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# THE DUAL SPACE OF $\beta(\Gamma)$

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**ABSTRACT.** The space of Boehmians with $\Delta$–convergence is a complete topological vector space in which the topology is induced by an invariant metric. We show that the dual space of the space of periodic Boehmians can be identified with the class of trigonometric polynomials.

**KEY WORDS AND PHRASES:** Boehmian, dual space, generalized function.
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## 1. INTRODUCTION

In this note we will consider a class $\beta$ of generalized functions known as Boehmians. The class of Boehmians is a generalization of regular operators, Schwartz distributions, and other spaces of generalized functions [1].

Mikusiński [2] has shown that $\beta$ with $\Delta$–convergence is an F–space (i.e. a complete topological vector space, not necessarily convex, such that the topology is given by an invariant metric). Hence, a natural question arises; can we characterize $\beta'$ the class of all continuous linear functionals on $\beta$?

T. K. Boehme proved (unpublished) that the class $\beta'(\mathcal{R})$ of all continuous linear functionals on $\beta(\mathcal{R})$ (Boehmians on the real line) consists only of the trivial linear functional. That is, there are no nontrivial continuous linear functionals on $\beta(\mathcal{R})$. This is not the case for $\beta'(\Gamma)$ (the dual space of the class of periodic Boehmians). Indeed, there are enough continuous linear functionals to separate points.

In this note, we will show that $\beta'(\Gamma)$ can be identified with the class of all trigonometric polynomials.

Let $\mathcal{A}$ denote the class of trigonometric polynomials. That is, $p\in\mathcal{A}$ if for some $m\in N$,

$$p(x) = \sum_{n=-m}^{m}\alpha_n e^{inx}, \text{ where } \alpha_n\in\mathcal{C} \text{ for } n=0, \pm1, \dots, \pm m.$$

**THEOREM 1.1.** For any $T\in\beta'(\Gamma)$, there exists a unique $p(x) = \sum_{n=-m}^{m}\alpha_n e^{inx} \in \mathcal{A}$ (for some $m\in N$) such that

$$T(F) = \sum_{n=-m}^{m}\alpha_n \hat{F}(n), \text{ for all } F\in\beta(\Gamma). \tag{1.1}$$

Conversely, any $p(x) = \sum_{n=-m}^{m} \alpha_n e^{inx} \in \mathcal{A}$ defines a continuous linear functional on $\beta(\Gamma)$ via (1.1).

## 2. PRELIMINARIES

We use the same notation as in [3]. For other results concerning $\beta(\Gamma)$ see [4,5]. In [3] the construction of $\beta(\Gamma)$ differs from the construction in [4,5], but the space $\beta(\Gamma)$ in both cases can be shown to be the same.

The collection of all continuous complex-valued functions on $\mathcal{R}$ will be denoted by $C(\mathcal{R})$. The support of a continuous function f, denoted by supp f, is the complement of the largest open set on which f is zero.

The *convolution* of two continuous functions, where at least one has compact support, is given by $(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$.

A sequence of continuous nonnegative functions $\{\delta_n\}$ will be called a *delta sequence* if (i) $\int_{-\infty}^{\infty} \delta_n(x)\,dx = 1$ for n=1, 2, . . . , and (ii) supp $\delta_n \subset (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$.

A pair of sequences $(f_n, \delta_n)$ is called a *quotient of sequences*, and denoted by $f_n/\delta_n$, if $f_n \in C(\mathcal{R})$ (n=1, 2, . . .), $\{\delta_n\}$ is a delta sequence, and $f_n*\delta_m = f_m*\delta_n$ for all m and n. Two quotients of sequences $f_n/\delta_n$ and $g_n/\sigma_n$ are equivalent if $f_n*\sigma_m = g_m*\delta_n$ for all m and n. The equivalence classes are called *Boehmians*. The space of all Boehmians will be denoted by $\beta(\mathcal{R})$, and a typical element of $\beta(\mathcal{R})$ will be written as $F = f_n/\delta_n$. By defining a natural addition and scalar multiplication on $\beta(\mathcal{R})$, i.e. $f_n/\delta_n + g_n/\sigma_n = (f_n*\sigma_n + g_n*\delta_n)/(\delta_n*\sigma_n)$ and $\alpha(f_n/\delta_n) = \alpha f_n/\delta_n$, where $\alpha$ is a complex number, $\beta(\mathcal{R})$ becomes a vector space.

A sequence $\{F_n\}$ of Boehmians is said to be *Δ-convergent* to F, denoted by $\Delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\delta_n\}$ such that for each n, $(F_n - F)*\delta_n \in C(\mathcal{R})$ and $(F_n - F)*\delta_n \to 0$ uniformly on compact sets as $n \to \infty$.

If either f is a periodic function of period $2\pi$ or supp $f \subset (-\pi, \pi)$, the $n^{th}$ *Fourier coefficient* $\hat{f}(n)$ of f is defined as $\hat{f}(n) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$, for n=0, ±1, ±2, . . . By a simple calculation we see that $(f*\delta)^{\wedge}(n) = 2\pi\hat{f}(n)\hat{\delta}(n)$ for all n.

Let $\beta(\Gamma) = \{F \in \beta(\mathcal{R}) : F = \sum_{-\infty}^{\infty} a_n e^{inx}$, for some sequence $\{a_n\}$ of complex numbers$\}$.

That is $F = \Delta\text{-}\lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^{ikx}$. For $F \in \beta(\Gamma)$ such that $F = \sum_{-\infty}^{\infty} a_n e^{inx}$, the $n^{th}$ *Fourier coefficient* of F, denoted by $\hat{F}(n)$, is $\hat{F}(n) = a_n$.

The proof of the next proposition is straightforward and hence is left to the reader.

**PROPOSITION 2.1.** Suppose that $F_n \in \beta(\Gamma)$ for n=1, 2, . . . If $\Delta\text{-}\lim_{n\to\infty} F_n = F$, then $\lim_{n\to\infty} \hat{F}_n(k) = \hat{F}(k)$ for each k.

**PROPOSITION 2.2** [3, Lemma 3.5]. Let $\{n_k\}$ be a subsequence of positive integers such that $\sum_{k=1}^{\infty} 1/n_k < \infty$. If $\{a_n\}$ is any sequence of complex numbers such that $a_n = 0$ for $n \neq n_k$ (k=1, 2, . . .), then there is a Boehmian $F \in \beta(\Gamma)$ such that $\hat{F}(n) = a_n$ for all n.

The conclusion of Proposition 2.2 is also valid for a subsequence $\{n_k\}$ of negative integers such that $\sum_{k=1}^{\infty} 1/n_k > -\infty$.

## 3. PROOF OF THE THEOREM

Suppose that T has the form of (1.1), then clearly T is linear. The continuity of T follows from Proposition 2.1.

For the other direction, suppose that T is any continuous linear functional on $\beta(\Gamma)$.

Let $F = \sum_{-\infty}^{\infty} a_n e^{inx} \in \beta(\Gamma)$. Then, $T(F) = \sum_{-\infty}^{\infty} a_n T(e^{inx})$.

Hence to finish the proof, it suffices to show that for only a finite number of n's, $T(e^{inx})$ does not vanish.

Suppose that this is not the case. Thus, there are infinitely many positive (negative) n's such that $T(e^{inx})$ does not vanish.

Now, pick a subsequence $\{n_k\}$ from the above n's such that $n_k \geq 2^k$, for $k \in N$.

Let $\alpha_{n_k} = (T(e^{in_k x}))^{-1}$, for $k \in N$.

By Proposition 2.2, there there exists a $G \in \beta(\Gamma)$ such that $\hat{G}(n_k) = \alpha_{n_k}$ for k=1, 2, . . . and zero otherwise. But this gives $T(G) = \sum_{-\infty}^{\infty} \hat{G}(n) T(e^{inx}) = \infty$.

Thus, for only a finite number of n's, $T(e^{inx})$ must not vanish and the proof is complete.

## 4. SOME CONCLUDING REMARKS

The dual space of $\beta(\Gamma)$ can be identified with spaces other than the space of trigonometric polynomials, but these spaces are all isomorphic. For example, we could have identified $\beta'(\Gamma)$ with the subspace of $C^Z$ of finite sequences.

It is known that $\beta(\Gamma)$ with $\Delta$–convergence is not a Banach space. Can the characterization of $\beta'(\Gamma)$ in this paper be used to determine whether or not $\beta(\Gamma)$ is a Fréchet space?

Does the strong convergence in $\beta'(\Gamma)$ have a nice description in terms of some convergence of trigonometric polynomials?

Let $\omega$ be a real–valued even function defined on the integers Z such that

$$0 = \omega(0) \leq \omega(n+m) \leq \omega(n) + \omega(m) \text{ for all } n,m \in Z \text{ and } \sum_{n=1}^{\infty} \frac{\omega(n)}{n^2} < \infty.$$

By using Theorems 3.1 and 3.2 in [5], we obtain the following interesting connection between weak convergence and $\Delta$–convergence in $\beta(\Gamma)$.

**COROLLARY 4.1.** Suppose that the set of positive integers is partitioned into two disjoint sets $\{t_n\}$ and $\{s_n\}$ such that $\sum_{n=1}^{\infty} 1/t_n < \infty$. Let $\{F_n\}$ be a sequence of Boehmians such that the set $\{e^{-\omega(s_k)} \hat{F}_n(\pm s_k)\}_{n=1}^{\infty}$ is uniformly bounded. Then a necessary and sufficient condition for the sequence $\{F_n\}$ to be $\Delta$–convergent to zero is that $\{F_n\}$ converges weakly to zero.

The space of periodic Beurling distributions $P_\omega'$ can be identified with a subspace of Boehmians. By using Corollary 4.1 and the results in [6], one can show that the convergence structure in $P_\omega'$ is stronger than the convergence $P_\omega'$ inherits from $\beta(\Gamma)$.

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