Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 443-450

CONVERGENCE IN MEAN OF WEIGHTED SUMS OF $\{a_{nk}\}$-COMPACTLY UNIFORMLY INTEGRABLE RANDOM ELEMENTS IN BANACH SPACES

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(Received October 23, 1995 and in revised form March 14, 1996)

ABSTRACT. The convergence in mean of a weighted sum $\sum_k a_{nk}(X_k - EX_k)$ of random elements in a separable Banach space is studied under a new hypothesis which relates the random elements with their respective weights in the sum: the $\{a_{nk}\}$-compactly uniform integrability of $\{X_n\}$. This condition, which is implied by the tightness of $\{X_n\}$ and the $\{a_{nk}\}$-uniform integrability of $\{\|X_n\|\}$, is weaker than the compactly uniform integrability of $\{X_n\}$ and leads to a result of convergence in mean which is strictly stronger than a recent result of Wang, Rao and Deli.

KEYWORDS AND PHRASES: Weighted sums, random elements in separable Banach spaces, compactly uniform integrability, $\{a_{nk}\}$-compactly uniform integrability, tightness, $\{a_{nk}\}$-uniform integrability, convergence in mean.
1991 AMS SUBJECT CLASSIFICATION CODES: 60B12, 60F05.

1. INTRODUCTION.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, and let $\{X_n\}, n \in \mathbf{N}$, be a sequence of random elements in a separable Banach space $(X, \|.\|)$, i.e., a sequence of functions from Ω into Xwhich are $\mathcal{A}$-measurable with respect to the Borel subsets of X.

Let $\{a_{nk}\}, k, n \in \mathbf{N}$, be an array of real numbers with $\sup_n \sum_k |a_{nk}| < \infty$.

In this paper, we deal with the convergence in mean of the sequence of weighted sums $S_n = \sum_k a_{nk}(X_k - EX_k)$.

Habitually, this problem of weak convergence, as well as the problem of strong convergence, has been studied by considering separately the conditions on the sequence $\{X_n\}$ (relaxing more and more the initial hypothesis of independence and identical distribution) and the conditions on the array $\{a_{nk}\}$.

Ordóñez ([1]) obtains results of convergence in mean and convergence in r-mean ($r \in (0,1)$) for weighted sums of random variables (random elements in $\mathbf{R}$) by requiring a condition which relates the random variables X_k to their respective weights a_{nk}: the $\{a_{nk}\}$-uniform integrability of $\{X_n\}$. This condition is weaker than the uniform integrability of $\{X_n\}$ and leads to the

Cesàro uniform integrability (see [2]) as a particular case. A counterexample shows that the result of convergence in mean does not hold, as stated, for weighted sums of random elements in a separable Banach space.

In this paper, we obtain such a result, under a new condition on the random elements X_k concerning their respective weights in S_n: $\{X_n\}$ is required to be $\{a_{nk}\}$-compactly uniformly integrable. This requeriment is weaker than the requirement of compactly uniform integrability used by Hoffmann-Jorgensen and Pisier ([3]) and Daffer and Taylor ([4]), and characterized by Wang and Rao ([5]); consequently, our result extends the result obtained by Wang et al. ([6], Theorem 3.2).

2. PRELIMINARIES.

In the proof of the main result, the embedding of a separable Banach space X in a Banach space with a Schauder basis will be used as in Taylor ([7]).

A sequence $\{b_n\}, n \in \mathbf{N}$, in X is a Schauder basis for X if for each $x \in X$ there exists a unique sequence of real numbers $\{t_n\}$ such that $x = \sum_n t_n b_n$.

When a Banach space X has a Schauder basis $\{b_n\}$, a sequence of continuous linear functionals $\{f_n\}$ on X can be defined by $f_n(x) = t_n, n \in \mathbf{N}$; these are called the coordinate functionals for the basis $\{b_n\}$.

The partial sum operator U_n on X is defined by $U_n(x) = \sum_{k=1}^{n} f_k(x) b_k$, and the residual operator Q_n on X by $Q_n(x) = x - U_n(x)$, for every $x \in X$. $\{U_n\}$ and $\{Q_n\}$ are two sequences of continuous linear operators on X satisfying $\lim_n U_n(x) = x$ and $\lim_n Q_n(x) = 0$ for every $x \in X$.

A sequence $\{X_n\}$ of random elements in a Banach space X is said to be tight if for each $\varepsilon > 0$ there exists a compact subset K of X such that

$$\sup_n P[X_n \notin K] < \varepsilon.$$

Let $p > 0$. A sequence $\{X_n\}, n \in N$, of random elements in a Banach space X is said to be compactly uniformly p-th order integrable if for every $\varepsilon > 0$ there exists a compact subset K of X such that

$$\sup_n E\|X_n\|^p I_{[X_n \notin K]} < \varepsilon.$$

(I_A denotes the indicator of the event A).

If p = 1, $\{X_n\}$ is said to be compactly uniformly integrable.

For the characterization of this concept, we refer to Wang and Rao ([5]) and Cuesta and Matrán ([8]): Let $\{X_n\}$ be a sequence of random elements in a separable Banach space, and let $p > 0$. Then, $\{X_n\}$ is compactly uniformly p-th order integrable if, and only if, $\{X_n\}$ is tight and $\{\|X_n\|^p\}$ is uniformly integrable.

For the relation between the notions of tightness, uniform integrability, boundedness of moments and domination in probability, we refer to Taylor ([7]).

3. COMPACTLY UNIFORMLY INTEGRABLE RANDOM ELEMENTS CONCERNING AN ARRAY.

Ordóñez ([1]) introduces the following concept:

DEFINITION 3.1. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants satisfying $\sup_n \sum_k |a_{nk}| < \infty$.

A sequence $\{X_n\}$ of integrable random variables is said to be $\{a_{nk}\}$-uniformly integrable (or uniformly integrable concerning the array $\{a_{nk}\}$) if

$$\lim_{a\to\infty} \sup_n \sum_k |a_{nk}| E|X_k| I_{[|X_k|>a]} = 0.$$

The following assertion is easy to check:

PROPOSITION 3.1. Let $\{X_n\}$ be a sequence of uniformly integrable random variables. Then, $\{X_n\}$ is $\{a_{nk}\}$-uniformly integrable for all arrays $\{a_{nk}\}$ such that $\sup_n \sum_k |a_{nk}| < \infty$.

A sequence of random elements in a separable Banach space being compactly uniformly integrable is the natural extension of a sequence of random variables being uniformly integrable (both definitions are equivalent when the Banach space is finite dimensional). To this effect, we introduce the following notion:

DEFINITION 3.2. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants satisfying $\sup_n \sum_k |a_{nk}| < \infty$. Let $p > 0$. A sequence $\{X_n\}, n \in N$, of random elements in a separable Banach space X is said to be $\{a_{nk}\}$-compactly uniformly p-th order integrable if for every $\varepsilon > 0$ there exists a compact subset K of X such that

$$\sup_n \sum_k |a_{nk}| E\|X_k\|^p I_{[X_k \notin K]} < \varepsilon.$$

If $p = 1$, $\{X_n\}$ is said to be $\{a_{nk}\}$-compactly uniformly integrable.

The following theorem provides a sufficient condition for the $\{a_{nk}\}$-compactly uniform p-th order integrability of $\{X_n\}$:

THEOREM 3.1. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants with $\sup_n \sum_k |a_{nk}| < \infty$, and let $p > 0$. Let $\{X_n\}, n \in N$, be a sequence of random elements in a separable Banach space X, which is tight and such that the sequence $\{\|X_n\|^p\}$ is $\{a_{nk}\}$-uniformly integrable.

Then, $\{X_n\}$ is $\{a_{nk}\}$-compactly uniformly p-th order integrable.

PROOF. By Theorem 2 in [1], given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\{A_k\}$ is a sequence of events satisfying $\sup_n \sum_k |a_{nk}| P(A_k) < \delta$, then $\sup_n \sum_k |a_{nk}| E\|X_k\|^p I_{A_k} < \varepsilon$.

The tightness of $\{X_n\}$ implies the existence of a compact subset K of X such that $P[X_n \notin K] < \delta C^{-1}$ for every $n \in N$, where $C > 0$ is any constant such that $\sup_n \sum_k |a_{nk}| \le C$.

Then $\sup_n \sum_k |a_{nk}| P[X_k \notin K] < \delta$, and therefore:

$$\sup_n \sum_k |a_{nk}| E\|X_k\|^p I_{[X_k \notin K]} < \varepsilon$$

i.e., $\{X_n\}$ is $\{a_{nk}\}$-compactly uniformly p-th order integrable.

It is easy to check the following

PROPOSITION 3.2. Let $\{X_n\}$ be a sequence of compactly uniformly p-th order $(p > 0)$ integrable random elements in a separable Banach space. Then, $\{X_n\}$ is as in Theorem 3.1, and so $\{X_n\}$ is $\{a_{nk}\}$-compactly uniformly p-th order integrable for all arrays $\{a_{nk}\}$ such that $\sup_n \sum_k |a_{nk}| < \infty$.

REMARK. The characterization of $\{a_{nk}\}$-compactly uniform p-th order integrability of $\{X_n\}$ in terms of tightness of $\{X_n\}$ and $\{a_{nk}\}$-uniform integrability of $\{\|X_n\|^p\}$ is not available (i.e. the condition in Theorem 3.1 is not necessary), as is shown by considering the sequence of random elements in $l^1 = \{x \in \mathbf{R}^\infty : \|x\| = \sum_n |x_n| < \infty\}$ defined by $X_n = e_n$ with probability 1, where $\{e_n\}$ is the standard basis of l^1, and the array

$$a_{nk} = \begin{cases} \frac{1}{k^2} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n. \end{cases}.$$

Given $\varepsilon > 0$, take $m \in N$ such that $\frac{1}{m} < \varepsilon$, and let the compact $K = \{e_1, e_2, ..., e_m\}$. Then

$$\sup_n \sum_k |a_{nk}| E\|X_k\|^p I_{[X_k \notin K]} \leq \sum_{k>m} \frac{1}{k^2} \leq \frac{1}{m} < \varepsilon.$$

Therefore, $\{X_n\}$ is $\{a_{nk}\}$-compactly uniformly p-th order integrable $(p > 0)$, but $\{X_n\}$ is not tight.

4. CONVERGENCE IN MEAN.

Ordóñez ([1]) obtains the following result of convergence in mean for weighted sums of random variables:

THEOREM 4.1. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants satisfying:

a) $\sup_n \sum_k |a_{nk}| < \infty$

b) $\lim_n \sup_k |a_{nk}| = 0$.

Let $\{X_n\}, n \in N$, be a sequence of pairwise independent and $\{a_{nk}\}$-uniformly integrable random variables.

Then $S_n = \sum_k a_{nk}(X_k - EX_k) \to 0$ in mean.

This theorem does not hold, as stated, for random elements in separable Banach spaces (see [1]). Now, in Theorem 4.2, we prove that such an extension is possible for a sequence of random elements $\{X_n\}$ being $\{a_{nk}\}$-compactly uniformly integrable. Previously, we prove the following lemma:

LEMMA 4.1. Let X be a Banach space with a Schauder basis $\{b_n\}$. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants such that $\sup_n \sum_k |a_{nk}| < \infty$.

Let $\{X_n\}, n \in N$, be a sequence of random elements in X which is $\{a_{nk}\}$-compactly uniformly p-th order integrable for some $p \geq 1$.

Then:

$$\lim_t \sup_n \sum_k |a_{nk}| E\|Q_t(X_k - EX_k)\|^p = 0.$$

PROOF. Given $\varepsilon > 0$, there exists a compact subset K of X such that

$$\sup_n \sum_k |a_{nk}| E\|X_k\|^p I_{[X_k \notin K]} < \varepsilon 2^{-2p}(M+1)^{-p}$$

where M is the basis constant of the Schauder basis $\{b_n\}$.

We define, for each $n \in N$:

$$W_n = X_n I_{[X_n \in K]} \qquad Y_n = X_n I_{[X_n \notin K]} = X_n - W_n.$$

The compactness of K implies (see [7]) that there exists $t_0 \in N$ such that $\|Q_t(W_k)\| < \frac{1}{4}\varepsilon^{\frac{1}{p}} C^{-\frac{1}{p}}$ for every $k \in N$ and $t \geq t_0$, where $C > 0$ is a constant such that $\sup_n \sum_k |a_{nk}| \leq C$.

Then, for every $k \in N$:

$$\begin{aligned} E\|Q_t(W_k - EW_k)\|^p &= E\|Q_t(W_k) - EQ_t(W_k)\|^p \\ &\leq 2^{p-1}\left(E\|Q_t(W_k)\|^p + E^p\|Q_t(W_k)\|\right) \leq 2^p E\|Q_t(W_k)\|^p < \varepsilon 2^{-p} C^{-1}. \end{aligned}$$

On the other hand:

$$\begin{aligned} \sum_k |a_{nk}| E\|Q_t(Y_k - EY_k)\|^p &\leq 2^{p-1}(M+1)^p \sum_k |a_{nk}| \left(E\|Y_k\|^p + E^p\|Y_k\|\right) \\ &\leq 2^p (M+1)^p \sum_k |a_{nk}| E\|Y_k\|^p < \varepsilon 2^{-p}. \end{aligned}$$

Therefore, for every $t \geq t_0$:

$$\sup_n \sum_k |a_{nk}| E\|Q_t(X_k - EX_k)\|^p \leq 2^{p-1}(\varepsilon 2^{-p} + \varepsilon 2^{-p}) = \varepsilon.$$

In a similar manner, the following lemma, where the restriction $p \geq 1$ is omitted, can be proved:

LEMMA 4.2. Let X, $\{a_{nk}\}$ and $\{X_n\}$ be as in Lemma 1, with $p > 0$. Then:

$$\lim_t \sup_n \sum_k |a_{nk}| E\|Q_t(X_k)\|^p = 0.$$

THEOREM 4.2. Let X be a separable Banach space. Let $\{a_{nk}\}, k, n \in N$, be an array of real constants satisfying:

a) $\sup_n \sum_k |a_{nk}| < \infty$

b) $\lim_n \sup_k |a_{nk}| = 0$.

Let $\{X_n\}, n \in N$, be a sequence of pairwise independent and $\{a_{nk}\}$-compactly uniformly integrable random elements in X.

Then $E\|\sum_k a_{nk}(X_k - EX_k)\| \to 0 \quad$ as $\quad n \to \infty$.

PROOF. The $\{a_{nk}\}$-compactly uniform integrability of $\{X_n\}$, the consideration of boundedness of the corresponding compact K and the condition a) yield $\sum_k |a_{nk}|E\|X_k - EX_k\| < \infty$ for each $n \in N$, and so the almost sure convergence of S_n for each $n \in N$.

Since X can be isometrically embedded in a Banach space with a Schauder basis, it can be assumed, without loss of generality, that X has a Schauder basis $\{b_n\}$; let M be the basis constant.

For each fixed $t \in N$, and for every $n \in N$:

$$E\|\sum_k a_{nk}(X_k - EX_k)\| \le E\|\sum_{i=1}^{t} f_i\left(\sum_k a_{nk}(X_k - EX_k)\right) b_i\| + E\|Q_t\left(\sum_k a_{nk}(X_k - EX_k)\right)\|.$$

According to Lemma 4.1, given $\varepsilon > 0$, there exists $t \in N$ such that

$$E\|Q_t\left(\sum_k a_{nk}(X_k - EX_k)\right)\| < \frac{\varepsilon}{4} \quad \text{for every } n \in N.$$

We fix such a $t \in N$: let $m = \max_{1\le i\le t} \|f_i\|\|b_i\|$.
There exists a compact subset K of X such that:

$$\sup_n \sum_k |a_{nk}|E\|X_k\|I_{[X_k \notin K]} < \frac{\varepsilon}{4mt}.$$

We define, for each $n \in N$:

$$W_n = X_n I_{[X_n \in K]} \qquad Y_n = X_n I_{[X_n \notin K]} = X_n - W_n.$$

We have:

$$E\|\sum_{i=1}^{t} f_i\left(\sum_k a_{nk}(X_k - EX_k)\right) b_i\|$$
$$\le \sum_{i=1}^{t}\left(E\left|f_i\left(\sum_k a_{nk}(W_k - EW_k)\right)\right| + E\left|f_i\left(\sum_k a_{nk}(Y_k - EY_k)\right)\right|\right)\|b_i\|.$$

$\{f_i(W_k - EW_k)\}$, $k \in N$, is, for each $i \in N$, a sequence of pairwise independent random variables with mean 0, and, consequently:

$$E\left|f_i\left(\sum_k a_{nk}(W_k - EW_k)\right)\right|\|b_i\| \le E^{\frac{1}{2}}\left(\sum_k a_{nk}f_i(W_k - EW_k)\right)^2\|b_i\|$$
$$\le m\left(\sum_k |a_{nk}|^2 E\|W_k - EW_k\|^2\right)^{\frac{1}{2}} \le A\left(\sum_k |a_{nk}|^2\right)^{\frac{1}{2}}.$$

by the boundedness of K (A is a constant).

As $\sum_k |a_{nk}|^2 \le (\sup_k |a_{nk}|) \sum_k |a_{nk}| \to 0$ when $n \to \infty$, we can choose $n_0 \in N$ such that for every $n \ge n_0$ and $1 \le i \le t$:

$$E\left| f_i \left(\sum_k a_{nk}(W_k - EW_k) \right) \right| \|b_i\| < \frac{\varepsilon}{4t}.$$

On the other hand:

$$E\left| f_i \left(\sum_k a_{nk}(Y_k - EY_k) \right) \right| \|b_i\| \le 2m \sum_k |a_{nk}| E\|Y_k\|.$$

Therefore

$$E\| \sum_k a_{nk}(X_k - EX_k)\| < t\frac{\varepsilon}{4t} + 2mt\frac{\varepsilon}{4mt} + \frac{\varepsilon}{4} = \varepsilon.$$

for every $n \ge n_0$.

EXAMPLE 4.1. The following example (suggested by another one in [9]) shows that the conditions in Theorem 3.1, and therefore the $\{a_{nk}\}$-compactly uniform p-th order integrability, are weaker than the compactly uniform p-th order integrability. So, our Theorem 4.2 is strictly stronger than Theorem 3.2 in [6]:

Consider the separable Banach space l^1, and let $\{e_n\}$ be the standard basis.

Let $\{X_n\}, n \in N$, be the sequence of independent random elements in l^1 defined by

$$X_n = \begin{cases} ne_n & \text{with probability } \frac{1}{2n} \\ -ne_n & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

If K is any compact subset of l^1, then K contains at most finitely many elements of the set $\{\pm ne_n, n \in N\}$, and so $\sup_n E\|X_n\| I_{[X_n \notin K]} = 1$ which implies that $\{X_n\}$ is not compactly uniformly integrable.

Now, given $\varepsilon > 0$, we choose $n_0 \in N$ such that $\frac{1}{n_0} < \varepsilon$ and let $K = \{0, \pm ne_n : n = 1, 2, ..., n_0\}$.

K is a compact subset of l^1, and

$$P[X_n \notin K] = \begin{cases} 0 & \text{if } n \le n_0 \\ \frac{1}{n} & \text{if } n > n_0 \end{cases}$$

Thus, $\{X_n\}$ is tight.

Let $\{a_{nk}\}$ be an array of real constants, and let $a > n_0$. Then, for every $n \in N$:

$$\begin{aligned} \sum_k |a_{nk}| E\|X_k\| I_{[\|X_k\| > a]} &\le \sum_k |a_{nk}| E\|X_k\| I_{[X_k \notin K]} \\ &\le \sum_k |a_{nk}| k P[X_k \notin K] < \varepsilon \sum_k k|a_{nk}|, \end{aligned}$$

and so $\{\|X_n\|\}$ is $\{a_{nk}\}$-uniformly integrable for any array $\{a_{nk}\}$ such that $\sup_n \sum_k k|a_{nk}| < \infty$; for instance, $a_{nk} = \begin{cases} \frac{1}{n^2} & \text{if } 1 \le k \le n \\ 0 & \text{if } k > n. \end{cases}$

Therefore, $\{X_n\}$ is $\{a_{nk}\}$-compactly uniformly integrable for such an array $\{a_{nk}\}$.

Note that the sequence $\{X_n\}$ does not verify the hypothesis of compactly uniform integrability in [6], Theorem 3.2; nevertheless our Theorem 4.2 shows that the thesis is true for the array $\{a_{nk}\}$ in the example.

REMARK. The definitions and results of this paper can be formulated by considering an array $\{X_{nk}, 1 \leq k \leq k_n \leq \infty, n \geq 1\}$ of random elements, and, basically, nothing would be changed in the proofs. We have preferred the formulation for a sequence $\{X_n\}, n \in N$, in order to stay within the framework of the classical WLLN and make easier the comparison of our results with the classical ones in the literature.

For recents results on the WLLN for arrays of random variables we refer to Gut ([10]) and Hong and Oh ([11]).

ACKNOWLEDGMENT. The author is grateful to the referee for his comments and his suggestions. This work is supported in part by DGICYT grant PB93-0926 and Junta de Andalucía.

REFERENCES

[1] ORDÓÑEZ CABRERA. M., "Convergence of weighted sums of random variables and uniform integrability concerning the weights", *Collect. Math.* 45 (2) (1994), 121-132

[2] CHANDRA, T.K., "Uniform integrability in the Cesàro sense and the weak law of large numbers", *Sankhya. Ser. A* 51 (1989), 309-317

[3] HOFFMANN-JORGENSEN, J. and PISIER, G., "The law of large numbers and the central limit theorem in Banach spaces", *Ann. Probab.* 4 (1976), 587-599.

[4] DAFFER, P.Z. and TAYLOR, R.L., "Tightness and strong laws of large numbers in Banach spaces", *Bull. Inst. Math. Acad. Sinica* 10 (1982), 251-263.

[5] WANG, X.C. and RAO, M.B., "Some results on the convergence of weighted sums of random elements in separable Banach spaces", *Studia Math.* 86 (2) (1987), 131-153.

[6] WANG, X.C., RAO, M.B. and DELI, L., "Convergence in the r-th mean of weighted sums of random elements in Banach spaces of type p", *Northeast Math. J.* 8 (3) (1992), 349-356.

[7] TAYLOR, R.L., *"Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces"*, Lecture Notes in Mathematics, 672, Springer Berlin, 1978.

[8] CUESTA, J.A. and MATRÁN, C., "Strong convergence of weighted sums of random elements through the equivalence of sequences of distributions", *J. Multivariate Anal.* 25 (1988), 311-322.

[9] ADLER, A., ROSALSKY, A. and TAYLOR, R.L., "Some strong laws of large numbers for sums of random elements", *Bull. Inst. Math. Acad. Sinica* 20 (4) (1992), 335-357. .

[10] GUT, A., "The weak law of large numbers for arrays", *Stat. Probab. Lett.* 14 (1992), 49-52.

[11] HONG, D.H. and OH, K.S., "On the weak law of large numbers for arrays", *Stat. Probab. Lett.* 22 (1995), 55-57.