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MATRIX SPREAD SETS OF p-PRIMITIVE SEMIFIELD PLANES

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ABSTRACT. In this article we present the matrix spread sets of the p-primitive planes of order p^4 where $p = 3, 5, 7, 11$.

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1. INTRODUCTION

The p-primitive semifield planes are precisely the semifield planes of order p^4 and kernel $GF(p^2)$ which are obtained when the construction method of Hiramine, Matsumoto and Oyama [1] is applied to the Desarguesian plane of order p^2 (see Johnson [2]). If π is a p-primitive semifield plane, then π has a matrix spread set of the form

$$\left\{ \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} : u, v \in GF(p^2) \right\}$$

where $f(v) = f_0 v + f_1 v^p$ for some $f_0, f_1 \in GF(p^2)$. We denote this plane by $\pi(f)$ or $\pi(f_0, f_1)$ In [3] we began our study of this class of planes which we continued on [4]-[6]. First we studied necessary and sufficient conditions on the function f that give isomorphic planes. Also we showed on Theorem 4.2 [4] that there are $\left(\frac{p+1}{2}\right)^2$ nonisomorphic p-primitive semifield planes for every prime $p > 2$. Of these $\frac{p+1}{2}$ are of the type introduced by Hughes-Kleinfeld in [7]; one is a Dickson semifield plane (see Dembowski [8]) and $(p-1)/4$ or $(p-3)/4$ are Boerner-Lantz [9] semifield planes (according as -1 is a square or a nonsquare in $GF(p)$, respectively, $p > 3$. For $p = 3$, the Boerner-Lantz semifield plane of order 81 is p-primitive). In a joint work with R. Figueroa [10] we showed that the remaining planes and their duals do not belong to any of the known classes of semifield planes. In this article we present the results of a search done with the aid of the computer to determine explicitly the matrix spread set of a representative of each isomorphism class of these new semifield planes of order p^4 for $p = 3, 5, 7$ and 11

2. p-PRIMITIVE PLANES FOR $p \leq 11$

We recall the following result.

PROPOSITION 2.1 (see Cordero [3]) Let $f : GF(p^2) \to GF(p^2)$ be given by $f(u) = f_0 u + f_1 u^P$ where $f_0 = a_0 + a_1 t$, $f_1 = b_2 + b_1 t$, a_0, a_1, b_0, $b_1 \in GF(p)$ and let θ be a nonsquare in $GF(p)$ such that $t^2 = \theta$. Then f defines a matrix spread set

$$\left\{ \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} : u, v \in GF(p^2) \right\}$$

of a p-primitive semifield plane if and only if $a_0^2 - (a_1^2 - b_1^2)\theta$ is a nonsquare in $GF(p)$

For $p = 3, 5, 7$ and 11 all the functions f in $GF(p^2)$ that satisfy the condition on (2 1) were determined employing the computer program PRIMITIVE The input for this program is NONSQ which contains first the prime p, then an arbitrary but fixed nonsquare θ in $GF(p)$ and then all the nonsquares in $GF(p)$. PRIMITIVE determines all the sets a_0, a_1, b_1 that satisfy the condition above for the given value of θ In the output we get these coefficients a_0, a_1, b_0, b_1 where b_0 is any element in $GF(p)$

After obtaining all such functions f we divided the planes determined by these into isomorphism classes For this, we first used a computer program called ISO_B that determines which planes are isomorphic via the isomorphism given by

$$\Gamma = {}_{\sigma}\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

where $B = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$, $b \in GF(p^2) - \{0\}$ and σ is an automorphism of $GF(p^2)$. Notice that if Γ is an isomorphism from $\pi(f_0, f_1)$ into $\pi(F_0, F_1)$ then $F_0 = b^2 f_0$ and $F_1 = b^{p+1} f_1$ or $F_0 = b^2 f_0^2$ and $F_1 = b^{p+1} f_1^p$ and therefore many planes will be found to be isomorphic via this isomorphism (see Cordero [4])

When these programs were run, the following was obtained·

Prime p	Nonsquare θ	How Many Solutions	How Many Nonisomorphic
3	2	13	4
5	2	200	11
7	6	882	23
11	10	6050	58

After obtaining all the possible isomorphic planes with this type of isomorphism we analyze the output and apply the isomorphism theorem for p-primitive semifield planes given in Cordero [4] to determine all the nonisomorphic p-primitive planes for $p = 3, 5, 7$ and 11

Case $p = 3$: From the output of PRIMITIVE, we obtain that there are 18 functions f that give matrix spread sets of p-primitive planes for $p = 3$. After running ISO_B with these as input we obtain that there are 4 isomorphism classes and no further collapsing is possible by Theorem 3.1 in Cordero [4].

Two of these planes have $f_0 = 0$ and by using Theorem 3.3 in Cordero [4] we conclude that they are Hughes-Kleinfeld semifield planes. Of the two remaining planes one has $f_1 = 0$ and therefore it is a Dickson semifield plane by Theorem 3.2 in [4] and the other is the semifield plane of Boerner-Lantz of order 81 by Theorem 3.5 in [4]. We present these results in the following table; the first column gives the coefficients a_0, a_1, b_0, b_1 of the function f in the matrix spread set of the plane

$$\left\{ \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} : u, v, \in GF(p^2) \right\}$$

where $f(v) = f_0 v + f_1 v^p$, $f_0 = a_0 + a_1 t$, $f_1 = b_1 + b_1 t$, $t \in GF(3)$, of one representative of each class.

Table 1. p-primitive planes for $p = 3$

Coefficients of f	Identification of the Class
0,0,0,1, 0,0,1,1	Hughes-Kleinfeld
1,1,0,0	Dickson
1,1,0	Boerner-Lantz

Case $p = 5$: There are 200 matrix spread sets of p-primitive planes of order 5^4 When we use these as input for ISO_B, we obtain 11 isomorphism classes: a representative of each class is given below (the plane number is the number that was assigned to the plane in the output of PRIMITIVE)

Plane	Coefficients of f $a_0\ a_1\ b_0\ b_1$
1	1 1 1 2
2	1 1 2 2
3	1 1 3 2
4	1 1 4 2
5	1 1 0 2
11	1 2 1 0
12	1 2 2 0
15	1 2 0 0
181	0 0 1 1
182	0 0 2 1
185	0 0 0 1

Planes #1-5 have $f_0 = 1 + t$ and $f_1 \neq 0$. Applying Theorem 3.4 in Cordero [4] to these, we get that plane #1 is isomorphic to plane #4 and plane #2 is isomorphic to plane #3. The next two planes have $f_0 = 1 + 2t$, but the f_1's do not have the necessary property for the planes to be isomorphic. Plane #15 has $f_1 = 0$ and it is not isomorphic to any other plane on the list by Theorem 3.2 in Cordero [4] and the last 3 planes have $f_0 = 0$ and are not isomorphic by Theorem 3.1 in [4] A plane with $f_0 = 1 + t$ cannot be isomorphic to a plane with $f_0 = 1 + 2t$ because this will imply that there exist $a \in GF(5)$ and $c \in GF(25)$ such that $1 + 2t = ac^{p-1}(1 + t)$ orl $+ 2t = ac^{p-1}(1 - t)$; in either case we will need $a^2 = 2$, which is impossible. Therefore, we conclude that there are 9 nonisomorphic p-primitive planes for $p = 5$. A p-primitive semifield plane $\pi(f_0, f_1)$ with $p \geq 5$ is said to be of **type IV** if $f_0 \neq 0$ and $f_1^{2(p-1)} \neq 0, 1$, and of **type V** if $f_0 \neq 0$ and $f_1^{2(p-1)} = 1$. In a joint work with R. Figueroa [10] we showed that if π is a p-primitive plane of type IV which is not a Boerner-Lantz semifield plane or is of type V then neither π nor its dual belong to any of the known classes of semifield planes. For $p = 5$ we have one plane of type IV which is not Boerner-Lantz and three nonisomorphic planes of type V.

In table 2 we give representatives of the p-primitive planes $p = 5$.

Table 2. p-primitive planes for $p = 5$

Coefficients of f	Identification of the Class
0,0,0,1; 0,1,2,1; 0,0,2,1	Hughes-Kleinfeld
1,2,0,0	Dickson
1,1,2,2	Boerner-Lantz
1,1,1,2	Type IV
1,1,0,2; 1,2,1,0; 1,2,2,0	Type V

Case $p = 7$. When $p = 7$ there are 822 functions f that give matrix spread sets of p-primitive planes of order 7^4 With ISO_B these are reduced to 23 isomorphism classes and by using similar arguments as in the case when $p = 5$ we get that there are 16 nonisomorphic p-primitive planes for $p = 7$ These are presented in the following table.

Table 3. p-primitive planes for $p = 7$

Coefficients of f	Identification of the Class
0,0,0,1; 0,0,1,1; 0,0,2,1, 0,0,3,1	Hughes-Kleinfeld
1,2,0,0	Dickson
1,1,4,2	Boerner-Lantz
1,1,1,2; 1,1,2,2; 1,2,1,3; 1,2,2,3; 2,2,3,3	Type IV
1,1,0,2; 1,2,0,3; 1,2,1,0; 1,2,2,0; 1,2,3,0	Type V

Case $p = 11$. There are 6050 matrix spread sets of p-primitive planes of order 11^4 When these are used as input for ISO_B we obtain 58 isomorphism classes. To complete the analysis of these we need to determine if a plane with $f_0 = 1 + t$ can be isomorphic to a plane with $f_0 = 1 + 2t$. Suppose there exists $a \in GF(11) - \{0\}$ and $c \in GF(11^2)$ such that $1 + 2t = ac^{p-1}(1 + t)$ or $1 + 2t = ac^{p-1}(1 - t)$ In the first case we will have $\left(\frac{1+2t}{(1+t)c^{p-1}}\right)^{p+1} = a^{p+1}$ Since $t^2 = -1$ we must have that a satisfies the equation $a^2 = 8$, but 8 is a nonsquare in $GF(11)$. In the second case we will have that $\left(\frac{1+2t}{(1-t)c^{p-1}}\right)^{p+1} = a^{p+1}$ and again this implies that $a^2 = 8$. Therefore no plane with $f_0 = 1 + t$ can be isomorphic to a plane with $f_0 = 1 + 2t$. We conclude that there are 36 classes of nonisomorphic p-primitive planes for $p = 11$ Their function f is given in the following table.

Table 4. p-primitive planes for $p = 11$

Coefficients of f	Identification of the Class
0,0,0,1; 0,0,1,1; 0,0,2,1; 0,0,3,1; 0,0,4,1; 0,0,5,1	Hughes-Kleinfeld
1,1,0,0	Dickson
1,2,4,3; 1,2,4,5	Boerner-Lantz
1,1,1,4, 1,1,2,4; 1,1,3,4, 1,1,4,4, 1,1,5,4; 1,1,1,5, 1,1,2,5; 1,1,3,5; 1,1,4,5; 1,1,5,5, 1,2,1,3; 1,2,2,3; 1,2,3,3; 1,2,5,3; 1,2,1,5; 1,2,2,5, 1,2,3,5; 1,2,5,5	Type IV
1,1,0,4; 1,1,0,5; 1,1,1,0; 1,1,2,0; 1,1,3,0; 1,1,4,0; 1,1,5,0; 1,2,0,3; 1,2,0,5	Type V

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