Internat. J. Math. & Math. Sci.
VOL. 21 NO. 1 (1998) 171-182

FINITE BANDWIDTH, LONG WAVELENGTH CONVECTION WITH BOUNDARY IMPERFECTIONS: NEAR-RESONANT WAVELENGTH EXCITATION

D.N. RIAHI
Department of Theoretical and Applied Mechanics
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801, U.S A.

(Received June 12, 1996)

ABSTRACT. Finite amplitude thermal convection with continuous finite bandwidth of long wavelength modes in a porous layer between two horizontal poorly conducting walls is studied when spatially non-uniform temperature is prescribed at the lower wall. The weakly nonlinear problem is solved by using multiple scales and perturbation techniques. The preferred long wavelength flow solutions are determined by a stability analysis. The case of near resonant wavelength excitation is considered to determine the non-modal type of solutions. It is found that, under certain conditions on the form of the boundary imperfections, the preferred horizontal structure of the solutions is of the same spatial form as that of the total or some subset of the imperfection shape function It is composed of a multi-modal pattern with spatial variations over the fast variables and with non-modal amplitudes, which vary over the slow variables The preferred solutions have unusual properties and, in particular, exhibit 'kinks' in certain vertical planes which are parallel to the wave vectors of the boundary imperfections. Along certain vertical axes, where some of these vertical planes can intersect, the solutions exhibit multiple 'kinks'.

KEY WORDS AND PHRASES: Convection, imperfections, long wavelength, long wavelength convection, near-resonance, boundary imperfections, finite-bandwidth
1991 AMS SUBJECT CLASSIFICATION CODES: 76R10, 76R99, 76S05, 80A20, 76R05, 76D30.

1. INTRODUCTION

Recently Riahi [1] investigated the problem of finite amplitude discrete-modal convection in a horizontal layer with boundary imperfections due to spatially modulated boundary temperatures For resonant wavelength excitation case, regular or non-regular multi-modal pattern convection in the discrete-modal domain are found to be able to become preferred in some ranges of the amplitudes of the boundary temperatures, provided the wave vectors of such discrete-modal patterns are contained in certain subset of the wave vectors representing some portion of the modulated boundary temperatures The present paper extends the above work to the more complicated case of finite amplitude convection with continuous finite bandwidth of modes in a horizontal layer. In order to make the problem mathematically simpler, we consider a porous layer with boundary imperfections to be due to spatially non-uniform temperature at the lower wall only and assume that the horizontal walls are poor conductors of heat. Such later assumption provides the analytical method of approach due to Busse and Riahi [2] and Riahi [3] to be used in the present problem.

The present investigation of the continuous finite bandwidth of convection modes applies the method of approach due to Newell and Whitehead [4]. As was explained by Newell and Whitehead [4], continuous-modal analysis of convection leads to a wider class of solutions which can describe adequately the problem with the amplitude modulations which inevitably occur as a result of, for example, non-uniform boundary imperfections

Rees and Riley [5,6] investigated the effects of one-dimensional sinusoidal boundary imperfections on weakly nonlinear thermal convection in a porous medium and determined, in particular, the nonlinear system for the flow amplitudes, and the effects of the boundary modulations on the stability of different roll cells, and the evolution of the unstable rolls were studied Rees [7] investigated the effect of one-dimensional long wavelength thermal modulations on the onset of convection in a porous medium and predicted, in particular, the preference of a mode in the form of rectangular cells for certain ranges of values of the modulation wave number. The present paper is, in a sense, an extension of Rees and Riley's work [5] to more general two-dimensional imperfections and to more general and new three-dimensional flows and is an example of an imperfect bifurcation driven by imperfect heating and/or cooling. There have been literature associated with a number of authors whose works are all relevant to the present problem and were reviewed in Riahi [1].

The general problem under consideration can have practical values in that one might want to make the temperature of a boundary non-uniform if the transport processes are enhanced [1] or if the flow structure could be controlled.

The present paper considers continuous finite bandwidth of three-dimensional modes of convection and arbitrary non-uniform temperature boundary condition in the form of such continuous modes at the lower wall. We have found a number of interesting results. In particular, we found stable envelope solutions whose flow patterns can have quite unusual behavior. For example, depending on the form of the boundary imperfections, one such pattern couplets in different parts of space may be 180° out of phase, and the solutions can exhibit 'kinks' in the horizontal structure

2. GOVERNING SYSTEM

We consider an infinite horizontal porous layer of average depth d filled with fluid and bounded by two infinite half spaces with the thermal conductivity which is assumed to be small in comparison with that of the porous medium. In the steady static state, a constant heat flux traverses the system such that the mean temperatures $\overline{T}_l$ and $\overline{T}_u$ are attained at the lower and upper boundaries of the fluid. Introduce a cartesian system, with the origin at the centerplane of the layer and with the z-coordinate in the vertical direction anti-parallel to the gravity vector. We shall examine the effects of lower boundary temperature perturbation variations at a fixed value of $\Delta T = \overline{T}_l - \overline{T}_u$ and represent the magnitude of such variation relative to ΔT by δ. It is assumed that $\delta < o(1)$. We define a temperature relative to the conduction state by

$$T*(x,y,z,t) = \left(\frac{d}{2} - z\right)\frac{\Delta T}{d} + T(x,y,z,t) + \overline{T}_u, \qquad (2\ 1)$$

where x and y are the horizontal variables and t is the time variable It is convenient to use non-dimensional variables in which lengths, velocities, time and temperature T are scaled respectively by d, $\lambda/(d\rho_o c), d^2\rho_o c/\lambda$ and $\Delta T/R$. Here $R = \beta g K \Delta T d\rho_o c/(\mu\lambda)$ is the Rayleigh number, β is the coefficient of thermal expansion, g is acceleration due to gravity, K is the Darcy permeability coefficient, ρ_o is a reference (constant) fluid density, c is the specific heat at constant pressure, λ is the thermal conductivity of the porous medium (fluid-solid mixture) and μ is the kinematic viscosity. Then, with the usual Boussinesq approximation that density variations are taken into account only in the buoyancy term, the Darcy-Boussinesq-Oberbeck equations for momentum, continuity and heat in the limit of infinite prandtl-Darcy number [8] are obtained, which are given in Riahi [3]. These are equations for θ (dimensionless T), u (velocity vector) and P (modified deviation of pressure from its static value).

The governing equations are simplified by using the representation

$$\underset{\sim}{u} = \Omega\phi,\ \Omega \equiv \nabla \times \nabla \times \underset{\sim}{z}, \qquad (2\ 2)$$

for the divergence free $\underset{\sim}{u}$ [3] Here $\underset{\sim}{z}$ is a unit vector in the vertical direction. Taking the vertical component of double curl of the momentum equation and using (2 2) in the heat equation yield the following equations:

$$\Delta_2(\nabla^2\phi + \theta) = 0, \tag{2.3a}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)\theta - R\Delta_2\phi = \Omega\phi \cdot \nabla\theta, \tag{2.3b}$$

where Δ_2 is the horizontal Laplacian These equations must then be solved subject to the boundary conditions

$$\phi = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \tag{2 4a}$$

$$\frac{\partial\theta}{\partial z} = \eta\gamma^2[\theta - \delta Rh(x,y)] \quad \text{at} \quad z = -\frac{1}{2}, \tag{2 4b}$$

$$\frac{\partial\theta}{\partial z} + \eta\gamma^2\theta = 0 \quad \text{at} \quad z = \frac{1}{2}, \tag{2 4c}$$

where $h(x, y)$ is a given spatially non-uniform function of x and y. The formulation of the boundary conditions for θ follows from those due to Kelly and Pal [9] and Riahi [1] for isothermal boundaries and those due to Sparrow et al. [10], Busse and Riahi [2] and Riahi [3] for poorly conducting boundaries 'For further details regarding boundary condition alternative to (2.4b), the reader is referred to Riahi [1,11]. The parameter γ^2 given in (2 4b,c) is a Biot number, which is assumed to be small ($\gamma \ll 1$) in the present paper. For problems treated in Riahi [1,11], $\gamma = \infty$. Additional parameter η introduced in (2 4b,c) is related to the horizontal wave number for the classical linear problem (Busse and Riahi [2], Riahi [3]) by the relation

$$\alpha = \eta\gamma^{\frac{1}{2}}, \tag{2.5}$$

and its presence in (2.4b,c) is needed to cover the classical linear discrete modal results [3].

3. ANALYSIS

The case of near resonant wavelength excitation corresponds to the critical regime where $R \approx R_c$ and $\epsilon = o\left(\delta^{\frac{1}{3}}\right)$ [1,9]. Here R_c is the critical value of R below which there is no motion and ϵ is the amplitude of convection. We consider the following double series expansions for ϕ, θ, and R in powers of γ and $\delta^{\frac{1}{3}}$:

$$\left.\begin{aligned} \phi &= \sum_{n=0,m=1} \gamma^n\delta^{m/3}\phi_m^{(n)}, \\ \theta &= \sum_{n=0,m=1} \gamma^n\delta^{m/3}\theta_m^{(n)}, \\ R &= \sum_{n=0,m=0} \gamma^n\delta^{m/3}R_m^{(n)}. \end{aligned}\right\} \tag{3 1}$$

Because of the nature of the present thermal boundary conditions, it turns out that many of the coefficients vanish and only systems to orders $\delta^{1/3}, \gamma\delta^{1/3}, \gamma^2\delta^{1/3}$, $\gamma\delta^{2/3}$ and $\gamma^2\delta$ need to be analyzed in order to determine the nonlinear properties of the system in the double limit of small γ and small δ [2,3] Upon inserting (3.1) into (2.3)-(2 4) and disregarding the quadratic terms, we find the linear problem whose system is given in Riahi [3], and it is the classical linear system whose discrete modal solution was determined by Riahi [3] up to and including order γ^2. In order to formulate the problem for a continuous finite bandwidth of modes, we follow the method of approach due to Newell and Whitehead [4] However, these authors formulated the problem with isothermal boundaries where they allowed an $o(\epsilon)$ band of modes in the x direction versus an $o(\epsilon^{1/2})$ band of modes in the y direction centered around the critical two-dimensional mode in the form of rolls along the y-axis In the present problem, we need to formulate the case for poorly conducting boundaries where the critical mode at the onset of convection, based on the discrete modal analysis, is known to be three-dimensional mode in the form of square cells [2,3]. Hence, we need to allow an $o(\delta^{1/3})$ band of modes in the x direction and an $o(\delta^{1/3})$ band of modes in the y direction centered around the critical three-dimensional mode in the form of squares whose wave number vector is along the line $y = x$ in the horizontal plane. Thus, the general linear solution of such finite bandwidth modes can be written as

$$\left(\begin{aligned} \phi_l^{(0)} + \gamma\phi_1^{(1)} + 0(\gamma^2) &= f_1(z)\sum_{n=-N}^{N} W_n(x,y)A_n(x_s,y_s,t_s), \\ \theta_l^{(0)} + \gamma\theta_1^{(1)} + 0(\gamma^2) &= g_1(z)\sum_{n=-N}^{N} W_n(x,y)A_n(x_s,y_s,t_s), \end{aligned}\right\}\Bigg) \tag{3.2}$$

where A_n are functions of the slow variables formulated as

$$x_s = \gamma^{\frac{1}{2}}\delta^{\frac{1}{3}}x, \quad y_s = \gamma^{\frac{1}{2}}\delta^{\frac{1}{3}}y, \quad t_s = \gamma^2\delta^{\frac{2}{3}}t, \tag{3 3}$$

the function W_n has the representation

$$W_n \equiv \exp(i\underset{\sim}{K}_n \cdot \underset{\sim}{r}), \tag{3.4a}$$

and satisfies the relation

$$\Delta_2 W_n = -\alpha^2 W_n. \tag{3 4b}$$

Here $\underset{\sim}{r}$ is the horizontal position vector, $i = \sqrt{-1}$, α is the horizontal wave number of the flow structure, N is a positive integer, and the horizontal wave vectors $\underset{\sim}{K}_n = (K_{nx}, K_{ny})$ of the flow structure satisfy the properties

$$\underset{\sim}{K}_n \cdot \underset{\sim}{z} = 0, \quad |\underset{\sim}{K}| = \alpha, \quad \underset{\sim}{K}_{-n} = -\underset{\sim}{K}_n. \tag{3 5}$$

The amplitude functions A_n satisfy the condition

$$A_n^* = A_{-n}, \tag{3 6}$$

where the asterisk indicates the complex conjugate The expressions for $f_1(z)$ and $g_1(z)$ are given in Riahi [3], and the details of the results for R_o and its minimum R_c attained at $\alpha = \alpha_c$ can also be found in Riahi [3]. Here

$$R_o \equiv R_o^{(o)} + \gamma R_o^{(1)} + o(\gamma^2) \quad \text{and} \quad \alpha_c \equiv \eta_c\gamma^{\frac{1}{2}}, \quad \text{where} \quad \eta_c' = \left(\frac{21}{2}\right)^{\frac{1}{3}} \quad \text{and}$$

$$R_c = 12[1 + 3\gamma/\eta_c] + o(\gamma^2) \quad [3].$$

In the order $\gamma\delta^{\frac{2}{3}}$ equation (2.3b) yields

$$\frac{\partial^2\theta_2^{(1)}}{\partial z^2} - \gamma^{-1}R_1^{(0)}\Delta_2\phi_1^{(0)} - \gamma^{-1}\Omega\phi_1^{(0)} \cdot \nabla\theta_1^{(0)} = E_s\left(R_o^{(0)}\phi_1^{(0)} - \theta_1^{(0)}\right), \tag{3.7}$$

where

$$E_s \equiv 2\left(\frac{\partial^2}{\partial x\partial x_s} + \frac{\partial^2}{\partial y\partial y_s}\right).$$

The solvability condition for this equation is obtained by multiplying (3.7) with W_n^* and averaging over the whole layer Due to symmetric property of the layer and (2.4), this condition yields

$$R_1^{(0)} = 0. \tag{3 8}$$

In order to determine the solution for (3 7), the boundary condition

$$\overline{\theta_2^{(1)}} = 0 \quad \text{at} \quad z = \pm\frac{1}{2} \tag{3 9a}$$

has been used for the horizontally averaged component of $\theta_2^{(1)}$ because the horizontal mean of the boundary temperature is given as an external parameter of the problem The remaining component of $\theta_2^{(1)}$ satisfied the boundary condition

$$\frac{\partial}{\partial z}\left(\theta_2^{(1)} - \overline{\theta_2^{(1)}}\right) = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \tag{3 9b}$$

where an over-bar in (3.9) denotes horizontal average [2] The system (3 7) and (3 9) then yield

$$\theta_2^{(1)} = -D^2 g_2(z)\sum_{m=-N}^{N}|A_m|^2 - \sum_{\substack{l,P=-N\\ l\neq -P}}^{l,P=N} A_l A_p D^2 f_2(z,\widehat{\phi}_{lP})W_l W_P + g_3(z)\sum_{n=-N}^{N} D_{sn}A_n W_n, \tag{3 10a}$$

where

$$g_3(z) = -\frac{1}{2}z^4 + \frac{1}{4}z^2 - \frac{7}{480},\ D \equiv \frac{\partial}{\partial z},\ D_{sn} \equiv 2i\left(K_{nx}\frac{\partial}{\partial x_s} + K_{ny}\frac{\partial}{\partial y_s}\right),\ \widehat{\phi}_{lp} \equiv (\underset{\sim}{K}_l\cdot\underset{\sim}{K}_p)/\alpha^2 \tag{3 10b}$$

and the expressions for the functions f_2 and g_2 are given in Riahi [3]. In contrast to the discrete modal analysis case [1], there is a need to determine the solution $\phi_2^{(1)}$ for (2 3a) in the order $\gamma\delta^{2/3}$. It is

$$\phi_2^{(1)} = \sum_{\substack{l,P=-N\\ l\neq -P}}^{l,P=N} A_l A_P f_2(z,\widehat{\phi}_{lP})W_l W_P + g_2(z)\sum_{m=-N}^{N}|A_m|^2 + f_3(z)\sum_{m=-N}^{N} D_{sm}A_m W_m, \tag{3 10c}$$

where

$$f_3(z) = \frac{1}{60}z^6 + \frac{1}{48}z^4 - \frac{53}{960}z^2 + \frac{47}{3840}. \tag{3 10d}$$

In the order $\gamma^2\delta$ equations (2.3)-(2 4) yield the following system for $\theta_3^{(2)}$:

$$-\frac{\partial\theta^{(0)}}{\partial t_s} + \frac{\partial^2\theta_3^{(2)}}{\partial z^2} - \gamma^{-1}R_2^{(1)}\Delta_2\phi_1^{(0)} = \left(\Omega\phi_1^{(0)}\cdot\nabla\theta_2^{(1)} + \Omega\phi_2^{(1)}\cdot\nabla\theta_1^{(0)}\right)\gamma^{-1} + E_s\left(R_0^{(0)}\phi_2^{(1)} - \theta_2^{(1)}\right), \tag{3 11a}$$

$$\overline{\theta_3^{(2)}} = \frac{\partial}{\partial z}\left(\theta_3^{(2)} - \overline{\theta_3^{(2)}}\right) + \eta_c R_0^{(0)}h = 0 \quad \text{at} \quad z = -\frac{1}{2}, \tag{3 11b}$$

$$\overline{\theta_3^{(2)}} = \frac{\partial}{\partial z}\left(\theta_3^{(2)} - \overline{\theta_3^{(2)}}\right) = 0 \quad \text{at} \quad z = \frac{1}{2}. \tag{3 11c}$$

The function h given in (3.11b) actually is assumed to have the following arbitrary representation

$$h(x,y) = \left[\eta_c R_0^{(0)}\right]^{-1}\sum_{n=-N^{(b)}}^{N^{(b)}} L_n W_n^{(b)}, \tag{3.12a}$$

where

$$W_n^{(b)} \equiv \exp\left(i\underset{\sim}{K}_n^{(b)}\cdot\underset{\sim}{r}\right), \tag{3.12b}$$

L_n are functions of x_s and y_s, $N^{(b)}$ is a positive integer which may tend to infinity, and the horizontal wave number vectors $\underset{\sim}{K}_n^{(b)} = \left(K_{nx}^{(b)}, K_{ny}^{(b)}\right)$ for the boundary imperfections satisfy the properties

$$\underset{\sim}{K}_n^{(b)}\cdot\underset{\sim}{z} = 0,\quad |\underset{\sim}{K}_n^{(b)}| = \alpha_n^{(b)},\quad \underset{\sim}{K}_{-n}^{(b)} = -\underset{\sim}{K}_n^{(b)}. \tag{3.13}$$

The function L_n satisfy the condition

$$L_n^* = L_{-n}. \tag{3 14}$$

We shall assume that $\alpha_n^{(b)} = \alpha_c$

Multiplying (3.11a) by W_n^*, averaging over the whole layer and using (3 11b,c) yields

$$\frac{-\partial A_n}{\partial t_s} + \sum_{m=-N^{(b)}}^{N^{(b)}} L_m \langle W_n^* W_m^{(b)} \rangle + \frac{1}{2}\eta_c^2 R_2^{(1)} A_n = \frac{1}{720}\eta_c^4 \Bigg[A \sum_{m=-N}^{N} |A_m|^2 - 6 \sum_{l \neq -p} A_m A_l A_p \widehat{\phi}_{lp} \cdot \left(1 + \widehat{\phi}_{ml} + \widehat{\phi}_{mp}\right) \langle W_n^* W_m W_l W_p \rangle \Bigg] + \frac{2}{21} D_{sn}^2 A_n, \tag{3.15}$$

where an angular bracket indicates an average over the fluid layer.

Using (3 15), doing some scalings on t_s, L_m and A_m (for all possible m) and applying the conditions given in Riahi [3] for the non-zero average product, we end up with the following simplified form for (3 15)

$$\left[\frac{\partial}{\partial t_s} - R_2^{(1)} - \left(K_{nx}\frac{\partial}{\partial x} + K_{ny}\frac{\partial}{\partial y_s} \right)^2 \right] A_n = -A_n \left(\sum_{m \neq n} \widehat{\phi}_{mn}^2 |A_m|^2 + \sum_{m=-N}^{N} |A_m|^2 \right) + \sum_{m=-N^{(b)}}^{N^{(b)}} L_m \langle W_n^* W_m^{(b)} \rangle, \quad (n = -N, \cdots, -1, 1, \cdots, N). \tag{3 16}$$

The above system is a collection of $2N$ partial differential equations for the $2N$ unknown functions $A_n (n = -N, \cdots, -1, 1, \cdots, N)$

To distinguish the physically realizable solution (s) among all the steady solutions of (3 16), the stability of $A_m (m = -N, \cdots, -1, 1, \cdots, N)$ with respect to disturbances $B_m(x_s, y_x, t_s)$ are investigated The system of equations for the time dependent disturbances with addition of a time dependence of the form $\exp(\sigma t_s)$ are given by

$$\left[\sigma - R_2^{(1)} - \left(K_{nx}\frac{\partial}{\partial x_s} + K_{ny}\frac{\partial}{\partial y_s} \right)^2 \right] B_n + \sum_{m \neq n} \widehat{\phi}_{mn}^2 \left(A_n A_m B_m^* + A_n A_m^* B_m + B_n |A_m|^2 \right) + \sum_{m=-N}^{N} \left(A_n A_m B_m^* + A_n A_m^* B_m + B_n |A_m|^2 \right) = 0, \quad (n = -N, \cdots, -1, 1, \cdots, N), \tag{3 17}$$

where B_n satisfy conditions of the form (3.6) for $\sigma = \sigma^*$ Taking complex conjugate of (3.17) and replacing the subscript n by $-n$, it then can be seen after some rearrangement that $B_n \exp(\sigma t_s) = B_{-n}^* \exp(\sigma^* t_s)$ which implies that σ is real It is clear from (3.16)-(3 17) that the boundary imperfection affects the steady solutions directly as a source term in (3.16), while the imperfection affects the disturbances indirectly through the steady solutions

4. SOLUTIONS

We consider the system (3.16) for the cases where

$$A_n = a_n \tanh(x_n + y_n) + i b_n [\cosh(x_n + y_n)]^{-1} \equiv \overline{U_n} + i \overline{V_n}, \tag{4.1a}$$

where

$$x_n = x_s/(2K_{nx}), \quad y_n = y_s/(2K_{ny}), \tag{4.1b}$$

and $\overline{U_n}$ and $\overline{V_n}$ are the real and imaginary parts of A_n, respectively Such assumption (4.1) is suggested by the extension of the simple one-dimensional envelope solution due to Newell and Whitehead [4] in their studies of finite bandwidth, finite amplitude rolls convection in a layer with isothermal boundaries The coefficients a_n and b_n given in (4.1a) are real constants We are assuming here that (4.1) is suggested due to the following form of the boundary imperfection function L_m

$$L_m = g_m \tanh(x_n + y_n) + ih_m[\cosh(x_n + y_n)]^{-1}, \tag{4 2}$$

where g_m and h_m are real constants. The justification for such assumption was confirmed by using (4 1)-(4 2) into (3 16) which led to the following algebraic system.

$$R_2^{(1)} a_n = a_n \left[-2a_n^2 + \sum_{m=-N}^{N} \left(1 + \widehat{\phi}_{mn}^2\right) a_m^2 \right] - \sum_{m=-N^{(b)}}^{N^{(b)}} g_m \langle W_n^* W_m^{(b)} \rangle, \tag{4 3a}$$

$$\left(1 + R_2^{(1)}\right) b_n = b_n \left[-2a_n^2 + \sum_{m=-N}^{N} \left(1 + \widehat{\phi}_{mn}^2\right) a_m^2 \right] - \sum_{mn=-N^{(b)}}^{N^{(b)}} h_m \langle W_n^* W_m^{(b)} \rangle, \tag{4 3b}$$

$$a_n \left[2\left(1 + a_n^2 - b_n^2\right) + \sum_{m=-N}^{N} \left(1 + \widehat{\phi}_{mn}^2\right) \left(b_m^2 - a_m^2\right) \right] = 0, \tag{4 3c}$$

$$b_n \left[2\left(1 + a_n^2 - b_n^2\right) + \sum_{m=-N}^{N} \left(1 + \widehat{\phi}_{mn}^2\right) \left(b_m^2 - a_m^2\right) \right] = 0, \quad (n = -N, \cdots, -1, 1, \cdots, N). \tag{4 3d}$$

This is a system of $8N$ equations for $4\left(N + N^{(b)}\right)$ unknown coefficients a_m, b_m, g_m and h_m. Generally, 'solution for $N > N^{(b)}$ is not possible, unless $a_m = b_m = 0$ for $m > N^{(b)}$. Non-trivial solutions $(a_m \neq o, b_m \neq 0)$ are always possible for $N \leq N^{(b)}$ Of course we are assuming the cases of significant boundary imperfections, so that the last terms in the right hand sides of (4 3a,b) are non-zero

It should be noted that one could, in general, consider any solution of the form $A_n = F_n(x_n, y_n)$ for (3.16), for given functions F_n, where the functions L_m are to be so chosen to satisfy (3 16) However, detailed investigation of stability of such solution requires a knowledge of particular forms of F_n, although, as we shall see later in this section, all such type of solutions can become stable for sufficiently large $\left|R_2^{(1)}\right|$ and $R_2^{(1)} < 0$, provided the horizontal averages of functions involving the base flow and disturbance quantities and/or their first or higher derivatives remain finite. The main reason for assuming (4 1) is the rather unusual non-modal properties of the real parts $\overline{U_n}$ of such solutions Where $\overline{U_n}$ is weak namely at $x_n = y_n = 0$, $\overline{V_n}$ is strong and vice versa For large $|x_n|$, the solutions (3.2) for $y_n = o$ are of scale $2\pi/\alpha_c$ except that the sense of rotation of the corresponding pattern couplets due to $\overline{U_n}$ is reversed on the opposite sides of the $x_s = 0$ axis.

Next, we investigate stability of the solutions of the form (4.1) with respect to disturbances B_n of the general form

$$B_n = \widehat{U}_n + i\widehat{V}_n, \tag{4 4}$$

where $\widehat{U}_n$ and $\widehat{V}_n$, are the real and imaginary parts of B_n, respectively. Using (4.1) and (4.4) and (3 17) lead us the following systems for $\widehat{U}_n$ and $\widehat{V}_n$:

$$\left[\sigma - R_2^{(1)} + 3\overline{U}_n^2 + \overline{V}_n^2 - \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n} \right)^2 \right] \widehat{U}_n + 2\overline{U}_n \overline{V}_n \widehat{V}_n + \sum_{m \neq n} \left(1 + \widehat{\phi}_{mn}^2\right) \cdot \left[2\overline{U}_n \overline{U}_m \widehat{U}_m + 2\overline{U}_n \overline{V}_m \widehat{V}_m + \left(\overline{U}_m^2 + \overline{V}_m^2\right) \widehat{U}_n \right] = 0, \tag{4.5a}$$

$$\left[\sigma - R_2^{(1)} + 3\overline{V}_n^2 + \overline{U}_n^2 - \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n} \right)^2 \right] \widehat{V}_n + 2\overline{U}_n \overline{V}_n \widehat{V}_n + \sum_{m \neq n} \left(1 + \widehat{\phi}_{mn}^2\right) \cdot \left[2\overline{U}_n \overline{V}_m \widehat{U}_m + 2\overline{V}_n \overline{V}_m \widehat{V}_m + \left(\overline{U}_m^2 + \overline{V}_m^2\right) \widehat{V}_n \right] = 0, \quad (n = -N, \cdots, -1, 1, \cdots, N). \tag{4.5b}$$

Multiply (4.5a) by $\widehat{U}_n$, (4.5b) by $\widehat{V}_n$, add the two resulting systems of equations and average over the slow variables. The resulting system has the property that $\sigma < 0$ for sufficiently large $\left|R_2^{(1)}\right|$ and

$R_2^{(1)} < 0$, provided the horizontal averages which involve $\overline{U}_n$, $\overline{V}_n$, $\widehat{U}_n$, $\widehat{V}_n$ and their first or higher order derivatives remain finite This result holds for general base flow solutions which include those considered in (4 1), provided that the boundary imperfections are significant so that the last terms in the right hand sides of (4 3a,b) are non-zero. For the cases of insignificant boundary imperfections, where none of the wave vectors $\underset{\sim}{K}_n$ are along any of the wave vectors $\underset{\sim}{K}_m^{(b)}$, then the last terms in the right hand sides of (4.3a,b) vanish It is then not difficult to show from (4.3)-(4 5) that the base flow solutions with kinks $\overline{U}_n = a_n \tanh(x_n + y_n)$ and $\overline{V}_n = 0$ for all n, are unstable and $R_2^{(1)} = 2$ and $\sigma = 1$ follow For the cases of significant boundary imperfections, such solutions can be stable as the following examples indicate

Let us consider now the following specific examples for significant boundary imperfections cases in order to illustrate the inter-relations between the boundary imperfections and the resulting preferred flow patterns and to demonstrate specific conditions on $R_2^{(1)}$ under which absolute stability of different solutions with kinds are possible.

EXAMPLE 1 $N^{(b)} = 1$. Consider the case $N = 1$ first. Suppose $g_1 = 0$ and $h_1 \neq 0$ Then (4 3) implies that both a_1, and b_1, are non-zero and $R_2^{(1)} = 2(1 + h_1^2)$. Assume that the disturbance quantities $\widehat{U}_1$ and $\widehat{V}_1$, in general, are functions of slow variables. Multiply (4.5a) (for $n = 1$) by $\widehat{U}_1$, (4 5b) (for $n = 1$) by $\widehat{V}_1$, add the two resulting systems and average over the slow variables The resulting system then yield the result that no stable solution is possible for sufficiently large h_1, while stable solution may be possible for sufficiently small h_1 though such possibility cannot be proved rigorously It is implied from these results that smaller $R_2^{(1)}$ cases are favored over larger $R_2^{(1)}$ cases. Suppose now that $g_1 \neq 0$ and $h_1 = 0$. Then (4.3) implies that non-zero value for b_1 is possible for $R_2^{(1)} > 1$ since $R_2^{(1)} = 2a_1^2 - 1$ and $a_1^2 > 1$, while $R_2^{(1)}$ can be small for $b_1 = o$. For $b_1 = o$, we find

$$a_1^2 = 1, \quad R_2^{(1)} = 2 - g_1/a_1, \quad a_{-1} = -a_1. \tag{4 6}$$

Hence, $a_1 = -1$ is preferred for $g_1 < 0$, while $a_1 = 1$ is preferred for $g_1 > 0$ Forming again the integral system for the averaged amplitude of the disturbances, we find that $b_1 = 0$ solution for (4.5) is stable for $R_2^{(1)} < -1$. Apply the same method of approach as the one described above for the $g_1 \neq 0$ and $h_1 \neq 0$ case, we find

$$a_1^2 = 1 + b_1^2, \ R_2^{(1)} = 2a_1^2 - g_1/a_1, \ b_1 = a_1 h_1/(g_1 - a_1), \ a_{-1} = -a_1, \ b_{-1} = -b_1, \tag{4.7}$$

and this solution is stable for sufficiently large $\left|R_2^{(1)}\right|$ and $R_2^{(1)} < 0$. Again negative root for a_1 is preferred for $g_1 < 0$, while the opposite is true for $a_1 > 0$. As can be seen from (4.6)-(4.7), the solution for the case $g_1 \neq 0$ and $h_1 = 0$ is preferred over the one for non-zero g_1 and h_1 since it leads to smaller $R_2^{(1)}$ for given g_1 and h_1.

We now consider the case of arbitrary $N > 1$ for the particular values of g_1 and h_1 where (4.6) holds for $N = 1$, that is $g_1 \neq 0$ and $h_1 = 0$. Using (4.3a) for $n = 1$, it follows that $a_1 \neq 0$ since $g_1 \neq 0$ The equations (4.3a,c) then yield

$$R_2^{(1)} = 2 - \frac{g_1}{a_1} + 2b_1^2 + \sum_{m \neq \pm 1} \left(1 + \widehat{\phi}_{m1}\right) b_m^2, \quad a_1 \neq 0. \tag{4.8}$$

Comparing (4.8) to (4.6), it follows that the solution for $N = 1$ is preferred over the one for $N > 1$ if at least one $b_n \neq 0$ for one particular value of $n\,(n = 1, \cdots, N)$ since $R_2^{(1)}$ for $N = 1$ is less than the one for $N > 1$ in such situation. If all the coefficients b_n for $n = 1, \cdots, N$ are zero, then (4.3) implies that such case is possible only if all the coefficients a_n for $n = 2, \cdots, N$ are zero. But, this result then implies that no non-trivial solution for $N > 1$ case can be preferred. The results discussed above thus indicate that the preferred horizontal platform function $H(x, y; x_1, y)$, for either ϕ or θ, for the case, where the boundary imperfection function $h(x, y; x_1, y_1)$ is of the form

$$h = (6\eta_c)^{-1} g_1 \tanh(x_1 + y_1) \cos\left(\underset{\sim}{K}_1^{(b)} \cdot \underset{\sim}{r}\right) \tag{4 9a}$$

is the following function

$$H = \sum_{n=-1}^{1} W_n A_n = 2a_1 \tanh(x_1 + y_1) \cos(\underset{\sim}{K}_1 \cdot \underset{\sim}{r}), \quad a_1^2 = 1, \tag{4 9b}$$

where the wave vector $\underset{\sim}{K}_1$ is along the wave vector $\underset{\sim}{K}_1^{(b)}$ and $\alpha_1^{(b)} = \alpha = \alpha_c$. It is seen from (4 9) that the preferred horizontal structure of the solutions for $N^{(b)} = 1$ case is a copy of the boundary imperfection shape. It is a long wavelength pattern composed of a modal roll with periodicity over the fast variables (x, y) and a non-modal amplitude which varies over the slow variables (x_1, y_1) and exhibits kinks locally in the horizontal plane. The function H can exhibit unusual behavior due to the function $\tanh(x_1 + y_1)$. For example, the sign of this function for $(x_1 + y_1) \to \infty$ is opposite to that for $(x_1 + y_1) \to -\infty$, and $\tanh(x_1 + y_1) \to \pm 0$ as $(x_1 + y_1) \to \pm 0$ Thus, in the local regions where $(x_1 + y_1)$ is close to zero, H can be 180 degrees out of phase from one location to another one within such local regions.

EXAMPLE 2 $\mathbf{N^{(b)} = 2.}$ For significant imperfection case, we assume $\underset{\sim}{K}_1 = \underset{\sim}{K}_1^{(b)}$ and $\underset{\sim}{K}_2 = \underset{\sim}{K}_2^{(b)}$ Consider first $N = 1$. Using (4.3), we find the results (4 6)-(4.7) and, thus, the results of the type for $N^{(b)} = 1$ case (example 1) follow, so that two solutions satisfying (4 6) one for which (4.9) holds and another one for which (4 9) with $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_1^{(b)}$ replaced respectively by $\underset{\sim}{K}_2$ and $\underset{\sim}{K}_2^{(b)}$ holds

Following the results for $N = 1$, we consider the case $g_m \neq 0$ and $h_m = 0$ for arbitrary $N (m = 1, 2, \cdots, N)$. For the case where $N = 2$, (4.3) yield the following results for the solutions corresponding to the smallest $R_2^{(1)}$:

$$a_m^2 = \left(2 + \widehat{\phi}_{21}^2\right)^{-1}, \; b_m = 0, \; a_{-m} = -a_m, \; b_{-m} = -b_m, \; m = 1, 2 \tag{4 10a}$$

$$R_2^{(1)} = 2 - g_1/a_1 = 2 - g_2/a_2. \tag{4 10b}$$

These results imply that there is no solution, unless

$$g_1^2 = g_2^2. \tag{4 11}$$

We shall assume that the given constants g_1 and g_2 satisfy (4.11). Here $\widehat{\phi}_{21}$ is the cosine of the angle between the wave vectors $\underset{\sim}{K}_1^{(b)}$ and $\underset{\sim}{K}_2^{(b)}$. It is seen from (4.10) that the negative root for a_m is preferred for $g_m < 0$, while the positive root for a_m is preferred for $g_m > o\,(m = 1, 2)$ Using (4.6) and (4.10), we find that the solution for $N = 2$ corresponds to smaller value of $R_2^{(1)}$, for any value of $\widehat{\phi}_{21}$, than the one for $N = 1$ case Applying the same stability analysis as the one described in example 1, we find the solu-tion (4.10) is stable for $R_2^{(1)} < -1$. For the case $N = 1$, we found that the solution (4.6) is stable for $R_2^{(1)} < -1$. Of course, these stability conditions were determined based on upper bounding type approach, and it is possible that the solutions for $N = 1$ and $N = 2$ still remain stable if $R_2^{(1)}$ is bigger than -1 and -2, respectively. For the case where $N > 2$ and for significant boundary imperfection where $\underset{\sim}{K}_n = \underset{\sim}{K}_n^{(b)} (n = 1, 2)$, we find from (4.3) that no such solution can exist which admits value of $R_2^{(1)}$ smaller than those for $N = 1$ and $N = 2$ since $a_n = b_n = o(n = 3, \cdots, N)$ follow. The results discussed above indicate that the preferred solution corresponds to $N = 2$ case if (4.11) holds and to $N = 1$ case if (4.11) does not hold. In the later case the results (4.9) follow, while in the former case we have

$$g_1^2 = g_2^2, \; h = (6\eta_c)^{-1} \left[g_1 \tanh(x_1 + y_1) \cos\left(\underset{\sim}{K}_1^{(b)} \cdot \underset{\sim}{r}\right) + g_2 \tanh(x_2 + y_2) \cos\left(\underset{\sim}{K}_2^{(b)} \cdot r\right)\right], \tag{4.12a}$$

$$a_1^2 = a_2^2 = \left(2 + \widehat{\phi}_{21}^2\right)^{-1}, \; H = 2[a_1 \tanh(x_1 + y_1) \cos(\underset{\sim}{K}_1 \cdot \underset{\sim}{r}) + a_2 \tanh(x_2 + y_2) \cos(\underset{\sim}{K}_2 \cdot \underset{\sim}{r})], \tag{4.12b}$$

where $\underset{\sim}{K}_n = \underset{\sim}{K}_n^{(b)} (n = 1, 2)$. Again, the same results as in the case $N^{(b)} = 1$ presented in example 1 follow The preferred horizontal structure of the solutions is a copy of the total imperfection shape (case where (4 11) holds) or copy of a subset of the imperfection shape (case where (4 11) does not hold) The long wavelength pattern exhibited by (4 12) is composed of a modal rectangular pattern with periodicity over the fast variables and with non-modal amplitudes, which vary over the slow variables and exhibit 'kinks' locally

The two examples presented above indicate a general theory for arbitrary $N^{(b)}$ and for the case where the wave vectors of the flow pattern coincide with a subset of the wave vectors of the boundary imperfection. Such a theory, to be discussed below, is consistent with the results for $N^{(b)} = 1, 2$ presented in the above two examples. For significant boundary imperfection, $\underset{\sim}{K}_m = \underset{\sim}{K}_m^b$ (for $m = 1, \cdots, N^{(b)}$), and for $h_m = o\left(m = 1, \cdots, N^{(b)}\right)$ and given real constants $g_i \left(i = 1, \cdots, N^{(b)}\right)$, we have without loss of generality, the following relation

$$g_1^2 = g_i^2, \quad i = 1, \cdots, M^{(b)}, \tag{4 13}$$

where $1 \leq M^{(b)} \leq N^{(b)}$ If (4 13) holds for $M^{(b)} = N^{(b)}$, then the preferred horizontal structure of the solutions is of the same spatial form as that of the total imperfection shape function. It is composed of a multi-modal $\left(N = N^{(b)}\right)$ pattern with spatial variations over the fast variables (x, y) and with non-modal amplitudes, which vary over the slow variables (x_n, y_n) $(n = 1, \cdots, N)$. If (4.13) holds for $M^{(b)} < N^{(b)}$, then the preferred horizontal structure of the solutions is of the same spatial form as that of a subset of the imperfection shape function. It is composed of a multi-modal $\left(N = M^{(b)}\right)$ pattern with spatial variations over the fast variables (x, y) and with non-modal amplitudes which vary over the slow variables (x_n, y_n) $\left(n = 1, \cdots, M^{(b)}\right)$ Such solutions can exhibit kinks in spatial locations where $(x_n + y_n) \to o\left(n = 1, \cdots, M^{(b)}\right)$ These kinks are in certain vertical planes which are parallel to significant wave vectors of the boundary imperfections The preferred solutions are stable for sufficiently large $\left|R_2^{(1)}\right|$ and $R_2^{(1)} < 0$.

5. DISCUSSION

Due to the fact that the present investigation is based on the assumption that the amplitude of convection is of order $\delta^{\frac{1}{3}}$ and $\delta \ll 1$, the present results do not change qualitatively from those for the problem where the lower boundary's location is at $z = -\frac{1}{2} + \delta h(x, y)$. This conclusion actually proved by Riahi [11] for the discrete-modal case and appears to be followed here, as well. The boundary corrugated problem, whose location is described above, can incorporate the effects of roughness elements of arbitrary shape h, provided $N^{(b)}$ may tend to infinity for arbitrary functions L_m and that $\alpha_m^{(b)}$ may not all have the same value as α_c. The discussion and results presented in Riahi [11] indicate that the case with $\alpha_m^{(b)} > 2\alpha_c$ is expected to lead to zero contribution on various flow features and, thus, is irrelevant for the present problem. However, the case with $\alpha_m^{(b)} < 2\alpha_c$ is expected to be relevant for the non-resonant wavelength excitation system which is presently under investigation by the present author, and the results will be reported in the near future.

An important demonstration carried out from the present investigation is that the convective flow can be admitted, by the boundary imperfections, certain solutions which exhibit kinks in certain vertical planes within the fluid layer. Each of these vertical planes is parallel to one active wave vector of the boundary imperfections. Here by an active wave vector, we mean one which coincides with one wave vector of the preferred flow solution. All such vertical planes intersect each other at oz-axis Thus along oz-axis there are multiple kinks in the solutions. It is possible to increase the complexity in the solutions by replacing the argument $(x_n + y_n)$ in (4.1a) and (4.2) with the expression $(x_n + y_n + c_n)$, where c_n are some arbitrary chosen real constants. With such new forms of the arguments (for $n = 1, \cdots, N$) we find the same types of results as before, except that the preferred solutions now exhibit kinks in different vertical planes, parallel to the active wave vectors, and these planes can intersect each other at vertical axes Along these vertical axis solutions can exhibit multiple kinks of various degrees S, where S indicates the number of planes that intersect each other at one vertical axis

The results presented in this paper regarding the preferred solutions, their stability and the roles played by the boundary imperfections indicate that the constants g_i, representing the maximum amplitude of the imperfection components, control the stability of the solution, that is sufficiently high values of $|g_i|$ lead to stability The imperfection wave vectors $K_m^{(b)}$ control the directions of the wave number vectors of the flow solutions. The spatial forms of the amplitude functions L_m for the boundary imperfection lead to unusual behavior of the solutions with kinks. The importance of problems of the type considered here, thus, should not be underestimated In addition, to demonstrate existence and preference of new and unusual types of solutions, we provided a way to control instabilities and the flow structures which can be of significance in flow control applications.

The problem studied here deals with poorly conducting boundaries. This problem, as we have shown in this paper, admits slow horizontal variables x_s and y_s of orders $\gamma^{\frac{1}{2}}\delta^{\frac{1}{3}}$ due to the fact that for R just beyond R_c, in the absence of imperfections, three-dimensional solutions in the form of squares are preferred [2,3] The resulting amplitude system is then a system of non-linear partial differential equations where each equation is second order in derivative with respect to either x_s or y_s. Another equally important problem is one for the case of high conducting boundaries. This problem, as Newell and Whitehead [4] demonstrated, has the property that it admits slow horizontal variables x_s and y_s of orders $\delta^{\frac{1}{3}}$ and $\delta^{\frac{1}{6}}$, respectively, and x_s dependence is more important than y_s dependence. This property is due to the fact that for R just beyond R_c, in the absence of imperfections, two-dimensional rolls are preferred [3,4], where it is assumed that x-axis is along these rolls The resulting amplitude system will then be a system of nonlinear partial differential equations where each equation is second order in derivative with respect to x_s and fourth order in derivative with respect to y_s Although the results for such a system will be reported elsewhere, it is of interest to note here that such a system can admit non-modal solutions with kinks, different from those discussed in the present paper, and the resulting preferred patterns will be affected accordingly.

REFERENCES

[1] RIAHI, D.N., Finite amplitude thermal convection with spatially modulated boundary temperatures, *Proc. R. Soc. Lond. A.* **449** (1995), 459-478

[2] BUSSE, F H and RIAHI, N., Nonlinear convection in a layer with nearly insulating boundaries, *J. Fluid Mech.* **96** (1980), 243-256.

[3] RIAHI, N., Nonlinear convection in a porous layer with finite conducting boundaries, *J. Fluid Mech.* **129** (1983), 153-171.

[4] NEWELL, A.C. and WHITEHEAD, J.A., Finite bandwidth, finite amplitude convection, *J. Fluid Mech.* **39** (1969), 279-303.

[5] REES, D.A.S. and RILEY, D.S., The effects of boundary imperfections on convection in a saturated porous layer: near-resonant wavelength excitation, *J. Fluid Mech.* **199** (1989a), 133-153.

[6] REES, D A.S. and RILEY, D.S., The effects of boundary imperfections on convection in a saturated porous layer: non-resonant wavelength excitation, *Proc. R. Soc. Lond. A.* **421** (1989b), 303-339

[7] REES, D.A.S., The effect of long wavelength thermal modulations on the onset of convection in an infinite porous layer heated from below, *Q. J. Mech. Appl. Math.* **43** (1990), 189-214.

[8] JOSEPH, D.D., *Stability of Fluid Motions I and II* (1976), Springer.

[9] KELLY, R.E. and PAL, D., Thermal convection with spatially periodic boundary conditions resonant wavelength excitation, *J. Fluid Mech.* **86** (1978), 433-456.

[10] SPARROW, E.M., GODSTEIN, R.J. and JONSSON, V.H., Thermal instability in a horizontal fluid layer: effect of boundary conditions and nonlinear temperature profile, *J. Fluid Mech.* **18** (1964), 513-528.

[11] RIAHI, N., Preferred pattern of convection in a porous layer with a spatially non-uniform boundary temperature, *J. Fluid Mech.* **246** (1993), 529-543.