Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 209-218

RELATIONS BETWEEN EXPONENTIAL LAWS FOR SPACES OF C^∞-FUNCTIONS

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(Received December 6, 1992 and in revised form June 1, 1995)

ABSTRACT. In this paper we prove that some new as well as some already existing exponential laws for spaces of C^∞- and holomorphic functions can all be generated from one general exponential law.

KEY WORDS AND PHRASES. Exponential law, holomorphy, convergence structure.

1992 AMS SUBJECT CLASSIFICATION CODE. 58C25.

1. INTRODUCTION AND PRELIMINARIES.

In [3] an exponential law $C^\infty_{(c)}(U \times V, G) \cong C^\infty_{(c)}(U, C^\infty_{(c)}(V, G))$ was proved for spaces of C^∞-functions (in the Bastiani sense [1]) defined on c-open sets in arbitrary convergence vector spaces and taking values in L_c-embedded spaces. Here, convergence in $C^\infty_{(c)}(V, G)$ means convergence with respect to all derivatives, where the linear spaces are equipped with continuous convergence. The idea in this paper is to use this exponential law in order to construct exponential laws within other differentiability theories. We obtain such laws in three different theories; a similar theory as that in [3] but with the continuous convergence replaced with its equable structure, the differential calculus by Seip [13,14] for compactly generated spaces and finally the concept of holomorphy by Bjon [5,6]; Bjon and Lindström [7].

A *convergence space* X [8] is a set, on which with each point $x \in X$ is associated a set of filters, which are said to converge to x, such that the following conditions hold:

1) The trivial ultrafilter associated with x always converges to x;

2) If $\mathcal{F} \geq \mathcal{G}$ and $\mathcal{G}$ converges to x, then $\mathcal{F}$ converges to x;

3) If $\mathcal{F}$ and $\mathcal{G}$ converge to x, then $\mathcal{F} \cap \mathcal{G}$ converges to x.

A *convergence vector space* (cvs) [8] is a convergence space with a vector structure, such that the vector operations are continuous (a map is continuous if it preserves convergence). All vector spaces in this paper have the scalar field $\mathbb{K}(= \mathbf{R}$ or $\mathbf{C})$.

A cvs E is said to be *equable* [10] if each filter which converges to zero in E contains a filter $\mathcal{G}$, such that $\mathbf{V}\mathcal{G} = \mathcal{G}$ and $\mathcal{G}$ converges to zero in E. Here $\mathbf{V}$ denotes the zero-neighbourhood filter of $\mathbb{K}$. Clearly there exists on E a coarsest equable vector convergence structure finer than the original structure on E. The vector space E endowed with this equable structure is denoted E^e. All topological vector spaces are equable.

For X a convergence space and E a cvs $C_c(X, E)$ denotes the vector space of all continuous functions $f : X \to E$ endowed with continuous convergence [2]. A filter $\mathcal{F}$ converges to zero in

$C_c(X,E)$ iff for each $x \in X$ and each filter $\mathcal{G}$ which converges to x, the filter $\mathcal{F}(\mathcal{G})$ converges to zero in E. We write $C_e(X,E)$ and $L_e(E,F)$ instead of $(C_c(X,E))^e$ and $(L_c(E,F))^e$.

A cvs E is said to be *L_a-embedded* if the mapping $j_E : E \to L_aL_aE$ $(a = c, e)$, $j_E(x)l = l(x)$, into the second dual is an embedding [4]. If E is L_a-embedded, the same applies for the space $C_a(X,E)$ $(a = c,e)$ by Bjon [4]. All Hausdorff locally convex topological vector spaces are L_c- and L_e-embedded and all polar bornological vector spaces are L_e-embedded. An L_e-embedded cvs is equable. The dual LE of an L_a-embedded cvs E separates the points of E.

A convergence space is said to be *locally compact* if every convergent filter contains some compact set. Let X be a convergence space. The inclusion mappings $i : K \to X$, where K ranges over all compact subsets of X, induce a final convergence structure on X. The set X equipped with this structure is denoted X^{lc}. A filter $\mathcal{F}$ converges in X^{lc} iff $\mathcal{F}$ converges in X and $\mathcal{F}$ contains a compact set [2,9]. Thus X^{lc} is the coarsest locally compact convergence space with a finer structure than X. We can consider lc as a functor from the category of convergence spaces to the full subcategory of locally compact convergence spaces. This functor is coreflective, so $(X \times Y)^{lc} = X^{lc} \times Y^{lc}$ for convergence spaces X and Y [9].

The open sets of a convergence space X define a topology $t(X)$ on X [8]. This t is a functor between the categories of convergence and topological spaces. Let $k = t \circ lc$ be the idempotent composed functor. We call a separated topological space X *compactly generated* if $X = kX$. All separated locally compact topological spaces and all metrizable spaces are compactly generated [15].

If X is a topological space, then $id : kX \to X$ is continuous, which however not always is true when X is a convergence space. For topological spaces X and Y one usually writes $X \times_k Y$ - the *k-product* of X and Y - instead of $k(X \times Y)$. Now, $X \times_k Y$ do not in general equal $kX \times kY$, not even if X and Y are compactly generated. If however X is compactly generated and Y is a locally compact topological space, then $X \times_k Y = X \times Y$ [15].

A major advantage when dealing with the category of compactly generated spaces compared to the category of topological spaces, is that the former is Cartesian closed whereas the latter is not. By Gabriel - Zisman (see [13]) we have

$$f \in C(X, kC_{co}(Y,Z)) \iff \tilde{f} \in C(X \times_k Y, Z),$$

for compactly generated spaces X, Y and Z. It is mostly because of this type of universal mapping property, that U. Seip has been able to construct his exponential law for spaces of C^∞-functions in [13] and [14].

2. SPACES OF DIFFERENTIABLE FUNCTIONS.

Let E be a cvs. A set $U \subset E$ is said to be *c-open* in E if it is a union of circled open sets in E, or equivalently that to every $x \in U$ there exists a circled open set U_x such that $x + U_x \subset U$. Every open set in a topological vector space is c-open.

Let E and F be cvs and U a c-open set in E. We say that a function $f : U \to F$ is *differentiable at* $x \in U$ if there exists a continuous linear operator $Df(x) : E \to F$, such that the mapping $\varepsilon_f : S \times U_x \to F$ $(S = \{\lambda \in \mathbb{K} : |\lambda| < 1\})$, defined by

$$\varepsilon_f(0,h) = 0, \quad \varepsilon_f(s,h) = \frac{f(x+sh) - f(x)}{s} - Df(x)h \quad \text{for } s \neq 0,$$

is continuous. As usual $f : U \to F$ is said to be *differentiable* if it is differentiable at every point in U and *continuously differentiable* or of class C_a^1 if it is differentiable and the derivative $Df : U \to L_a(E, F)$ is continuous ($a = c, e$). Higher order derivatives are defined recursively by using the notations:

$$L_a({}^0E, F) = F, \quad L_a({}^{k+1}E, F) = L_a(E, L_a({}^kE, F)) \quad \text{for } k = 0, 1, \ldots$$

We say that f is of class C_a^p if $D^p f : U \to L_a({}^pE, F)$ $(p = 0, 1, \ldots)$ exists and is continuous. For $p \geq 1$ this is equivalent by saying that Df is of class C_a^{p-1}. If f has derivatives of all orders it is said to be of class C_a^∞.

It is a straightforward exercise to verify that the functions $f \circ g$, $f \times g - f \times g(x, y) = (f(x), g(y))$ – and $[f, g]$ – $[f, g](x) = (f(x), g(x))$ – when defined, are of the same class C_a^p as f and g are, likewise that continuous multilinear maps are of class C_a^∞. In normed spaces a function is of class C_e^p whenever it is p times continuously Fréchet differentiable. In locally convex topological vector spaces the class C_e^p agrees with the one denoted by C_Π^p in [11].

Let $C_a^p(U, F)$ denote the subspace of $C(U, F)$ consisting of all functions $f : U \to F$ of class C_a^p and $C_{(a)}^p(U, F)$ the set $C_a^p(U, F)$ equipped with the initial convergence structure induced by the differential operators

$$D^k : C_a^p(U, F) \to C_a(U, L_a({}^kE, F)) \quad k = 0, 1, \ldots, p,$$

(k varies over $\mathbf{N}$ for $p = \infty$). Thus convergence in $C_{(a)}^p(U, F)$ means convergence with respect to each existing derivative.

In order to compare the spaces $C_{(e)}^p(U, F)$ and $C_{(c)}^p(U, F)$ with each other, we need some information about the spaces $L_a({}^kE, F)$. Since embeddings turn out to be useful, we briefly skiss some general features of embeddings in function spaces.

If, for two cvs F and G, F is embedded in G (denoted $F \hookrightarrow G$), we identify F as a subspace of G. Then also $C_a(X, F) \hookrightarrow C_a(X, G)$ $(a = c, e)$ for every convergence space X, as well as $L_a(E, F) \hookrightarrow L_a(E, G)$ $(a = c, e)$ for each cvs E. By Bjon [6] we have $C_e(X, F^e) \hookrightarrow C_e(X, F)$ for X a convergence space and F a cvs. Obviously $L_e(E, F^e) \hookrightarrow L_e(E, F)$ for cvs E and F.

LEMMA 2.1. Let E and F be two cvs. Then there exist embeddings

$$L_e({}^kE, F) \hookrightarrow (L_c({}^kE, F))^e, \ L_e({}^kE, F^e) \hookrightarrow L_e({}^kE, F) \quad \text{for } k \geq 1. \tag{2.1}$$

Moreover, if E is equable, the embeddings above are equalities

$$L_e({}^kE, F) = (L_c({}^kE, F))^e, \ L_e({}^kE, F^e) = L_e({}^kE, F) \quad \text{for } k \geq 1. \tag{2.2}$$

PROOF. The embeddings in (2.1) can both be proven by induction, the first because of

$$L_e(E, L_e({}^kE, F)) \hookrightarrow L_e(E, (L_c({}^kE, F))^e) \hookrightarrow L_e(E, L_c({}^kE, F))$$

and the second similarily. The very same induction procedure can be applied to (2.2), since $L_e(E, F^e) = L_e(E, F)$ for equable E.

We now have the tools needed for proving the following proposition.

PROPOSITION 2.2. Let E and F be arbitrary cvs and U a c-open subset of E. Then

$$C_{(e)}^p(U, F^e) \hookrightarrow C_{(e)}^p(U, F) \hookrightarrow (C_{(c)}^p(U, F))^e, \ p = 0, 1, \ldots, \infty.$$

PROOF. Since the associated equable structure is finer than the original one, we obviously have $C_e^p(U, F^e) \subset C_e^p(U, F) \subset C_c^p(U, F)$. Then the first embedding is more or less a direct consequence of $C_e(U, L_e(^kE, F^e)) \hookrightarrow C_e(U, L_e(^kE, F))$, guaranteed by (2.1). The second embedding, on the other hand, emerges from $C_e(U, L_e(^kE, F)) \hookrightarrow C_e(U, L_c(^kE, F))$, also because of (2.1).

As shown in [3], the mapping $f \mapsto (f, Df, ..., D^pf)$ defines an embedding

$$C_{(a)}^p(U, F) \hookrightarrow \prod_{k=0}^{p} C_a(U, L_a(^kE, F)) \ (a = c, e).$$

By Bjon [4], the L_c- as well as the L_e-embedded spaces are closed under formating of arbitrary products and subspaces. Consequently we obtain:

PROPOSITION 2.3. Let U be a c-open subset of a cvs E and F an L_a-embedded cvs. Then also $C_{(a)}^p(U, F)$ is L_a-embedded $(a = c, e)$ for $p = 0, 1, ..., \infty$.

3. EQUABLE EXPONENTIAL LAW AND COMPLETENESS.

Let G be a cvs and let U be a c-open subset of an equable cvs E. In general $C(U, G^e)$ is a proper subset of $C(U, G)$. However, $L(E, G) = L(E, G^e)$. Since the spaces $C_e^p(U, G)$ somewhat are between $C(U, G)$ and $L(E, F)$, it seems reasonable to question whether $C_e^p(U, G^e) = C_e^p(U, G)$ for some large p.

PROPOSITION 3.1. Let G be an L_c-embedded cvs and U a c-open subset of an equable cvs E. Then

$$C_{(e)}^p(U, G^e) = C_{(e)}^p(U, G) \quad \text{for } p = 2, 3, ..., \infty.$$

PROOF. According to Proposition 2.2, the space $C_{(e)}^p(U, G^e)$ is embedded into $C_{(e)}^p(U, G)$. Thus we only have to verify that $C_e^p(U, G^e) = C_e^p(U, G)$ for $p = 2, ..., \infty$. Now E is equable, so by (2.2) there is an equality $C(U, L_e(^kE, G^e)) = C(U, L_e(^kE, G))$ for $k \geq 1$. Since $C_e^2(U, G) \subset C_c^2(U, G)$, it remains to be shown that $f : U \to G^e$ is differentiable for $f \in C_c^2(U, G)$. This is the case when $\varepsilon_f : S \times U_x \to G^e$ is continuous for each $x \in U$ $(S = \{\lambda \in \mathbb{K} : |\lambda| < 1\})$. Certainly $S \times U_x$ is a c-open subset of the equable space $\mathbb{K} \times E$. By proposition 2.3 in [6], $\varepsilon_f : S \times U_x \to G^e$ is continuous if $\varepsilon_f : S \times U_x \to G$ is differentiable. As in theorem 3.4 in [6] we write

$$\varepsilon_f(s, h) = \int_0^1 \gamma(t, s, h) dt,$$

where $\gamma(t, s, h) = Df(x + tsh)h - Df(x)h$. Now, $\gamma(t, -, -)$ is differentiable with the derivative continuous as a function of all three variables (t, s, h). Thus, by theorem 3.3 in [6], ε_f is differentiable and the proposition is proved.

PROPOSITION 3.2. Let G be an L_c-embedded cvs and U a c-open subset of an equable cvs E. Then

$$C_c^{p+1}(U, G) \subset C_e^p(U, G) \quad \text{for } p = 0, 1, ...$$

PROOF. Certainly the statement is true for $p = 0$. We proceed by induction and assume that the assertion holds for $p \geq 0$. Let $f \in C_c^{p+2}(U, G)$. We have to show that $f \in C_e^{p+1}(U, G)$. By the induction hypothesis $D^pf : U \to L_e(^pE, G)$ exists and is continuous. Let $x \in U$. The function $D^pf : U \to L_e(^pE, G)$ is differentiable at x if $\varepsilon_{D^pf} : S \times U_x \to L_e(^pE, G)$ is continuous. As in the proof of Proposition 3.1, we write

$$\varepsilon_{D^pf}(s, h) = \int_0^1 \gamma(t, s, h) dt,$$

where $\gamma(t,-,-) = \omega \circ [D^{p+1}f \circ \zeta(t), pr_2] + D^{p+1}f(x) \circ pr_2$, and where

$$\omega : L_c({}^{p+1}E, G) \times E \to L_c({}^pE, G), \quad \omega(g,x) = g(x),$$

$$\zeta(t) : S \times U_x \to E, \quad \zeta(t)(s,h) = x + tsh,$$

$$pr_2 : S \times U_x \to E, \quad pr_2(s,h) = h,$$

$$D^{p+1}f : U \to L_c({}^{p+1}E, G).$$

Since $L_c({}^pE, G)$ is L_c-embedded, we obtain in a similar manner as in Proposition 3.1 that $\varepsilon_{D^pf} : S \times U_x \to L_c({}^pE, G)$ is differentiable. Thus $D^pf : U \to L_e({}^pE, G)$ is differentiable and even continuously differentiable as a consequence of (2.2) and the hypothesis that $D^{p+1}f : U \to L_c({}^{p+1}E, G)$ is differentiable. Hence the assertion is true for $p+1$ and thus for all $p = 0, 1, ...$

Combining the proposition above and Proposition 2.2 we obtain:

COROLLARY 3.3. Let G be an L_c-embedded cvs and U a c-open subset of an equable cvs E. Then

$$C^\infty_{(e)}(U,G) = (C^\infty_{(c)}(U,G))^e.$$

We are now ready for the exponential law.

THEOREM 3.4. Let G be an L_c- or an L_e-embedded cvs and U and V c-open subsets of equable cvs E and F repectively. Then there exists a natural isomorphism

$$C^\infty_{(e)}(U \times V, G) \cong C^\infty_{(e)}(U, C^\infty_{(e)}(V,G)). \tag{3.1}$$

PROOF. Let G be L_e-embedded and consider its L_c-embedded reflection G_c (i.e. the canonical image of G in L_cL_cG). According to [3] there is an exponential law

$$C^\infty_{(c)}(U \times V, G_c) \cong C^\infty_{(c)}(U, C^\infty_{(c)}(V,G_c)), \tag{3.2}$$

and thus

$$\left(C^\infty_{(c)}(U \times V, G_c)\right)^e \cong \left(C^\infty_{(c)}(U, C^\infty_{(c)}(V,G_c))\right)^e.$$

The set $U \times V$ is c-open in the equable cvs $E \times F$ and $C^\infty_{(c)}(V, G_c)$ is L_c-embedded by Proposition 2.3, so by Corollary 3.3:

$$C^\infty_{(e)}(U \times V, G_c) \cong C^\infty_{(e)}(U, C^\infty_{(c)}(V,G_c)). \tag{3.3}$$

Thus, by Proposition 3.1 and Corollary 3.3 applied on the right member of (3.3), we get

$$C^\infty_{(e)}(U \times V, G_c) \cong C^\infty_{(e)}(U, C^\infty_{(e)}(V,G_c)). \tag{3.4}$$

Since $H_c = H$ for L_c-embedded H, the assertion follows for the case that the range space is L_c-embedded. Using Proposition 3.1 and the fact that $(G_c)^e = G$ by Bjon [4], the isomorphism in (3.4) finally turns into

$$C^\infty_{(e)}(U \times V, G) \cong C^\infty_{(e)}(U, C^\infty_{(e)}(V,G)).$$

REMARK. In general for equable convergence spaces X,Y and Z

$$C(X \times Y, Z) \not\cong C(X, C_e(Y,Z)).$$

Therefore the exponential law for equable structures in the theorem above do not fit into the very general setting of differential calculus, based on category theory, by Nel [12].

We conclude this section by giving some completeness results.

PROPOSITION 3.5. Let F be a complete L_c-embedded cvs and U a c-open subset of a cvs E. Then $C^p_{(c)}(U,F)$ is complete for $p = 0,1,...,\infty$.

PROOF. By Bjon and Lindström [7], $C_c(U,F)$ is complete if F is L_c-embedded and complete. Thus $C_c(U, L_c({}^kE, F))$ is complete for all $k \geq 0$. Let p be finite (the case when $p = \infty$ can be treated in the same manner) and let (f_ι) be a Cauchy-net in $C^p_{(c)}(U,F)$. Hence there are mappings $f_k \in C(U, L_c({}^kE, F))$ $(k = 0,1,...,p)$ with

$$D^k f_\iota \downarrow f_k \quad \text{in } C_c(U, L_c({}^kE, F)) \quad \text{for } k = 0,1,...,p.$$

However, by theorem 3.4 in [6], $f_k = D(f_{k-1})$ for $k = 1,...,p$, by which

$$D^k f_\iota \downarrow D^k f_0 \quad \text{in } C_c(U, L_c({}^kE, F)) \quad \text{for } k = 0,1,...,p.$$

Thus $f_\iota \downarrow f_0$ in $C^p_{(c)}(U,F)$ and the proof is complete.

The completeness of $C^p_{(e)}(U,F)$ is obvious under much more restrictive conditions.

PROPOSITION 3.6. Let F be a complete L_a-embedded cvs ($a = c, e$) and U a c-open subset of an equable cvs E. Then the space $C^\infty_{(e)}(U,F)$ is complete.

PROOF. First suppose that F is L_c-embedded and complete and let (f_ι) be a Cauchy-net in $C^\infty_{(e)}(U,F)$. Although we do not know whether the spaces $C_e(U, L_e({}^kE, F))$ are complete, we have the embedding

$$C_e(U, L_e({}^kE, F)) \hookrightarrow C_e(U, L_c({}^kE, F)),$$

and thus $(D^k f_\iota)$ is a Cauchy-net in the space $C_e(U, L_c({}^kE, F))$ that is complete by Bjon and Lindström [7]. As in the proposition above we obtain a function $f_0 \in C^\infty_c(U,F)$ with

$$f_\iota \downarrow f_0 \quad \text{in } \left(C^\infty_{(c)}(U,F)\right)^e.$$

The completeness of $C^\infty_{(e)}(U,F)$ then follows from Corollary 3.3. Now assume that F is L_e-embedded and complete. Then F is closed in L_eL_eF and conséquently $C^\infty_{(e)}(U,F)$ can be considered as a closed subspace of $C^\infty_{(e)}(U, L_eL_eF)$. Let $i : L_eL_eF \to C^\infty_{(e)}(L_eF, \mathbb{K})$ be the inclusion. Its associated mapping

$$\tilde{i} : L_eL_eF \times L_eF \to \mathbb{K}$$

is bilinear and continuous, hence of class C^∞_e. By Theorem 3.4 also i is of class C^∞_e, and especially it is continuous. Since L_eL_eF is closed in $C_e(L_eF)$, we thus obtain that L_eL_eF is closed in $C^\infty_{(e)}(L_eF, \mathbb{K})$ too. Consequently $C^\infty_{(e)}(U, L_eL_eF)$ is a closed subspace of $C^\infty_{(e)}(U, C^\infty_{(e)}(L_eF, \mathbb{K}))$. However, since L_eF is equable, $C^\infty_{(e)}(U, C^\infty_{(e)}(L_eF, \mathbb{K}))$ is isomorphic to $C^\infty_{(e)}(U \times L_eF, \mathbb{K})$ by Theorem 3.4. But this last space is complete according to the first part of this proof. Because closed sets in complete spaces are complete, we have thus shown the completeness of the cvs $C^\infty_{(e)}(U,F)$.

4. COMPACTLY GENERATED SPACES.

A compactly generated vector space E is said to be a *k-vector space* if the vector operations $+ : E \times_k E \to E$ and $\cdot : \mathbb{K} \times_k E \to E$ are continuous. If E is a topological vector space, then kE

is a k-vector space. A k-vector space is not always a topological vector space, since the structure of the k-product in general is finer than that of the topological one. If, however, E is a k-vector space, then E^{lc} is a cvs.

To every k-vector space E can be associated a locally convex topological vector space cE, where the topology is generated by the convex zero - neighbourhoods in E [13]. A k-vector space E with $E = kcE$, where cE is sequentially complete, we call a *convenient k-vector space.* These convenient k-vector spaces (containing all Fréchet spaces) constitute the basis for the differential calculus by Seip [13,14].

Let E and F be convenient k-vector spaces and U an open subset of E. A function $f : U \to F$ is said to be of class C_k^1 if there is a continuous mapping $Df : U \to kL_{co}(E,F)$, such that the map $\bar{\varepsilon}_0 : \tilde{O} \to F$ ($\tilde{O} = \{(s,x,h) \in \mathbb{K} \times_k U \times_k E : x + sh \in U\}$), defined by

$$\bar{\varepsilon}_0(0,x,h) = 0, \quad \bar{\varepsilon}_0(s,x,h) = \frac{f(x+sh) - f(x)}{s} - Df(x)h \quad \text{for } s \neq 0,$$

is continuous. Further we say that $f : U \to F$ is of class C_k^p if $Df : U \to kL_{co}(E,F)$ is of class C_k^{p-1}. By $C_k^\infty(U,F)$ we mean the space of all C_k^∞-functions $f : U \to F$. We use the following notations

$$kL_{co}({}^0E,F) = F, \quad kL_{co}({}^{p+1}E,F) = kL_{co}(E, kL_{co}({}^pE,F)) \quad \text{for } p = 0,1,...$$

The set $C_k^\infty(U,F)$, equipped with the initial structure induced by the differential operators

$$D^p : C_k^\infty(U,F) \to kC_{co}(U, kL_{co}({}^pE,F)), \quad p = 0,1,2,\dots,$$

we denote by $init(C_k^\infty(U,F))$. Let $C_{(k)}^\infty(U,F) = k(init(C_k^\infty(U,F)))$. In [13] and [14] it is shown that $C_{(k)}^\infty(U,F)$ is a convenient k-vector space and further that there exists an exponential law

$$C_{(k)}^\infty(U \times_k V, G) \cong C_{(k)}^\infty(U, C_{(k)}^\infty(V,G)) \tag{4.1}$$

for convenient k-vector spaces E, F and G with open sets U and V in E and F respectively.

In order to compare the classes of C_k^p- and C_c^p-functions, we need the following lemma.

LEMMA 4.1. Let X be compactly generated and Y a topological space. Then

$$C(X,Y) = C(X,kY) = C(X^{lc},kY) = C(X^{lc},Y), \tag{4.1}$$

$$kC_{co}(X,Y) = kC_{co}(X,kY). \tag{4.2}$$

Moreover, if Y is a separated topological vector space, then

$$C_{co}(X,Y) = C_c(X^{lc},Y). \tag{4.3}$$

PROOF. Since the verification of (4.1) is straightforward and thus left to the reader and (4.2) is a theorem due to Steenrod [15], we only have to concern about the structure in (4.3). Let $\mathcal{F} \downarrow 0$ in $C_{co}(X,Y)$, $x \in X$ and $\mathcal{G}$ be a filter with $\mathcal{G} \downarrow x$ in X^{lc}. Then there is a compact set $K \in \mathcal{G}$. Hence, to every zero - neighbourhood U of Y there exists an $F \in \mathcal{F}$ with $F(K) \subset U$ and thus $\mathcal{F}(\mathcal{G}) \downarrow 0$ in $C_c(X^{lc},Y)$. On the contrary, let $\mathcal{F} \downarrow 0$ in $C_c(X^{lc},Y)$, K be a compact set in X and U as above. Let further $x \in K$ and $\mathcal{G}$ be a filter with $\mathcal{G} \downarrow x$ in X^{lc}. Thus there exist $F_x \in \mathcal{F}$ and $G_x \in \mathcal{G}$ with $F_x(G_x) \subset U$. Since K is compact in X^{lc}, it can be covered by finitely many sets G_x, that is $K \subset G_{x_1} \cup \ldots \cup G_{x_n}$. Let $F = F_{x_1} \cap \ldots \cap F_{x_n}$. Then $F \in \mathcal{F}$ and $F(K) \subset U$, so $\mathcal{F} \downarrow 0$ in $C_{co}(X,Y)$.

In comparing C_k^∞- and C_c^∞-differentiability, the equality (4.3) will be of fundamental importance. The C_k^∞-functions are defined on open sets whereas the C_c^∞-functions require c-open domains. Now, it is easy to see that every open set in a k-vector space also is c-open. Since X and X^{lc} has exactly the same open sets, U^{lc} is c-open in E^{lc} if U is an open subset of a k-vector space E.

PROPOSITION 4.2. Let U be an open subset of a k-vector space E and F a separated locally convex topological vector space. Then

$$C_k^\infty(U,F) = C_c^\infty(U^{lc},F) = C_k^\infty(U,kF). \tag{4.4}$$

PROOF. By repeated use of Lemma 4.1 we arrive in

$$kL_{co}({}^pE,F) = kL_c({}^pE^{lc},F) = kL_{co}({}^pE,kF), \quad \text{for } p = 1,2,\ldots \tag{4.5}$$

By definition, $f \in C_k^\infty(U,F)$ iff for each $p = 0,1,\ldots$ there exist continuous mappings $D^{p+1}f : U \to kL_{co}({}^{p+1}E,F)$ and $\bar{\varepsilon}_p : \tilde{O} \to kL_{co}({}^pE,F)$ ($\tilde{O} = \{(s,x,h) \in \mathbb{K} \times_k U \times_k E : x + sh \in U\}$), where the latter is defined by

$$\bar{\varepsilon}_p(0,x,h) = 0, \; \bar{\varepsilon}_p(s,x,h) = \frac{D^pf(x+sh) - D^pf(x)}{s} - D^{p+1}f(x)h \quad \text{for } s \neq 0.$$

Take a function $f \in C_k^\infty(U,F)$. Since

$$C(U,kL_{co}({}^pE,F)) = C(U^{lc},L_c({}^pE^{lc},F)),$$

also $\bar{\varepsilon}_p : (\tilde{O})^{lc} \to L_c({}^pE^{lc},F)$ is continuous for every p, and hence obviously $f \in C_c^\infty(U^{lc},F)$. It then remains to be shown that $C_c^\infty(U^{lc},F) \subset C_k^\infty(U,F)$. Since the procedure is the same for all derivatives, it suffices to be verified that $C_c^1(U^{lc},F) \subset C_k^1(U,F)$. Let $f \in C_c^1(U^{lc},F)$. As in theorem 3.4 in [6] we write

$$\bar{\varepsilon}_0(s,x,h) = \int_0^1 (Df(x+tsh)h - Df(x)h)dt.$$

Since F is L_c-embedded, by lemma 3.2 in [6] $\bar{\varepsilon}_0 : \tilde{O} \to F$ is continuous, where $\tilde{O}$ is considered to be a subspace of $\mathbb{K}^{lc} \times U^{lc} \times E^{lc} = (\mathbb{K} \times U \times E)^{lc}$, and thus also when $\tilde{O}$ is a subspace of $\mathbb{K} \times_k U \times_k E$. Hence $f \in C_k^1(U,F)$, which completes the proof.

PROPOSITION 4.3. Let U be an open subset of a k-vector space E and F a separated locally convex topological vector space. Then

$$C_{(k)}^\infty(U,F) = C_{(k)}^\infty(U,kF), \tag{4.6}$$

$$kC_{(c)}^\infty(U^{lc},F) = C_{(k)}^\infty(U,F). \tag{4.7}$$

PROOF. Since $kC_{co}(U,kL_{co}({}^pE,F)) = kC_{co}(U,kL_{co}({}^pE,kF))$, (4.6) follows directly from (4.4). Further (4.7) is a consequence of (4.4) and the relation

$$kC_c(U^{lc},L_c({}^pE^{lc},F)) = kC_{co}(U,kL_{co}({}^pE,F)).$$

THEOREM 4.4. Let E, F and G be convenient k-vector spaces and U as well as V open subsets of E and F respectively. Then there is an exponential law

$$C^\infty_{(k)}(U \times_k V, G) \cong C^\infty_{(k)}(U, C^\infty_{(k)}(V, G)). \tag{4.8}$$

PROOF. Since $G = kcG$, where cG is a separated locally convex topological space, cG is L_c-embedded and hence, by theorem 3.9 in [3], there is an exponential law

$$C^\infty_{(c)}(U^{lc} \times V^{lc}, cG) \cong C^\infty_{(c)}(U^{lc}, C^\infty_{(c)}(V^{lc}, cG)). \tag{4.9}$$

The law in (4.8) then follows from that in (4.9) by repeated use of (4.6) and (4.7), remembering that $C^\infty_{(c)}(V^{lc}, cG)$ is topological and L_c-embedded and thus a separated locally convex topological vector space.

5. HOLOMORPHY.

an open subset of E. A function $f : U \to F$ is said to be *Gateaux holomorphic* if the function $\lambda \mapsto l(f(x + \lambda h))$ is holomorphic in a neighbourhood of zero in $\mathbf{C}$ for each $x \in U$, $h \in E$ and $l \in LF$. It is *holomorphic*, if it is Gateaux holomorphic and continuous. Let $H(U, F)$ denote the set of all holomorphic functions $f : U \to F$.

Let U be a τ-open subset (i.e. to each $x \in U$ there is a circled convex open set U_x with $x + U_x \subset U$) of a cvs E and F an L_c-embedded sequentially complete cvs. Bjon and Lindström [7] show that a holomorphic function $f : U \to F$ has an expansion

$$f(x + h) = \sum_{m=0}^{\infty} \frac{\hat{d}^m f(x)}{m!} h, \tag{5.1}$$

where the mappings $\hat{d}^m f(x) : E \to F$, defined by

$$\hat{d}^m f(x) h = \frac{m!}{2\pi i} \int_{|\lambda|=1} \frac{f(x + \lambda h)}{\lambda^{m+1}} d\lambda \quad (m = 0, 1, \dots), \tag{5.2}$$

are continuous m-homogeneous polynomials.

Certainly a differentiable function is holomorphic. On the contrary, by corollary 4.1.1 in [6], a holomorphic function $f : U \to F$, where U is a τ-open subset of a cvs E (respectively an equable cvs E) and F is a sequentially complete L_c-embedded (L_e-embedded) cvs, is of class C^∞_c (C^∞_e) with the derivative given by (5.2) for $m = 1$; that is,

$$Df(x)h = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{f(x + \lambda h)}{\lambda^2} d\lambda. \tag{5.3}$$

Thus

$$C^p_a(U, F) = H(U, F) \quad (a = c, e),\ p = 1, 2, ..., \infty. \tag{5.4}$$

Let (f_ι) be a net with $f_\iota \downarrow 0$ in $H_c(U, F)$. By proposition 4.5 in [7], the mapping $\hat{d} : H_c(U, F) \times U \times E \to F$, where $\hat{d}(f, x, h) = \hat{d}f(x)h$, is continuous. Thus $Df_\iota \downarrow 0$ in $C_c(U, L_c(E, F))$. From (5.4) we get that $Df_\iota \in H(U, L_c(E, F))$ and hence $D^2 f_\iota \downarrow 0$ in $C_c(U, L_c(^2E, F))$ etc. If E is equable and F is L_e-embedded and sequentially complete, we can use theorem 4.3 in [6] and get

$$f_\iota \downarrow 0 \text{ in } H_e(U, F) \Rightarrow D^p f_\iota \downarrow 0 \text{ in } C_e(U, L_e(^pE, F)).$$

Thus we have

PROPOSITION 5.1. Let U be a τ-open subset of a cvs E (an equable cvs E) and F an L_c-embedded (L_e-embedded) sequentially complete cvs F. Then

$$C^p_{(c)}(U,F) = H_c(U,F), \quad (C^p_{(e)}(U,F) = H_e(U,F))\ p = 1, 2, ..., \infty.$$

As is easily seen, completeness can be replaced with sequentially completeness in Propositions 3.5 and 3.6. Thus the theorem below follows directly from Proposition 5.1.

THEOREM 5.2. Let G be a sequentially complete L_a-embedded cvs ($a = c, e$) and U and V τ-open subsets of two cvs E and F respectively, which are equable when $a = e$. Then

$$H_a(U \times V, G) \cong H_a(U, H_a(V, G)) \quad (a = c, e). \tag{5.5}$$

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