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LOCALIZATION AND SUMMABILITY OF MULTIPLE HERMITE SERIES

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ABSTRACT. The multiple Hermite series in $\mathbf{R}^n$ are investigated by the Riesz summability method of order $\alpha > (n-1)/2$. More precisely, localization theorems for some classes of functions are proved and sharp sufficient conditions are given. Thus the classical Szegö results are extended to the $n-$dimensional case. In particular, for these classes of functions the localization principle and summability on the Lebesgue set are established.

KEY WORDS AND PHRASES: Riesz summability, multiple Hermite series

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1 Statement of the main results

Let f be locally in $L^1(\mathbf{R}^n), n \geq 2$, and consider the multiple Hermite series

$$f(y) \sim \sum f_k e^{-y^2/2}\tilde{H}_k(y)\ ,\ f_k = \int_{\mathbf{R}^n} f(y)e^{-y^2/2}\tilde{H}_k(y)dy,$$

where $\tilde{H}_k(y) = \tilde{H}_{k_1}(y_1)...\tilde{H}_{k_n}(y_n), k = (k_1,..,k_n), k_i \geq 0, y = (y_1,..,y_n)$, is a product of the normalized Hermitian polynomials. Here and later on y^2 stands for the scalar product (y,y) in $\mathbf{R}^n$ and for simplicity we shall write xy instead of (x,y). The corresponding spherical partial sum has the form

$$E_\lambda f(y) = \sum_{\mu_k<\lambda} f_k\phi_k(y),$$

where $\mu_k = 2|k| + n$ and $\phi_k(x) = e^{-x^2/2}\tilde{H}_k(x)$ are the eigenvalues and orthonormalized eigenfunctions of the operator $A = -\Delta + x^2$ in $L^2(\mathbf{R}^n)$. Let

$$E_\lambda^\alpha f(y) = \sum_{\mu_k<\lambda}(1-\mu_k/\lambda)^\alpha f_k\phi_k(y)$$

be the corresponding Riesz means of order $\alpha > 0$.We shall prove the convergence

$$R_\lambda^\alpha \stackrel{\text{def}}{=} E_\lambda^\alpha f(y) - \int_{|x-y|<\delta} f(x)I^\alpha e(\lambda,x,y)dx = o(1), \tag{1.1}$$

as $\lambda \to +\infty$, locally uniformly with respect to y, where $\delta > 0$ and

$$I^\alpha e(\lambda,x,y) = \int_0^\lambda (1-\mu/\lambda)^\alpha de(\mu,x,y) \tag{1.2}$$

is the Riesz kernel, under some conditions at infinity for the function f, including a system of sharp sufficient conditions. Thus the classical Szegö results [18] are extended to the n-dimensional case. In particular, for these classes of functions the localization principle and summability on the Lebesgue set are established. For other results see, for example, [1]-[4], [6]-[8], [10]-[19] and the bibliography in [15], [19]. Here

$$e(\lambda, x, y) = \sum_{\mu_k<\lambda} \phi_k(x)\phi_k(y)$$

is the spectral function of A.

In stating the main results we use the following notations. Let $a(\lambda, x)$ be the characteristic function of the set $\{x \in \mathbf{R}^n : A < x^2 < \lambda - \lambda^{1/3}\}$ and $b(\lambda, x)$ - the characteristic function of the set $\{x \in \mathbf{R}^n : |x^2 - \lambda| < \lambda^{1/3+\epsilon}\}$ for some small $\epsilon > 0$ and large $A > 0$.

Theorem 1. If $\alpha > (n-1)/2$ and

$$\int_{\mathbf{R}^n} a(\lambda, x)(1 - x^2/\lambda)^{-1/4}|x|^{-(n+1)/2-\alpha}|f(x)|dx = o(\lambda^{\alpha/2-(n-1)/4}) \qquad (H_1)$$

$$\int_{\mathbf{R}^n} b(\lambda, x)|f(x)|dx = o(\lambda^{\alpha+1/3}), \qquad (H_2)$$

then the convergence relation (1.1) is fulfilled.

Remark 1. The condition (H_2) is exact. Namely, it is satisfied by the function $f(x) = |x|^\beta$, $\beta > 0$ for every $\beta < 2\alpha - n + 2$, but not for $\beta \geq 2\alpha - n + 2$. On the other hand, $R^\alpha_\lambda f(0)$ is divergent if $\beta \geq 2\alpha - n + 2, \alpha > (n-1)/2$.

For the functions which are differentiable at infinity we can improve the condition (H_1).

Theorem 2 Let the function f be differentiable at infinity and satisfy for $\alpha > (n-1)/2$ the condition $f(x) = O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 2\alpha - n + 2$ and

$$\int_{\mathbf{R}^n} a(\lambda, x)(1 - x^2/\lambda)^{-3/4}|x|^{-(n+1)/2-\alpha-1}|\nabla f(x)|dx = o(\lambda^{\alpha/2-(n-1)/4}). \qquad (H_1')$$

Then the convergence relation (1.1) is valid.

Corollary 1. Let the function f be differentiable at infinity and $f, \nabla f = O(|x|^\beta)$ as $|x| \to \infty$, where $\beta < 2\alpha - n + 2, \alpha > (n-1)/2$. Then the relation (1.1) is true.

It is natural to "interpolate" between conditions (H_1) and (H_1') . Define

$$\omega(x, f) = \sum_{i=1}^{n} sup_{0\leq h_i\leq 1}|f(x + H) - f(x + H_i)|,$$

where $H = (h_1, .., h_n), H_i = (h_1, .., h_{i-1}, 0, h_{i+1}, .., h_n)$.

Theorem 3. Let the function f satisfy for $\alpha > (n-1)/2$ the condition $f(x) = O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 2\alpha - n + 2$ and

$$\int_{\mathbf{R}^n} a(\lambda, x)(1 - x^2/\lambda)^{-3/4}|x|^{-(n+1)/2-\alpha}\omega(x, f)dx = o(\lambda^{\alpha/2-(n-1)/4}). \qquad (H_1'')$$

Then the convergence relation (1.1) is fulfilled.

Remark 2. The conditions of theorem 3 are satisfied by the function $f(x) = |x|^\beta, \beta > 0$, if $\beta < 2\alpha - n + 2$, and they are not satisfied if $\beta \geq 2\alpha - n + 2$. Therefore, according to remark 1, theorem 3 provides a system of sharp sufficient conditions.

Corollary 2 (localization principle). Let $y \in R^n, \delta > 0$ be fixed. Then under the conditions of theorems 1,2,3 respectively we have

$$E^\alpha_\lambda f(y) \to 0 \text{ if } f(x) = 0 \text{ for } |x - y| < \delta.$$

As a consequence of theorems 1,2,3,4 and corollary 4.16 [16] we obtain

Corollary 3. Under the conditions of theorems 1,2,3 respectively we have $E^\alpha_\lambda f(y) \to f(y)$ on the Lebesgue set of the function f.

The further organisation of the paper is as follows. The results about the asymptotics of the Riesz kernels are formulated in section 2, while the proofs are given in sections 7-10. These asymptotics are used to prove theorems 1-3 in sections 3-5 respectively. Finally, remark 1 is proved in section 6.

2 Asymptotics of Riesz kernels

Here we state the uniform asymptotics of the Riesz kernels which we need. Since

$$E_\lambda^\alpha f(y) = \int_{\mathbf{R}^n} I^\alpha e(\lambda, x, y) f(x) dx, \ \alpha > 0, \tag{2.1}$$

we have to find the asymptotics or bounds for the Riesz kernels $I^\alpha e(\lambda, x, y)$ as$\lambda \to \infty$, which must be uniform with respect to the parameters $x \in \boldsymbol{R}^n, y^2 < A$. It is convenient to consider also the functions

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y), \ E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda} x, \sqrt{\lambda} y). \tag{2.2}$$

Theorem 4. If $x^2 + y^2 < A$ and $\alpha \geq (n-1)/2$ then

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\lambda^{(n-1)/2} G_\alpha(\sqrt{\lambda}|x-y|), \tag{2.3}$$

where

$$G_\alpha(s) = (1+s)^{-(n+1)/2-\alpha}, s \geq 0, \ e^o(\lambda, x, y) = (2\pi)^{-n} \int_{\xi^2 \leq \lambda} e^{i(x-y)\xi} d\xi$$

and for $d_\alpha = (2\pi)^{-n/2} 2^\alpha \Gamma(\alpha+1)$,

$$I^\alpha e^0(\lambda, x, y) = \lambda^{n/2} F_\alpha(\sqrt{\lambda}|x-y|) \ , \ F_\alpha(s) = d_\alpha s^{-n/2-\alpha} J_{n/2+\alpha}(s).$$

Theorem 5. Let $A/\lambda < x^2 < 1-\delta, \ |y| < \epsilon |x|$ and $\alpha > 0$. Then for every small $\delta > 0, \epsilon > 0$ and $A > 0$ we have the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} b_k(\lambda, x, y, \alpha) e^{i\lambda\psi_k} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1}),$$

where

$$|b_k| \leq c|x|^{-(n+1)/2-\alpha}, \ |\nabla_x b_k| \leq c|x|^{-(n+1)/2-\alpha-1} \tag{2.4}$$

and

$$|\nabla \psi_k|^2 = 1 - x^2, \ |\Delta \psi_k|^2 \leq c(1-x^2)^{-1}. \tag{2.5}$$

Theorem 6. There exist $\delta > 0, \epsilon > 0$ such that the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y)\lambda^{-k-1/3} + b_{1k}(\lambda, x, y)\lambda^{-k-2/3}) \tag{2.6}$$

holds if $|x^2 - 1| < \delta, \ |y| < \epsilon |x|$, where

$$a_{1k} = (a_k e^{\lambda A} + b_k e^{-\lambda A}) Ai(\lambda^{2/3} B), \ b_{1k} = (c_k e^{\lambda A} + d_k e^{-\lambda A}) Ai'(\lambda^{2/3} B)$$

and the functions $\lambda \to a_k, \ b_k, \ c_k, \ d_k$, or their derivatives with respect to x are bounded. Here Ai is the Airy function and the smooth functions $A = A(x,y), B = B(x,y)$ have the following properties: $Re\, A = 0, \ Im\, B = 0$. Moreover, let $x = |x|\omega$ and

$$a^2 = (1-y^2)(1-(\omega y)^2)^{-1}. \tag{2.7}$$

Then

$$B(x,y) < 0 \text{ if } x^2 < a^2, \tag{2.8}$$

$$B(x,y) \sim c(y,\omega)(x^2 - a^2) \text{ as } x^2 \to a^2, \ c(y,\omega) > 0. \tag{2.9}$$

From theorem 6, the asymptotics of the Airy function and (2.8),(2.9) it follows

Corollary 4. There exists $\delta > 0$ such that

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} (a_k(a^2 - x^2)^{-1/4} + b_k(a^2 - x^2)^{1/4}) \exp i\lambda\psi_k + (a^2 - x^2)^{-1} O(\lambda^{-1}),$$

uniformly with respect to x, y if $1 - \delta < x^2 < 1 - \lambda^{-2/3+\epsilon}, y^2 < A/2\lambda$, where $\epsilon > 0, A > 0$ are fixed The functions $\lambda \longrightarrow a_k, b_k$ and their derivatives over x are bounded and ψ_k satisfy (2.5).

Theorem 7. Let $x^2 > 1 + \delta, y^2 < \epsilon$. Then we have the uniform estimate

$$|E_\alpha(\lambda, x, y)| \leq c(x^2 - 1)^{-1/4} \lambda^{-1/2} \exp(-c\delta\lambda(x^2 - 1)^{1/2}),$$

where c is a positive constant.

As a consequence of theorems 6 and 7 it follows

Corollary 5. If $x^2 > \lambda + \lambda^{1/3+\epsilon},\ \epsilon > 0,\ y^2 < A$, then

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp\left(-c\lambda^{1/3}(x^2/\lambda - 1)^{1/2}\right).$$

3 Proof of theorem 1

Let $y^2 < A/2, \delta > 0, \alpha > (n-1)/2, n \geq 2$, According to (1.1) and (2.1)

$$R_\lambda^\alpha f(y) = \int_{|x-y|>\delta} f(x) I^\alpha e(\lambda, x, y) dx.$$

From theorem 4 and the asymptotics of the Bessel functions it follows

$$R_\lambda^\alpha f(y) = \int_{x^2>A} f(x) I^\alpha e(\lambda, x, y) dx + o(1). \tag{3.1}$$

Therefore it is sufficient to prove the relations

$$K_j(\lambda, y) = \int_{\mathbf{R}^n} a_j(\lambda, x) f(x) I^\alpha e(\lambda, x, y) dx = o(1), \tag{3.2}$$

for $1 \leq j \leq 4, y^2 < A/2, \alpha > (n-1)/2, n \geq 2$, where $a_1(\lambda, x)$ is the characteristic function of the set $\{x \in \mathbf{R}^n : A < x^2 < \lambda(1-\delta)\}$, $a_2(\lambda, x)-$ the characteristic function of the set $\{x \in \mathbf{R}^n : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}\}$, $a_3(\lambda, x) = b(\lambda, x)$ and $a_4(\lambda, x)$ is the characteristic function of the set $\{x \in \mathbf{R}^n : x^2 > \lambda + \lambda^{1/3+\epsilon}\}$ for some small $\epsilon > 0, \delta > 0$.

a. Estimate of K_1. It is not hard to see that theorem 5 implies the bound

$$|I^\alpha e(\lambda, x, y)| \leq c(1 - x^2/\lambda)^{-1/4} |x|^{-(n+1)/2-\alpha} \lambda^{(n-1)/4-\alpha/2} \tag{3.3}$$

if $A < x^2 < (1-\delta)\lambda, y^2 < A/2, \alpha > 0$. So the hypothesis (H_1) gives (3.2) for K_1.

b. Estimate of K_2. Now we can use corollary 4. Since $a^2 - x^2/\lambda > (1 - x^2/\lambda)/2$ for large λ we see that the estimate (3.3) is fulfilled if $(1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}, y^2 < A/2$. Thus (H_1) shows (3.2) for K_2.

c. Estimate of K_3. From theorem 6 and (H_2) we get (3.2) for K_3.

d. Estimate of K_4. Using corollary 5 we obtain

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c\lambda^{\epsilon/2}) \text{ if } x^2 > \lambda + \lambda^{1/3+\epsilon} \tag{3.4}$$

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c|x|^{1/2}) \text{ if } x^2 > \lambda^2. \tag{3.5}$$

On the other hand (H_1) gives $\int_{|x|>1} |x|^{-N} |f(x)| dx < \infty$ for large N, so the last three estimates and (H_1) imply (3.2) for K_4. Theorem 1 is proved.

4 Proof of theorem 2

As in the proof of theorem 1 we have to estimate the integrals $K_j(\lambda, y)$ given by (3.2) for $y^2 < A/2$. It is clear that the estimate (3.2) for K_3 and K_4 are valid again. Thus it remains to bound K_1 and K_2. Consider also the integrals $(j = 1, 2)$:

$$B_j(\lambda, y) = \lambda^{n/2-\alpha} \int_{\mathbf{R}^n} a_j(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) E_\alpha(\lambda, x, y) dx.$$

a. Estimate of K_1. According to theorem 5 we have the following asymptotics for $\alpha > 0$

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^{4} b_k e^{i\lambda\psi_k} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1}),$$

uniformly in the domain $\{x, y \in \mathbf{R}^n : A/\lambda < x^2 < 1 - \delta,\ y^2 < A/2\lambda\}$, where b_k satisfy (2.4).

Using the estimate $f(x) = O(|x|^\beta), \beta > 0$ as $|x| \to \infty$, we obtain for $\alpha > (n-1)/2$:

$$B_1(\lambda, y) = \lambda^{(n-1)/2-\alpha} \sum_{k=1}^{4} \int_{\mathbf{R}^n} e^{i\lambda\psi_k} a_1 b_k f(\sqrt{\lambda}x) dx + \tag{4.1}$$

$$O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda + \lambda^{-1/2}).$$

Let $I(\lambda)$ be the integral in (4.1) together with the factor $\lambda^{(n-1)/2-\alpha}$. We shall integrate by parts using the operator L_k, where its transpose is given by $\sum \partial_j \psi_k |\nabla \psi_k|^{-2} \partial_j$, $1 \le j \le n$, and $\partial_j = \partial/\partial x_j$. Taking into account (2.5) we get

$$I(\lambda) = \lambda^{(n-1)/4-\alpha/2} \Big(\int_{\mathbf{R}^n} a_1(\lambda, x) |x|^{-(n+1)/2-\alpha-1} |\nabla f(x)| dx + \lambda^{-1/2} B \Big) + \tag{4.2}$$

$$O(\lambda^{-1/2} + \lambda^{\beta/2+n/2-\alpha-3/2}),$$

where

$$B = \int_{\mathbf{R}^n} a_1(\lambda, x) |x|^{-(n+1)/2-\alpha-1} |f(x)| dx = \tag{4.3}$$

$$O(\lambda^{\beta/2+(n-1)/4-\alpha/2-1/2} \log \lambda).$$

Since $\beta < 2\alpha + 2 - n$, (4.1)-(4.3) and (H_1') give $K_1 = o(1)$.

b. Estimate of K_2. Using corollary 4 and $1 - 2A/\lambda < a^2 < 1$ for $y^2 < A/2\lambda$ and large λ we obtain:

$$B_2(\lambda, y) = \lambda^{(n-1)/2-\alpha} \sum_{k=1}^{4} \int_{\mathbf{R}^n} e^{i\lambda\psi_k} a_2(\lambda, \sqrt{\lambda}x) g_k f(\sqrt{\lambda}x) dx + \tag{4.4}$$

$$O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda),$$

where $g_k = a_k(\lambda, x, y)(a^2 - x^2)^{-1/4} + b_k(\lambda, x, y)(a^2 - x^2)^{1/4}$. Integrating by parts as at the estimate of K_1 and taking into account (2.5), (H_1') we get

$$K_2 = I + O(\lambda^{\beta/2+n/2-\alpha-1}) + o(1), \tag{4.5}$$

where

$$I = \int_{\mathbf{R}^n} a_2(\lambda, x)(1 - x^2/\lambda)^{-7/4} |f(x)| dx\ O(\lambda^{-\alpha-3/2}),$$

Since $(1 - x^2/\lambda)^{-3/4} < \lambda^{1/2}$ in the integral I and $f(x) = O(|x|^\beta)$ as $|x| \to \infty$ we find

$$I = O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda). \tag{4.6}$$

Hence (4.5), (4.6) imply $K_2 = o(1)$ since $\beta < 2\alpha + 2 - n$. Theorem 2 is proved.

5 Proof of theorem 3

As in the proof of theorems 1 and 2 it is sufficient to estimate the integrals $K_j = K_j(f), 1 \leq j \leq 4$. For $j = 3, 4$ we have the bound (3.2). Further let

$$f_1(x) = \int_0^1 ... \int_0^1 f(x+h)dh, \ f_0(x) = f(x) - f_1(x).$$

Then $f_j(x) = O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 2\alpha - n + 2$ and

$$|\nabla f_1(x)| \leq \omega(x, f), |f_0(x)| \leq \omega(x, f),$$

therefore f_0 satisfies (H_1) and f_1 satisfies (H_1'). Evidently, $K_j(f) = K_j(f_0) + K_j(f_1), j = 1, 2$. As in the proof of theorems 1 and 2 we obtain $K_j(f) = o(1), j = 1, 2$. Thus theorem 3 is proved.

6 Proof of remark 1

It is not hard to see that for $\alpha > (n-1)/2$ remark 1 will follow from (1.1), theorem 4, corollary 4.16 [16] and the asymptotics

$$E_\lambda^\alpha f(0) = \lambda^{n/2+\beta/2-\alpha-1}(a(\lambda) + O(\lambda^{-1})) + O(\lambda^{-\beta/2}), \tag{6.1}$$

where $f(x) = |x|^\beta$, $\alpha > 0, \beta > 0, n \geq 2$ and $a(\lambda) = a_+(\lambda) + a_-(\lambda) + a_o(\lambda)$,

$$a_\pm(\lambda) = c\sum_{k\geq 1}(-1)^{kn}|\pm\pi/4 + k\pi|^{-\alpha-1}\sin(\lambda\pi(k+1/4) - (\alpha+n/2)\pi/2),$$

$$a_o(\lambda) = c(\pi/4)^{-\alpha-1}\sin(\lambda\pi/4 - (\alpha+n/2)\pi/2),$$

c being a positive constant.

To prove (6.1) we shall use the formula

$$e_\alpha(\lambda, x, y) = \Gamma(\alpha+1)(2\pi i)^{-1}\int_S e^{\lambda p}V(p,x,y)H_\alpha(\lambda+n,p)dp, \tag{6.2}$$

where $S = (\delta - i\pi/2, \delta + i\pi/2), \delta > 0, \alpha > 0$ and the function $s \to H_\alpha(s,p)$ is 2-periodic,

$$H_\alpha(s,p) = \sum_{k=-\infty}^{+\infty} e^{isk\pi}(p+ik\pi)^{-\alpha-1}, p \in S, \alpha > 0.$$

For proving (6.2) we notice that $p^{-\alpha-1}\Gamma(\alpha+1)V(p,x,y)$ is the Laplace transform of the function $\lambda \to e_\alpha(\lambda, x, y)$, where

$$V(p,x,y) = \int_0^\infty e^{-\lambda p}de(\lambda,x,y), Re\, p > 0,$$

in particular, $V(p+ik\pi,x,y) = e^{ik\pi n}V(p,x,y)$. Applying the inverse Laplace formula we get (6.2).

Since (see, for example, [18], [19])

$$V(p,x,y) = (2\pi\sinh 2p)^{-n/2}\exp\left(-\frac{x^2+y^2}{2}\coth 2p + \frac{xy}{\sinh 2p}\right), \tag{6.3}$$

we can write

$$E_\lambda^\alpha f(0) = \Gamma(\alpha+1)(2\pi i)^{-1}\lambda^{-\alpha}\int_S e^{\lambda p}H_\alpha(\lambda+n,p)u(p,0)dp, \tag{6.4}$$

where

$$u(p,0) = (2\pi\sinh 2p)^{-n/2}\int_{\mathbf{R}^n}|x|^\beta\exp(-2^{-1}x^2\coth 2p)\,dx, Re\, p > 0.$$

The integrand in (6.4) has singularities only at the points $p = 0, \pm i\pi/2$ and $p = \pm i\pi/4$. To find the asymptotics of the function (6.4) we shall apply the method of the stationary phase. Let

$1 = g_1(p) + g_2(p) + g_3(p)$ for $p \in S$, where $g_j \in C^\infty$ and $supp\, g_1 \subset \{p . |Im\, p| < \pi/4\}, supp\, g_2 \subset \{p : 0 < |Im\, p| < \pi/2\}, g_3$ being $i\pi-$periodic function. Then

$$E_\lambda^\alpha f(0) = I_{1\delta}(\lambda) + I_{2\delta}(\lambda) + I_{3\delta}(\lambda), \tag{6.5}$$

$$I_{j,\delta}(\lambda) = \lambda^{-\alpha}\Gamma(\alpha+1)(2\pi i)^{-1}\int_{S_j} e^{\lambda p} H_\alpha(\lambda+n,p)u(p,0)g_j dp,$$

where $S_1 = S_2 = S, S_3 = (\delta + i0, \delta + i\pi), g_j \in C_0^\infty(S_j), j = 1,2,3$. In obtaining the third integral we have used the periodicity of the integrand in (6.4). Since

$$u(p,0) = c_\beta(p^{-1}\sinh 2p)^{\beta/2}(\cosh 2p)^{-\beta/2-n/2}p^{\beta/2}$$

we have

$$I_{1\delta}(\lambda) = \lambda^{-\alpha}\left(\int_{S_1} e^{\lambda p}p^{-\alpha-1+\beta/2}q_1(p)dp + \int_{S_1} e^{\lambda p}p^{\beta/2}q_2(p)dp\right), \tag{6.6}$$

where $q_j \in C_0^\infty(S_1)$.

On the other hand, we obtain

$$I_{2\delta}(\lambda) = \lambda^{n/2+\beta/2-\alpha}\int e^{\lambda\phi(p,\sigma)}q(p,\sigma)dpd\sigma + e^{\lambda\delta}O(\lambda^{-\infty}) \tag{6.7}$$

where $q \in C_0^\infty(S_2 \times (0,\infty)), \phi(p,\sigma) = p - 2^{-1}\sigma^2\coth 2p$. Here we have integrated by parts and used the bound $|\partial_p\phi| \geq c > 0$ for $\sigma \sim 0$ or $\sigma \sim \infty$. Consequently (6.5)-(6.7) give

$$E_\lambda^\alpha f(0) = I(\lambda) + O(\lambda^{-\beta/2}), \tag{6.8}$$

where

$$I(\lambda) = \lambda^{n/2+\beta/2-\alpha}\int e^{i\lambda\phi(t,\sigma)}q(t,\sigma)dtd\sigma, \phi(t,\sigma) = t + 2^{-1}\sigma^2\cot 2t,$$

$$q(t,\sigma) = 2^{-n/2}\pi^{-3/2}/\Gamma(n/2-1/2)H_\alpha(\lambda+n,it)g_2(it)(i\sin 2t)^{-n/2}\sigma^{n-1+\beta}g(\sigma),$$

and $g \in C_0^\infty(0,\infty)$.

Now the method of the stationary phase implies

$$I(\lambda) = \lambda^{n/2+\beta/2-\alpha-1}(a(\lambda) + O(\lambda^{-1})). \tag{6.9}$$

Evidently (6.1) follows from (6.8), (6.9).

7 Proof of theorem 4

Starting with the formula (6.2) and having in mind the singularities at the points $p = 0, p = \pm i\pi/2$, we write

$$e_\alpha(\lambda,x,y) = \sum_{j=1}^{3} e_j(\lambda,x,y,\delta), \tag{7.1}$$

where

$$e_j(\lambda,x,y,\delta) = b\int_S e^{\lambda p}V(p,x,y)H_\alpha(\lambda+n,p)g_j(p)dp, \tag{7.2}$$

g_j are C^∞ functions, $g_1(p)+g_2(p)+g_3(p) = 1$ for $p \in S$, $supp\, g_1 \subset \{p \in S : |Im\, p| < \epsilon\}$, $supp\, g_2 \subset \{p \in S : |Im\, p| > \pi/2 - \epsilon\}$ for some small $\epsilon > 0$, and g_2 is $i\pi-$ periodic. Here $b = \Gamma(\alpha+1)(2\pi i)^{-1}$.

If $j = 1$ we shall use the representations:

$$V(p,x,y) = a(p,x,y)\int_{\mathbf{R}^n}\exp(-\xi^2 p + i(x-y)\xi)d\xi\,(2\pi)^{-n},\ Re\, p > 0,$$

where $a(0,x,y) = 1,\ p \to a(p,x,y)$ is smooth for $p \in S$ and

$$\int_{\mathbf{R}^2} e^{-p\eta^2}\eta^{2\alpha}d\eta = \pi\Gamma(\alpha+1)p^{-\alpha-1},\ Re\, p > 0.$$

Since

$$H_\alpha(\lambda+n,p)=p^{-\alpha-1}+h_\alpha(\lambda+n,p) \tag{7.3}$$

and h_α has no singularities on S, we have

$$e_1(\lambda,x,y,\delta)=\lambda^{n/2+\alpha+1}I_1+\lambda^{n/2}I_2, \tag{7.4}$$

where

$$I_1=-ic_1\int_{S\times\mathbf{R}^n\times\mathbf{R}^2}e^{\lambda(1-\xi^2-\eta^2)p+i\sqrt{\lambda}(x-y)\xi}a(p,x,y)g_1(p)\eta^{2\alpha}dpd\xi d\eta, \tag{7.5}$$

$$I_2=c_2\int_{S\times\mathbf{R}^n}e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x-y)\xi}a(p,x,y)h_\alpha(\lambda+n,p)g_1(p)dpd\xi, \tag{7.6}$$

$c_1=(2\pi)^{-n-1}\pi^{-1}$ and c_2 is a constant.

In both integrals I_1,I_2 we can suppose that the integration with respect to (ξ,η) or ξ is taken over a ball, the rest being estimated with $O(e^{c\delta\lambda}\lambda^{-\infty}),c>0$.

To represent e_2 we first use $i\pi-$ periodicity of the integrand in (7.2) and conclude that we can suppose $g_2\in C_0^\infty$, $supp\, g_2\subset\{p=\delta+it:|t-\pi/2|<\epsilon\}$. The translation $p\to p+i\pi/2$ finally gives

$$e_2(\lambda,x,y,\delta)=b\int_{\delta+i\mathbf{R}}e^{\lambda(p+i\pi/2)}V(p+i\pi/2,x,y)H_\alpha(\lambda+n,p+i\pi/2)g_2(p)dp,$$

where $g_2\in C_0^\infty$, $supp\, g_2\subset\{p=\delta+it:|t|<\epsilon\}$.

According to (6.3)

$$V(p+i\pi/2,x,y)=(-2\pi\sinh 2p)^{-n/2}\exp\left(-\frac{x^2+y^2}{2}\coth 2p-\frac{xy}{\sinh 2p}\right),$$

whence

$$V(p+i\pi/2,x,y)=b(p,x,y)\int_{\mathbf{R}^n}\exp\left(-\xi^2p+i(x+y)\xi\right)d\xi,\ Re\, p>0$$

and $b(0,x,y)=(2\pi)^{-n/2}e^{-i\pi n/2}$. Thus

$$e_2(\lambda,x,y,\delta)=\lambda^{n/2}\int_{S\times\mathbf{R}^n}e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x+y)\xi}q_2(\lambda,p,x,y)dpd\xi, \tag{7.7}$$

$$q_2=c_3b(p,x,y)H_\alpha(\lambda+n,p+i\pi/2)g_2(p)e^{i\lambda\pi/2}$$

for some constant c_3.

Analogously,

$$e_3(\lambda,x,y,\delta)=\lambda^{n/2}\int_{S\times\mathbf{R}^n}e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x-y)\xi}q_3(\lambda,p,x,y)dpd\xi, \tag{7.8}$$

$q_3=c_4a(p,x,y)H_\alpha(\lambda+n,p)g_3(p)$.

Since the functions $p\to q_j$ are C_0^∞ we can integrate by parts in the integrals $e_j, j=2,3$. So the integration with respect to ξ is over a ball, the rest being estimated with $O(e^{-c\delta\lambda}\lambda^{-\infty}),c>0$. Now letting $\delta\to 0$ in (7.1), (7.4)-(7.8) we obtain

$$e_\alpha(\lambda,x,y)=\sum_{j=1}^{3}e_j(\lambda,x,y)+O(\lambda^{-\infty}), \tag{7.9}$$

$$e_1=\lambda^{n/2+\alpha+1}I_1+\lambda^{n/2}I_2, \tag{7.10}$$

$$I_1=\int_{\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^2}e^{i\lambda\phi_1}q_1dtd\xi d\eta, \tag{7.11}$$

$$I_2=\int_{\mathbf{R}\times\mathbf{R}^n}e^{i\lambda\phi_2}q_2dtd\xi, \tag{7.12}$$

where $\phi_1=(1-\xi^2-\eta^2)t+\lambda^{-1/2}(x-y)\xi$, $\phi_2=(1-\xi^2)t+\lambda^{-1/2}(x-y)\xi$ and $q_1=c_1a(it,x,y)g_1(it)g(\xi,\eta)$, g being a cutoff function, and $q_2\in C_0^\infty$.

Notice that $e_j(\lambda, x, y), j = 2, 3$ have the same form as $\lambda^{n/2} I_2(\lambda, x, y)$, therefore it suffices to find the asymptotics of the integrals $I_j, j = 1, 2$.

To find the asymptotics of I_1 we use polar coordinates $(\xi, \eta) = \sigma(\omega, \theta), \omega \in \mathbf{R}^n$, $\sigma > 0$, $\omega^2 + \theta^2 = 1$ and the equality

$$\int_{\omega^2+\theta^2=1} e^{i\sqrt{\lambda}(x-y)\omega\sigma} \theta^{2\alpha} d(\omega, \theta) = c(\sqrt{\lambda}|x-y|\sigma)^{-n/2-\alpha} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|\sigma),$$

$c = (2\pi)^{n/2} 2^{\alpha+1} \pi \Gamma(\alpha+1)$. Therefore

$$I_1 = (\sqrt{\lambda}|x-y|)^{-n/2-\alpha} \int_{\mathbf{R}} \int_0^\infty e^{i\lambda(1-\sigma^2)t} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|\sigma) q dt d\sigma, \tag{7.13}$$

where

$$t \to q \in C_0^\infty \text{ and } q(0,1) = (2\pi)^{-n/2-1} 2^{\alpha+1} \Gamma(\alpha+1). \tag{7.14}$$

Integrating by parts in the integral (7.13) with respect to t when σ is close to zero, we can suppose that $q(t, \sigma)$ has a compact support in $\mathbf{R} \times (0, \infty)$, the rest being estimated with $O(\lambda^{-\infty})$.

Let $\sqrt{\lambda}|x-y| > 1$. Then we shall use the formula [20, p. 168].

$$J_{n/2+\alpha}(r) = r^{-1/2}(e^{ir} f(r) + e^{-ir} \bar{f}(r)),\ r > 0 \tag{7.15}$$

and the bound

$$|\partial_\sigma^k f(\sqrt{\lambda}|x-y|\sigma)| \leq c_k \text{ if } \sqrt{\lambda}|x-y| > 1,\ 0 < a \leq \sigma \leq b.$$

Consider the phase function ψ :

$$\psi(t, \sigma, x, y) = (1-\sigma^2)t \pm |x-y|\lambda^{-1/2}\sigma. \tag{7.16}$$

The critical points $(t, 1)$ are nondegenerate and

$$q(t, 1) = q(0, 1) + |x-y| O(\lambda^{-1/2}). \tag{7.17}$$

Hence the method of the stationary phase implies for $\sqrt{\lambda}|x-y| > 1$:

$$I_1 = (\sqrt{\lambda}|x-y|)^{-n/2-\alpha}(d_\alpha \lambda^{-1} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|) + |x-y|^{-1/2} O(\lambda^{-7/4})), \tag{7.18}$$

where $d_\alpha = (2\pi)^{-n/2} 2^\alpha \Gamma(\alpha+1)$.

Consider now the case $\sqrt{\lambda}|x-y| < 1$. Then we can write

$$I_1 = \int_{\mathbf{R}} \int_0^\infty e^{i\lambda(1-\sigma^2)t} g(t, \sigma, \lambda) dt d\sigma,$$

$$g(t, \sigma, \lambda) = c_1 \int_{\omega^2+\theta^2=1} e^{i\sqrt{\lambda}(x-y)\omega\sigma} \theta^{2\alpha} d(\omega, \theta)\ \sigma^{n+2\alpha+1} q_1(t, \sigma).$$

The method of the stationary phase shows that

$$I_1 = \pi c_1 \lambda^{-1} \int_{\omega^2+\theta^2=1} e^{i\sqrt{\lambda}(x-y)\omega} \theta^{2\alpha} d(\omega, \theta) + O(\lambda^{-2}),$$

whence

$$I_1 = d_\alpha(\sqrt{\lambda}|x-y|)^{-n/2-\alpha} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|)\lambda^{-1} + O(\lambda^{-2}) \tag{7.19}$$

if $\sqrt{\lambda}|x-y| < 1$.

On the other hand, in polar coordinates $\xi = \sigma\omega$,

$$I_2 = \int_{\mathbf{R}\times(0,\infty)\times S^{n-1}} e^{i\lambda(1-\sigma^2)t + i\sqrt{\lambda}(x-y)\omega\sigma} \sigma^{n-1} q_2(t, \sigma) dt d\sigma d\omega,$$

whence the stationary phase method gives

$$I_2 = O(\lambda^{-1}). \tag{7.20}$$

Thus (2.2), (7.9), (7.10), (7.18)-(7.20) imply (2.3) for $\alpha \geq (n-1)/2$. Theorem 4 is proved.

8 Proof of theorem 5

Starting with (6.2),(6.3) we see that the phase function $p \to \phi(p,x,y)$, given by

$$\phi(p,x,y) = p - 2^{-1}(x^2+y^2)\coth 2p + xy(\sinh 2p)^{-1}, \tag{8.1}$$

has the critical points $p_\pm = it_\pm$ and $\bar{p}_\pm$, where $\cos 2t_\pm = xy \mp d$ and $d^2 = (xy)^2 + 1 - x^2 - y^2$. If $x = y$ then $p_- = 0$ and the integrand in (6.2) is not holomorphic function in a neighborhood of the critical points. So we have to expand the singularities. Analogously to (7.1), (7.2) we can write

$$E_\alpha(\lambda,x,y) = \sum_{j=1}^{3} E_j(\lambda,x,y,\delta),$$

where

$$E_j = (2\pi i)^{-1}\Gamma(\alpha+1)\int_S e^{\lambda p} V(p,\sqrt{\lambda}x,\sqrt{\lambda}y) H_\alpha(\lambda+n,p) g_j(p)\,dp. \tag{8.2}$$

Further,

$$V(p,x,y) = (2\pi)^{-n}\int_{\mathbf{R}^n} \exp\left(-\frac{x^2+y^2}{2}\tanh p - \frac{\xi^2}{2}\sinh 2p + i(x-y)\xi\right)d\xi,$$

and for $Re\, p > 0$,

$$1 = c_\alpha(\sinh 2p)^{\alpha+1}\int_{\mathbf{R}^2} \eta^{2\alpha}\exp\left(-\frac{1}{2}\eta^2\sinh 2p\right)d\eta.$$

Analogously to (7.3) we have

$$E_1(\lambda,x,y,\delta) \sim \lambda^{n/2+\alpha+1} I_1(\lambda,x,y,\delta) + I_2(\lambda,x,y,\delta), \tag{8.3}$$

where now

$$I_1 = \int_{\mathbf{R}^n\times\mathbf{R}^2\times S} e^{\lambda\phi_1} q_1\, d\xi\, d\eta\, dp, \tag{8.4}$$

$$\phi_1 = p - 2^{-1}(\xi^2+\eta^2)\sinh 2p - 2^{-1}(x^2+y^2)\tanh p + i(x-y)\xi,$$

$q_1 = (p/\sinh 2p)^{-\alpha-1} g(\xi,\eta)\eta^{2\alpha} g_1(p)$ and

$$I_2 = \lambda^{n/2}\int_{\mathbf{R}^n\times S} e^{\lambda\phi_2} q_2\, d\xi\, dp, \tag{8.5}$$

$$\phi_2 = p - 2^{-1}\xi^2\sinh 2p - 2^{-1}(x^2+y^2)\tanh p + i(x-y)\xi,$$

$q_2 = h_\alpha(\lambda,p)g(\xi)g_1(p)$ for some cutoff function g.

To represent E_2 we use the periodicity of the integrand in (6.2) and the formula

$$V(p+i\frac{\pi}{2},x,y) = a(p)\int_{\mathbf{R}^n} \exp\left(-\frac{\xi^2}{2}\tanh 2p + xy\tanh + xy\tanh p + i(x+y)\xi\right)d\xi.$$

where $a(p) = (4\pi^2\cosh 2p)^{-n/2}$. Thus

$$E_2(\lambda,x,y,\delta) \sim \lambda^{n/2}\int_{\mathbf{R}^n\times S} e^{\lambda\psi} q\, d\xi\, dp, \tag{8.6}$$

$$\psi = p - 2^{-1}\xi^2\tanh 2p + xy\tanh p + i(x+y)\xi,$$

$$q = (\cosh 2p)^{-n/2} H_\alpha(\lambda+n, p+i\pi/2) g(\xi) g_2(p+i\pi/2) e^{i\lambda\pi/2}.$$

Therefore letting $\delta \to 0$ we obtain from (8.3)-(8.6)

$$E_\alpha(\lambda,x,y) = \sum E_j(\lambda,x,y) + O(\lambda^{-\infty}), \tag{8.7}$$

where

$$E_1 = \lambda^{n/2+\alpha+1} I_1 + \lambda^{n/2} I_2, \tag{8.8}$$

the integrals I_j being given by (7.11), (7.12), but now

$$\phi_1 = t - 2^{-1}(\xi^2 + \eta^2)\sin 2t - 2^{-1}(x^2+y^2)\tan t + (x-y)\xi,$$

$$\phi_2 = t - 2^{-1}\xi^2 \sin 2t - 2^{-1}(x^2+y^2)\tan t + (x-y)\xi,$$

$$q_1 = (t/\sin 2t)^{-\alpha-1} g(\xi,\eta)\eta^{2\alpha} g_1(it),\ q_2 = (\cos 2t)^{-n/2} h_\alpha(\lambda, it) g(\xi) g_1(it).$$

Further, E_2 is analogous to I_2 and

$$E_3 = \int_{\mathbf{R}} e^{i\lambda\phi_3} q_3 dt,\ \phi_3 = t + 2^{-1}(x^2+y^2)\cot 2t - xy(\sin 2t)^{-1},$$

$$q_3 = (2\pi)^{-1}\Gamma(\alpha+1)(2\pi i \sin 2t)^{-n/2} H_\alpha(\lambda+n, it) g_3(it).$$

To find the uniform asymptotics of the integrals E_j in the domain $\{x, y \in \mathbf{R}^n : A/\lambda < x^2 < 1-\delta,\ |y| < \epsilon|x|\}$ we shall apply the method of the stationary phase.

a. Asymptotics of I_1. Analogously to (7.13) we have

$$I_1 = (\lambda|x-y|)^{-n/2-\alpha} \int_{\mathbf{R}} \int_0^\infty e^{i\lambda\psi_o} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\lambda|x-y|\sigma) q(t,\sigma) dt d\sigma,$$

where $\psi_0 = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2+y^2)\tan t,\ q \in C_0^\infty(\mathbf{R}\times(0,\infty))$. Here we have integrated by parts using the estimate $|\partial_t \psi_o| \ge c > 0$ if σ is close to zero.

Since $|x-y| > c|x| > c\lambda^{-1/2}$ we have for $s = \lambda|x-y|\sigma$

$$J_{n/2+\alpha}(s) = \sum_{k=0}^{2} s^{-1/2-k} c_k \cos(s+b_k) + |x|^{-1/2} O(\lambda^{-2}),$$

where b_k is a constant. Therefore

$$I_1 = \sum_{k=0}^{2} (\lambda|x-y|)^{-(n+1)/2-\alpha-k} M_k + (\lambda|x|)^{-(n+1)/2-\alpha} O(\lambda^{-3/2}),$$

where

$$M_k = \int_{\mathbf{R}} \int_0^\infty e^{i\lambda\psi} \sigma^{(n+1)/2+\alpha-k} q_k(t,\sigma) dt d\sigma,\ q_k \in C_0^\infty,$$

$$\psi = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2+y^2)\tan t \pm |x-y|\sigma.$$

The critical points (t_j, σ_j) of ψ satisfy

$$\cos 2t_j = xy + (-1)^{j+1} d\ (j=1,2),\ t_3 = -t_1,\ t_4 = -t_2,\ \sigma_j \sin 2t_j = \pm|x-y|,$$

where $d^2 = (xy)^2 + 1 - x^2 - y^2$. Since $x^2 < 1-\delta$, $|y| < \epsilon|x|$ for small $\delta > 0, \epsilon > 0$ and the support of $t \to q_k(t,\sigma)$ is small enough, we have $d > c > 0$, $\det \psi'' = \pm 4d$ for the Hessian ψ'' in the critical points. Therefore the critical points are nondegenerate. Thus the stationary phase method implies

$$I_1 = \lambda^{-(n+1)/2-\alpha-1} \sum_{j=1}^{4} e^{i\lambda\psi_j} b_j g_1(it_j) + (\lambda|x|)^{-(n+1)/2-\alpha} O(\lambda^{-3/2}) \tag{8.9}$$

and b_j, ψ_j have the properties (2.4), (2.5) respectively.

b. Asymptotics of I_2. In polar coordinates $\xi = \sigma\omega$, $\sigma > 0$ we have

$$I_2 = \int_{\mathbf{R}} \int_0^\infty \int_{|\omega|=1} e^{i\lambda\psi} q(t,\sigma) dt d\sigma d\omega,\ q \in C_0^\infty,$$

$$\psi = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2+y^2)\tan t + (x-y)\omega\sigma.$$

Since the support of $t \to q(t,\sigma)$ is small, the critical points (t,σ) of ψ are nondegenerate if $x^2 < 1-\delta$, $|y| < \epsilon|x|$ for small $\delta > 0, \epsilon > 0$. Hence for large M,

$$I_2 = \sum \lambda^{-k} \int_{|\omega|=1} e^{i\lambda\phi_j} a_{kj}(\lambda, x, y, \omega) d\omega + O(\lambda^{-M-1}),$$

where $1 \leq k \leq M$, $1 \leq j \leq 4$ and $\phi_j = (x-y)\omega\, a_j(x,y,\omega)$, $a_j(x,x,\omega) = (1-x^2)^{1/2}$.

Since $\lambda|x-y| > c\sqrt{\lambda}$ the method of the stationary phase gives

$$I_2 = \lambda^{-(n+1)/2} \sum_{j=1}^{4} e^{i\lambda\psi_j} b_j g_1(it_j) + |x|^{-(n+1)/2} O(\lambda^{-(n+3)/2}). \tag{8.10}$$

Notice that E_2 has the same asymptotics (8.10), where g_1 is replaced by g_2.

c. Asymptotics of E_3. The critical points t_j of the phase function ϕ_3 satisfy $\cos 2t_j = xy + (-1)^{j+1}d$ and $\phi_3''(t_j,x,y) = (-1)^{j+1}4d(\sin 2t_j)^{-1}$, $1 \leq j \leq 4$. Therefore the stationary phase method implies

$$E_3 = \lambda^{-1/2} \sum_{j=1}^{4} b_j e^{i\lambda\psi_j} g_3(it_j) + O(\lambda^{-3/2}). \tag{8.11}$$

Evidently, theorem 5 follows from (8.7)-(8.11).

9 Proof of theorem 6

Starting with (6.2), (2.2), we can write

$$E_\alpha(\lambda,x,y) = \int_S e^{\lambda\phi} q\,dp, \tag{9.1}$$

where the function ϕ is given by (8.1) and

$$q(p) = \Gamma(\alpha+1)(2\pi i)^{-1}(2\pi \sinh 2p)^{-n/2} H_\alpha(\lambda+n,p).$$

Now the problem is to find the uniform asymptotics of the integral (9.1) as $\lambda \to \infty$. The critical points of the phase function $p \to \phi(p,x,y)$ satisfy the relation $\cosh 2p = xy + d$, where $d^2 = (xy)^2 + 1 - x^2 - y^2$. Let $x = r\omega$, $|\omega| = 1$. Then the critical points degenerate if $r = a$, where $a = a(y,\omega)$ is given by (2.7). We have two degenerate critical points: p_0 and $\bar{p}_0$, where $p_0 = it_0$ and $\cos 2t_0 = a\omega y$, $t_0 > 0$. In particular, $t_0 < \pi/2$ if $|y|$ is sufficiently small. Thus if $|x^2-1| < \delta$, $|y| < \epsilon|x|$ for some small $\delta > 0, \epsilon > 0$ there are only four critical points $p_\pm$, $\bar{p}_\pm$, where

$$p_\pm = it_\pm,\ \cos 2t_\pm = xy \pm d,\ 0 < t_\pm < \pi/2 \text{ if } x^2 < a^2,$$

$$p_\pm = \pm\delta + it,\ \cosh 2\delta\, \cos 2t = xy,\ 0 < t < \pi/2 \text{ if } x^2 > a^2,$$

and $2\cos^2 2t = x^2 + y^2 - ((x^2+y^2)^2 - 4(xy)^2)^{1/2}$.

Near these critical points the integrand in (9.1) is a holomorphic function and $\partial^3\phi/\partial p^3 = b_1(y,\omega)$, $\partial^2\phi/\partial p\partial r = -b_2(y,\omega)$ for $p = p_0$ or $p = \bar{p}_0$, where $b_1(o,\omega) = 8$, $b_2(0,\omega) = 2$. Therefore we can apply Lemma 2.3 in [5], p.343 and conclude that there exists a holomorphic change of variables $p = p(z,x,y)$, defined in a neighborhood of the points $z = 0$, $r = a$ such that

$$\phi(p(z,x,y),x,y) = A(x,y) - B(x,y)\,z + z^3/3,\ p(0,a\omega,y) = p_0, \tag{9.2}$$

for every fixed ω, y. In addition, the coefficients A, B are given by

$$A = \frac{1}{2}(\phi(p_+,x,y) + \phi_-,x,y)),$$

$$B = (\frac{3}{4}(\phi(p_+,x,y) - \phi(p_-,x,y)))^{2/3},$$

and $p(\pm\sqrt{B},x,y) = p_\pm$.

To use this change of variables in the integral (9.1), we notice first that

$$E_\alpha(\lambda,x,y) \sim \int_L e^{\lambda\phi} q\,dp,\ L = L_1 \cup L_2, \tag{9.3}$$

where L_1 is the segment $(\delta+i(t_0-2\delta, \delta+i(t_0+2\delta))$ and L_2- the segment $(\delta-i(t_0+2\delta), \delta+i(-t_0+2\delta))$ for $\delta > 0$ small enough. The equivalence relation "$a(\lambda, x, y) \sim b(\lambda, x, y)$" here means that $a - b = O(e^{-c\lambda}), c > 0$. Indeed, it is sufficient to notice the bound $Re\,\phi(p, x, y) \leq -c < 0$ for $p \in S \setminus L$, which follows from the definition (8.1) if $\delta > 0$ is small enough.

Now (9.1)-(9.3) yield

$$E_\alpha(\lambda, x, y) \sim \sum_{j=1}^{2} e^{\lambda A_j} \int_{L_j} e^{\lambda(-Bz+z^3/3)} q_j(z, \lambda) dz, \tag{9.4}$$

where $A_1 = A$, $A_2 = \bar{A}$, $q_1(z, \lambda) = q(p(z, x, y))\, \partial p/\partial z$, $q_2(z, \lambda) = q(\bar{p}(\bar{z}, x, y)) \times \partial \bar{p}/\partial z$, L_{1j} being the image of the segment L_j. Notice that $L_{1j} \subset \{z : Re\, z > 0\}$ and that the end points a_j, b_j of L_j satisfy $\arg a_j \in (-\pi/2, -\pi/6)$,
$\arg b_j \in (\pi/6, \pi/2)$.

Using the Weierstrass preparation theorem [9]:

$$q_j(z, \lambda) = r_j + r_{1j}\, z + (z^2 - B) q_{1j}(z, \lambda)$$

and the following representation of the Airy function

$$Ai(s) = (2\pi i)^{-1} \int_M e^{-sz+z^3/3} dz,\ M = M_1 \cup M_2,$$

$M_1 : z = re^{i\theta},\ r \in (+\infty, 0),\ \theta \in (-\pi/2, -\pi/6),\ M_2 : z = re^{i\theta},\ r \in (0, +\infty),\ \theta \in (\pi/6, \pi/2)$, in the integral (9.4), we obtain the uniform asymptotics (2.6), the rest being estimated as in [5], p. 348.

10 Proof of theorem 7

Now we use the formula (6.2) with $\delta = \delta(x, y) > 0$ such that $2\cosh^2 2\delta = x^2 + y^2 + ((x^2 + y^2)^2 - 4(xy)^2)^{1/2}$. The critical points $p(x, y) = \delta + it$ and $\bar{p}(x, y)$ are nondegenerate and $Re\,\phi(p, x, y) < Re\,\phi(p(x, y), x, y)$ if $0 \leq Im\, p \leq \pi/2,\ p \neq p(x, y)$; $Re\,\phi(p, x, y) < Re\,\phi(\bar{p}(x, y), x, y)$ if $-\pi/2 \leq Im\, p \leq 0,\ p \neq \bar{p}(x, y)$. In addition, $\partial^2\phi/\partial p^2(p(x, y), x, y) = 4d/\sinh 2p(x, y)$ and $Re\,\phi(p(x, y), x, y) = 2^{-1}(arcosh\beta - \beta\sqrt{\beta^2 - 1})$, $\beta = \cosh 2\delta$. Since $\beta^2 - 1 \geq c(x^2 - 1)$, $c > 0$ and $\beta\sqrt{\beta^2 - 1} - arcosh\beta \geq \gamma\sqrt{\beta^2 - 1}$ if $\beta^2 - 1 > \gamma$, for some $0 < \gamma < 1$, one obtains theorem 7 by the saddle-point method.

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