theta(n)}{n^{1/2}}, \quad n \in \mathbb{N}, \tag{8.4}$$

where $\theta(n) = (\pi - \omega(n)) n^{1/2}$, and estimating the series of $\theta(n)$, it follows that

$$\theta(n) = \left[\pi - \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1/2} \right] n^{1/2} > \theta = 1.1213^{+}. \tag{8.5}$$

Thus, result (8.4) yields

$$\omega(n) < \pi - \frac{\theta}{n^{1/2}}, \quad n \in \mathbb{N},\ \theta = 1.1213^{+}. \tag{8.6}$$

In view of (8.3), a strengthened version of (1.1) is given by

$$I < \left\{ \sum_{n=1}^{\infty} \left(\pi - \frac{\theta}{n^{1/2}} \right) a_n^2 \sum_{n=1}^{\infty} \left(\pi - \frac{\theta}{n^{1/2}} \right) b_n^2 \right\}^{1/2}. \tag{8.7}$$

Hsu and Wang [33] also raised an open question how to obtain the best value θ of (8.7). In 1992, Gao [34] gave the best value $\theta = \theta_0 = 1.281669^{+}$.

Xu and Gao [35] proved a strengthened version of (2.6) given by

$$\begin{aligned} I < {} & \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{p-1}{n^{1/p} + n^{-1/q}} \right] a_n^p \right\}^{1/p} \\ & \times \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{q-1}{n^{1/q} + n^{-1/p}} \right] b_n^q \right\}^{1/q}. \end{aligned} \tag{8.8}$$

In 1997, using the way of weight coefficient and the improved Euler-Maclaurin's summation formula, Yang and Gao [36, 37] showed that

$$I < \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1/p}} \right] a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \frac{1-\gamma}{n^{1/q}} \right] b_n^q \right\}^{1/q}, \tag{8.9}$$

where $1 - \gamma = 0.42278433^{+}$ (γ is the Euler constant).

In 1998, Yang and Debnath [38] gave another strengthened version of (2.6), which is an improvement of (8.8). We can also refer to some strengthened versions of (3.2) and (4.3) in papers of Yang [39] and Yang and Debnath [40].

9. Hilbert's Inequality with Independent Parameters

In 1998, using the optimized weight coefficients and introducing an independent parameter $\lambda \in (0, 1]$, Yang [27] provided an extension of (1.2) as follows.

If $0 < \int_0^\infty x^{1-\lambda} f^2(x)dx < \infty$ and $0 < \int_0^\infty x^{1-\lambda} g^2(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\, dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx \right\}^{1/2}, \tag{9.1}$$

where the constant factor $B(\lambda/2,\ \lambda/2)$ is the best possible. The proof of the best possible property of the constant factor was given by Yang [41], and the expressions of the beta function $B(u,v)$ are given in Wang and Guo [42]:

$$\begin{aligned} B(u,v) &= \int_0^\infty \frac{t^{u-1}dt}{(1+t)^{u+v}} = \int_0^1 (1-t)^{u-1}t^{v-1}dt \\ &= \int_1^\infty \frac{(t-1)^{u-1}dt}{t^{u+v}}, \quad (u,v>0). \end{aligned} \tag{9.2}$$

Some extensions of (4.1), (4.3), and (4.5) were given by Yang and Debnath [43–45] as follows. If $\lambda > 2-\min\{p,q\}$, then

$$\begin{aligned} \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy &< B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \\ &\quad\times \left\{\int_0^\infty x^{1-\lambda}f^p(x)dx\right\}^{1/p}\left\{\int_0^\infty x^{1-\lambda}g^q(x)dx\right\}^{1/q}. \end{aligned} \tag{9.3}$$

If $2-\min\{p,q\} < \lambda \le 2$, then

$$\begin{aligned} \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{(m+n)^\lambda} &< B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \\ &\quad\times \left\{\sum_{n=1}^\infty n^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^\infty n^{1-\lambda}b_n^q\right\}^{1/q}, \end{aligned} \tag{9.4}$$

$$\begin{aligned} \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{a_m b_n}{(m+n+1)^\lambda} &< B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \\ &\quad\times \left\{\sum_{n=0}^\infty \left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^\infty \left(n+\frac{1}{2}\right)^{1-\lambda}b_n^q\right\}^{1/q}, \end{aligned} \tag{9.5}$$

where the constant factor $B((p+\lambda-2)/p, (q+\lambda-2)/q)$ is the best possible. Yang [46] also proved that (9.4) is valid for $p = 2$ and $\lambda \in (0,4]$. Yang [47, 48] gave another extensions of (4.1) and (4.3) as follows: if $0 < \lambda \le \min\{p,q\}$, then

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda\sin(\pi/p)}\left\{\sum_{n=1}^\infty n^{(p-1)(1-\lambda)}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^\infty n^{(q-1)(1-\lambda)}b_n^q\right\}^{1/q} \tag{9.6}$$

and if $0 < \lambda \le 1$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+(1/2))^{\lambda}+(n+(1/2))^{\lambda}} < \frac{\pi}{\lambda\sin(\pi/p)}\left\{\sum_{n=0}^{\infty}(n+(1/2))^{p-1-\lambda}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+(1/2))^{q-1-\lambda}b_n^q\right\}^{1/q}. \tag{9.7}$$

In 2004, Yang [49] proved the following dual form of (4.1):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}n^{p-2}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q-2}b_n^q\right\}^{1/q}. \tag{9.8}$$

Inequality (9.8) reduces to (4.1) when $p = q = 2$. For $\lambda = 1$, (9.7) reduces to the dual form of (4.3) as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=0}^{\infty}(n+(1/2))^{p-2}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+(1/2))^{q-2}b_n^q\right\}^{1/q}. \tag{9.9}$$

We can find some extensions of the H-L-P inequalities with the best constant factors such as (5.5)–(5.16) (see [13, 50, 51]) by introducing some independent parameters.

In 2001, by introducing some parameters, Hong [52] gave a multiple integral inequality, which is an extension of (4.1). He et al. [53] gave a similar result for particular conjugate exponents. For making an improvement of their works, Yang [54] gave the following inequality, which is a best extension of (4.1): if $n \in \mathbf{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^{n} 1/p_i = 1$, $\lambda > n - \min_{1\le i\le n}\{p_i\}$, $f_i(t) \ge 0$ and $0 < \int_0^{\infty} t^{n-1-\lambda} f_i^{p_i}(t)dt < \infty$ $(i = 1, 2, \ldots, n)$, then we have

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{i=1}^{n}f_i(x_i)}{\left(\sum_{i=1}^{n}x_i\right)^{\lambda}}dx_1\cdots dx_n < \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\left(\frac{p_i+\lambda-n}{p_i}\right)\left\{\int_0^{\infty}t^{n-1-\lambda}f_i^{p_i}(t)dt\right\}^{1/p_i}, \tag{9.10}$$

where the constant factor $1/\Gamma(\lambda)\prod_{i=1}^{n}((p_i+\lambda-n)/p_i)$ is the best possible. In particular, for $\lambda = n-1$, it follows that

$$\int_0^{\infty}\cdots\int_0^{\infty}\frac{\prod_{i=1}^{n}f_i(x_i)}{\left(\sum_{i=1}^{n}x_i\right)^{n-1}}dx_1\cdots dx_n < \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\left(1-\frac{1}{p_i}\right)\left\{\int_0^{\infty}f_i^{p_i}(t)dt\right\}^{1/p_i}. \tag{9.11}$$

In 2003, Yang and Rassias [55] introduced the way of weight coefficient and considered its applications to Hilbert-type inequalities. They summarized how to use the way of weight coefficient to obtain some new improvements and generalizations of the Hilbert-type inequalities. Since then, a number of authors discussed this problem (see [56–77]). But how to give a uniform extension of inequalities (9.8) and (4.1) with a best possible constant factor was solved in 2004 by introducing two pairs of conjugate exponents.

10. Hilbert-Type Inequalities with Multiparameters

In 2004, by introducing an independent parameter $\lambda > 0$ and two pairs of conjugate exponents (p, q) and (r, s) with $(1/p) + (1/q) = (1/r) + (1/s) = 1$, Yang [78] gave an extension of (1.2) as follows: if $p,\ r > 1$ and the integrals of the right-hand side are positive, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx\, dy < \frac{\pi}{\lambda \sin(\pi/r)} \left\{ \int_0^\infty x^{p(1-(\lambda/r))-1} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty x^{q(1-(\lambda/s))-1} g^q(x) dx \right\}^{1/q}, \tag{10.1}$$

where the constant factor $\pi/\lambda \sin(\pi/r)$ is the best possible.

For $\lambda = 1$, $r = q$, $s = p$, inequality (10.1) reduces to (4.5); for $\lambda = 1$, $r = p$, $s = q$, inequality (10.1) reduces to the dual form of (4.5) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y} dx\, dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty x^{p-2} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty x^{q-2} g^q(x) dx \right\}^{1/q}. \tag{10.2}$$

In 2005, by introducing an independent parameter $\lambda > 0$, and two pairs of generalized conjugate exponents $(p_1, p_2, \dots, p_n)$ and $(r_1, r_2, \dots, r_n)$ with $\sum_{i=1}^n (1/p_i) = \sum_{i=1}^n (1/r_i) = 1$, Yang et al. [79] gave a multiple integral inequality as follows:

for $p_i, r_i > 1\ (i = 1, 2, \dots, n)$,

$$\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{\left(\sum_{i=1}^n x_i\right)^\lambda} dx_1 \cdots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right) \left\{ \int_0^\infty t^{p_i(1-(\lambda/r_i))-1} f_i^{p_i}(t) dt \right\}^{1/p_i}, \tag{10.3}$$

where the constant factor $1/\Gamma(\lambda) \prod_{i=1}^n (\lambda/r_i)$ is the best possible. For $n = 2$, $p_1 = p$, $p_2 = q$, $r_1 = r$, and $r_2 = s$, inequality (10.3) reduces to the following:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y)^\lambda} dx\, dy < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{ \int_0^\infty x^{p(1-(\lambda/r))-1} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty x^{q(1-(\lambda/s))-1} g^q(x) dx \right\}^{1/q}. \tag{10.4}$$

It is obvious that inequality (10.4) is another best extension of (4.5).

In 2006, using two pairs of conjugate exponents (p, q) and (r, s) with $p, r > 1$, Hong [80] gave a multivariable integral inequality as follows.

If $\mathbb{R}_+^n = \{x = (x_1, x_2, \ldots, x_n);\ \ x_i > 0,\ i = 1, 2, \ldots, n\}$, α, β, $\lambda > 0$, $\|x\|_\alpha = (\sum_{i=1}^n x_i^\alpha)^{1/\alpha}$, f, $g \geq 0$, $0 < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{p(n-(\beta\lambda/r))-n} f^p(x)dx < \infty$ and $0 < \int_{\mathbb{R}_+^n} \|x\|_\alpha^{q(n-(\beta\lambda/s))-n} g^q(x)dx < \infty$, then

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \frac{f(x)g(y)dx\,dy}{\left(\|x\|_\alpha^\beta + \|y\|_\alpha^\beta\right)^\lambda} < \frac{\Gamma^n(1/\alpha)}{\beta\alpha^{n-1}\Gamma(n/\alpha)} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{\int_{\mathbb{R}_+^n} \|x\|_\alpha^{p(n-(\beta\lambda/r))-n} f^p(x)dx\right\}^{1/p} \times \left\{\int_{\mathbb{R}_+^n} \|x\|_\alpha^{q(n-(\beta\lambda/s))-n} g^q(x)dx\right\}^{1/q}, \tag{10.5}$$

where the constant factor $\Gamma^n(1/\alpha)/(\beta\alpha^{n-1}\Gamma(n/\alpha))B(\lambda/r, \lambda/S)$ is the best possible. In particular, for $n = 1$, (10.5) reduces to Hong's work in [81]; for $n = \beta = 1$, (10.5) reduces to (10.4). In 2007, Zhong and Yang [82] generalized (10.5) to a general homogeneous kernel and proposed the reversion.

We can find another inequality with two parameters as follows (see Yang [83]):

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} < \frac{1}{\alpha} B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{\sum_{n=1}^\infty n^{p(1-(\alpha\lambda/r))-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^\infty n^{q(1-(\alpha\lambda/s))-1} b_n^q\right\}^{1/q}, \tag{10.6}$$

where α, $\lambda > 0$, $\alpha\lambda \leq \min\{r, s\}$. In particular, for $\alpha = 1$, we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m + n)^\lambda} < B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{\sum_{n=1}^\infty n^{p(1-(\lambda/r))-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^\infty n^{q(1-(\lambda/s))-1} b_n^q\right\}^{1/q}. \tag{10.7}$$

For $\lambda = 1$, $r = q$, inequality (10.7) reduces to (4.1), and for $\lambda = 1$, $r = p$, (10.7) reduces to (9.8). Also we can obtain the reverse form as follows (see Yang [84]):

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{(m + n + 1)^2} > 2\left\{\sum_{n=0}^\infty \left[1 - \frac{1}{4(n+1)^2}\right] \frac{a_n^p}{2n+1}\right\}^{1/p} \left\{\sum_{n=0}^\infty \frac{b_n^q}{2n+1}\right\}^{1/q}, \tag{10.8}$$

where $0 < p < 1$, $(1/p) + (1/q) = 1$. The other results on the reverse of the Hilbert-type inequalities are found in Xi [85] and Yang [86].

In 2006, Xin [87] gave a best extension of H-L-P integral inequality (5.10) as follows:

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y) f(x) g(y)}{x^\lambda - y^\lambda} dx\, dy < \left[\frac{\pi}{\sin(\pi/r)}\right]^2 \left\{\int_0^\infty x^{p(1-(\lambda/r))-1} f^p(x) dx\right\}^{1/p} \times \left\{\int_0^\infty x^{q(1-(\lambda/s))-1} g^q(x) dx\right\}^{1/q}. \tag{10.9}$$

In 2007, Zhong and Yang [88] gave an extension of another H-L-P integral inequality (5.5) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{\max\{x^\lambda, y^\lambda\}} dx\, dy < \frac{rs}{\lambda} \left\{\int_0^\infty x^{p(1-(\lambda/r))-1} f^p(x) dx\right\}^{1/p} \left\{\int_0^\infty x^{q(1-(\lambda/s))-1} g^q(x) dx\right\}^{1/q}. \tag{10.10}$$

Zhong and Yang [89] also gave the reverse form of (10.10).

Considering a particular kernel, Yang [90] proved

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(\sqrt{m} + \sqrt{n})\sqrt{\max\{m,n\}}} < 4 \ln 2 \left\{\sum_{n=1}^\infty n^{(p/2)-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^\infty n^{(q/2)-1} b_n^q\right\}^{1/q}. \tag{10.11}$$

Yang [91] also proved that

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m + an)^2 + n^2} < \left(\frac{\pi}{2} - \arctan a\right) \left\{\sum_{n=1}^\infty \frac{a_n^p}{n}\right\}^{1/p} \left\{\sum_{n=1}^\infty \frac{b_n^q}{n}\right\}^{1/q}, \quad (a \geq 0). \tag{10.12}$$

Using the residue theory, Yang [92] obtained the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x + ay)(x + by)(x + cy)} dx\, dy < k \left\{\int_0^\infty x^{(-p/2)-1} f^p(x) dx\right\}^{1/p} \left\{\int_0^\infty x^{(-q/2)-1} g^q(x) dx\right\}^{1/q}, \tag{10.13}$$

where $k = 1/((\sqrt{a}+\sqrt{b})(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{c}))(a, b, c > 0)$. The constant factors in the above new inequalities are all the best possible. Some other new results are proved by several authors (see [75, 93–97]).

11. Operator Expressions of Hilbert-Type Inequalities

Suppose that H is a separable Hilbert space and $T : H \to H$ is a bounded self-adjoint semipositive definite operator. In 2002, Zhang [32] proved the following inequality:

$$(a, Tb)^2 \leq \frac{\|T\|^2}{2}\left(\|a\|^2\|b\|^2 + (a,b)^2\right), \quad (a, b \in H), \tag{11.1}$$

where (a,b) is the inner product of a and b, and $\|a\| = \sqrt{(a,a)}$ is the norm of a. Since the Hilbert integral operator $\widetilde{T}$ defined by (2.5) satisfies the condition of (11.1) with $\|\widetilde{T}\| = \pi$, then inequality (1.2) may be improved as (7.6). Since the operator T_p defined by (4.7) (for $p = q = 2$) satisfies the condition of (11.1) (see Wilhelm [10]), we may improve (3.2) to the following form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sqrt{2}}\left\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2 + \left(\sum_{n=0}^{\infty}a_n b_n\right)^2\right\}^{1/2}. \tag{11.2}$$

The key of applying (11.1) is to obtain the norm of the operator and and to show the semidefinite property. Now, we consider the concept and the properties of Hilbert-type integral operator as follows.

Suppose that $p > 1$, $(1/p) + (1/q) = 1$, $L^r(\mathbb{R}_+)$ $(r = p, q)$ are real normal linear spaces and $k(x,y)$ is a nonnegative symmetric measurable function in $\mathbb{R}_+^2$ satisfying

$$\int_0^{\infty} k(x,t)\left(\frac{x}{t}\right)^{1/r} dt = k_0(p) \in \mathbb{R}, \quad (x > 0). \tag{11.3}$$

We define an integral operator as

$$T : L^r(\mathbb{R}_+) \longrightarrow L^r(\mathbb{R}_+), \quad (r = p, q), \tag{11.4}$$

for any $f(\geq 0) \in L^p(\mathbb{R}_+)$, there exists $h = Tf \in L^p(\mathbb{R}_+)$, such that

$$(Tf)(y) = h(y) := \int_0^{\infty} k(x,y) f(x)dx, \quad (y > 0). \tag{11.5}$$

Or, for any $g(\geq 0) \in L^q(\mathbb{R}_+)$, there exists $\widetilde{h} = Tg \in L^q(\mathbb{R}_+)$, such that

$$(Tg)(x) = \widetilde{h}(x) := \int_0^{\infty} k(x,y) g(y)dy, \quad (x > 0). \tag{11.6}$$

In 2006, Yang [98] proved that the operator T defined by (11.5) or (11.6) are bounded with $\|T\| \leq k_0(p)$. The following are some results in this paper: If $\varepsilon > 0$ is small enough and the integral $\int_0^{\infty} k(x,t)(x/t)^{(1+\varepsilon)/r} dt (r = p, q; x > 0)$ is convergent to a constant $k_\varepsilon(p)$ independent of x satisfying $k_\varepsilon(p) = k_0(p) + o(1)(\varepsilon \to 0^+)$, then $\|T\| = k_0(p)$. If $\|T\| > 0$, $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$, $\|f\|_p, \|g\|_q > 0$, then we have the following equivalent inequalities:

$$(Tf, g) < \|T\| \cdot \|f\|_p \|g\|_q, \tag{11.7}$$

$$\|Tf\|_p < \|T\| \cdot \|f\|_p. \tag{11.8}$$

Some particular cases are considered in this paper.

Yang [99] also considered some properties of Hilbert-type integral operator (for $p = q = 2$). For the homogeneous kernel of degree -1, Yang [100] found some sufficient conditions to obtain $\|T\| = k_0(p)$. We can see some properties of the discrete Hilbert-type operator in the discrete space in Yang [101–104]. Recently, Bényi and Oh [105] proved some new results concerning best constants for certain multilinear integral operators. In 2009, Yang [106] summarized the above part results. Some other works about Hilbert-type operators and inequalities with the general homogeneous kernel and multiparameters were provided by several other authors (see [107–114]).

12. Some Basic Hilbert-Type Inequalities

If the Hilbert-type inequality relates to a single symmetric homogeneous kernel of degree -1 (such as $1/(x+y)$ or $|\ln(x/y)|/(x+y)$) and the best constant factor is a more brief form, which does not relate to any conjugate exponents (such as (1.2)), then we call it basic Hilbert-type integral inequality. Its series analogue (if exists) is also called basic Hilbert-type inequality. If the simple homogeneous kernel is of degree $-\lambda(\lambda > 0)$ with a parameter λ and the inequality cannot be obtained by a simple transform to a basic Hilbert-type integral inequality, then we call it a basic Hilbert-type integral inequality with a parameter.

For examples, we call the following integral inequality, that is, (1.2) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \pi \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \tag{12.1}$$

and the following H-L-P inequalities (for $p = 2$ in (5.5) and (5.10)):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dx\, dy < 4 \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \tag{12.2}$$

$$\int_0^\infty \int_0^\infty \frac{\ln(x/y) f(x)g(y)}{x-y} dx\, dy < \pi^2 \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2} \tag{12.3}$$

basic Hilbert-type integral inequalities. In 2006, Yang [115] gave the following basic Hilbert-type integral inequality:

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)| f(x)g(y)}{\max\{x,y\}} dx\, dy < 8 \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}. \tag{12.4}$$

In 2011, Yang [116] gave the following basic Hilbert-type integral inequalities:

$$\int_0^\infty \int_0^\infty \frac{\left|\ln(x/y)\right| f(x)g(y)}{x+y} dx\,dy < c_0 \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{12.5}$$

$$\int_0^\infty \int_0^\infty \frac{\left\{\arctan\sqrt{x/y}\right\} f(x)g(y)}{x+y} dx\,dy < \frac{\pi^2}{4} \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{12.6}$$

where $c_0 = 8\sum_{n=1}^{\infty}(-1)^n/(2n-1)^2 = 7.3277^+$.

In 2005, Yang [115, 117–120] gave a basic Hilbert-type integral inequality with a parameter $\lambda \in (0,1)$:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\left|x-y\right|^\lambda} dx\,dy \\ &\quad < 2B\left(1-\lambda, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx \right\}^{1/2}. \end{aligned} \tag{12.7}$$

Similar to discrete inequality (3.7), the following inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx\,dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)dx \int_0^\infty x^{1-\lambda} g^2(x)dx \right\}^{1/2} \tag{12.8}$$

is called *basic Hilbert-type integral inequality* with a parameter $\lambda \in (0,\infty)$.

Also we find the following basic Hilbert-type inequalities:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right\}^{1/2}, \tag{12.9}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < 4 \left\{ \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right\}^{1/2}, \tag{12.10}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\ln(m/n) a_m b_n}{m-n} < \pi^2 \left\{ \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right\}^{1/2}. \tag{12.11}$$

It follows from (5.15) with $p = q = 2$ that

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{|\ln(m/n)| a_m b_n}{\max\{m,n\}} < 8 \left\{ \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right\}^{1/2}. \tag{12.12}$$

In 2010, Xin and Yang [20] proved the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|a_m b_n}{m+n} < c_0\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}, \tag{12.13}$$

where $c_0 = 8\sum_{n=1}^{\infty}(-1)^n/(2n-1)^2 = 7.3277^{+}$. Inequalities (12.12) and (12.13) are new basic Hilbert-type inequalities. We still have a basic Hilbert-type inequality with a parameter $\lambda \in (0,4]$ as follows (see Yang [47]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\sum_{n=1}^{\infty}n^{1-\lambda}a_n^2\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right\}^{1/2}. \tag{12.14}$$

13. Some Recent Work with Applications

In recent years, Pogány et al. [121–125] made some new important contributions to discrete Hilbert-type inequalities with nonhomogeneous kernels using special functions. On the other hand, in 2006–2011, Xie et al. [75, 95–97, 126, 127] have investigated many Hilbert-type integral inequalities with the particular kernels such as $|x+y|^{-\lambda}$ similar to inequality (4.15). In 2009–2012, Yang [128, 129] considered the compositions of two discrete Hilbert-type operators with two conjugate exponents and kernels $(m+n)$, and in 2010, Liu and Yang [108] also studied the compositions of two Hilbert-type integral operators with the general homogeneous kernel of negative degree and obtained some new results with applications. Hardy et al. [3] proved some results in Theorem 351 without any proof of the constant factors as best possible. Yang [130] introduced an interval variable to prove that the constant factor is the best possible. In 2011, Yang [131] proved the following half-discrete Hilbert-type inequalities with the best possible constant factor $B(\lambda_1,\lambda_2)$:

$$\int_0^{\infty} f(x)\left(\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda}}\right)dx < B(\lambda_1,\lambda_2)\|f\|_{p,\phi}\|a\|_{q,\psi}, \tag{13.1}$$

where $\lambda_1\lambda_2 > 0$, $0 \le \lambda_2 \le 1$, $\lambda_1+\lambda_2=\lambda$,

$$\begin{aligned}\|f\|_{p,\phi} &= \left\{\int_0^{\infty}\varphi(x)f^p(x)dx\right\}^{1/p} > 0,\\ \|a\|_{q,\psi} &= \left\{\sum_{n=1}^{\infty}a_n^q\right\}^{1/q} > 0, \quad \varphi(x) = x^{p(1-\lambda_1)-1},\end{aligned} \tag{13.2}$$

$\psi(n) = n^{q(1-\lambda_2)-1}$. Zhong et al. [15, 82, 88, 89, 132] have investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using the way of weight functions and the techniques of discrete and integral-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbb{R}$ is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_\lambda(x, n) a_n dx < k(\lambda_1) \|f\|_{p,\phi} \|a\|_{q,\psi}, \tag{13.3}$$

where $k(\lambda_1) = \int_0^{\infty} k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbb{R}_+$. This is an extension of the above particular result with the best constant factor $k(\lambda_1)$ (see Yang and Chen [133]). If the corresponding integral inequality of a half-discrete inequality is a basic Hilbert-type integral inequality, then we call it the basic half-discrete Hilbert-type inequality. Substituting some particular kernels in the main result found in [133] leads to some basic half-discrete Hilbert-type inequalities as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{x+n} dx < \frac{\pi}{2} \|f\|_2 \|a\|_2, \tag{13.4}$$

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{\ln(x/n) a_n}{x-n} dx < \left(\frac{\pi}{2}\right)^2 \|f\|_2 \|a\|_2, \tag{13.5}$$

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{\max\{x, n\}} dx < 4 \|f\|_2 \|a\|_2, \tag{13.6}$$

$$\int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{|\ln(x/n)| a_n}{\max\{x, n\}} dx < 16 \|f\|_2 \|a\|_2, \tag{13.7}$$

$$\int_1^{\infty} f(x) \sum_{n=1}^{\infty} \frac{|\ln(x/n)| a_n}{x+n} dx < c_0 \|f\|_2 \|a\|_2, \tag{13.8}$$

where $c_0 = 8 \sum_{n=1}^{\infty} (-1)^n / (2n-1)^2 = 7.3277^+$.

14. Concluding Remarks

(1) Many different kinds of Hilbert-type discrete and integral inequalities with applications are presented in this paper with references. Special attention is given to new results proved during 2009–2012. Included are many generalizations, extensions and refinements of Hilbert-type discrete and integral inequalities involving many special functions such as beta, gamma, hypergeometric, trigonometric, hyperbolic, zeta, Bernoulli's functions and Bernoulli's numbers, and Euler's constant.

(2) In his three books, Yang [116, 118, 119] presented many new results on integral and discrete-type operators with general homogeneous kernels of degree of real numbers and two pairs of conjugate exponents as well as the related inequalities. These research monographs contained recent developments of both discrete and integral types of operators and inequalities with proofs, examples, and applications.

Acknowledgments

The authors express their grateful thanks to Dr. Tibor Pogány and referees for many suggestions with recent references for improvement of the exposition of this paper. The work of the second author is supported by Guangdong Natural Science Foundation Grant no. 7004344.

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