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ON NEW STRENGTHENED HARDY-HILBERT'S INEQUALITY

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ABSTRACT. In this paper, a new inequality for the weight coefficient $\omega(q,n)$ in the form

$$\omega(q,n) := \sum_{m=1}^{\infty} \frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}} \left(q>1, \frac{1}{p}+\frac{1}{q}=1, n \in N\right)$$

is proved. This is followed by a strengthened version of the Hardy-Hilbert inequality.

KEY WORDS AND PHRASES: Hardy-Hilbert's inequality, weight coefficient, Holder's inequality.
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1. INTRODUCTION

If $a_n \geq 0$, $0 < \sum\limits_{n=1}^{\infty} n^2 a_n^2 < \infty$, then the Karlson's inequality is

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2, \tag{1.1}$$

where the constant π^2 cannot be made smaller. However, it can be strengthened (see Mikhlin [1], p. 7) as

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} \left(n-\frac{1}{2}\right)^2 a_n^2. \tag{1.2}$$

In recent years, considerable attention has been given to develop some types of strengthened inequality (see [2]-[10]) by estimating the weight coefficient $\omega(q,n)$ as

$$\omega(q,n) = \sum_{m=1}^{\infty} \frac{1}{(m+n)} \left(\frac{n}{m}\right)^{1/q} \left(q>1, p^{-1}+q^{-1}=1, n \in N\right). \tag{1.3}$$

Some improvement of Hardy-Hilbert's inequality (see Hardy et al. [11]) has been made in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{1/q}. \tag{1 4}$$

In their recent work, Xu and Gau [2] considered the following weight coefficient (1.3) and proved the following inequality

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{\eta_p}{n^{1/p}+n^{-1/q}}, \quad \eta_p = p-1. \tag{1.5}$$

Then a strengthened Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{p-1}{n^{1/p}+n^{-1/q}}\right]a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{q-1}{n^{1/q}+n^{-1/p}}\right]b_n^q\right\}^{1/q}. \tag{1.6}$$

was proved. The key is to estimate the corresponding weight coefficient effectively. Hsu and Wang [3] proved the following inequality

$$\omega(2,n) < \pi - \frac{\theta}{\sqrt{n}}, \quad \theta = \frac{3}{\sqrt{2}} - 1 = 1.12132^+ \quad (n \in N). \tag{1.7}$$

Then they gave a new strengthened Hilbert's inequality which is the same as (1.6) with $p = 2$. Since θ in (1.7) is not the best possible, Gau [5] obtained the best possible value of $\theta = \pi - \sum_{k=1}^{\infty}\frac{1}{(1+k)}\left(\frac{1}{\sqrt{k}}\right) = 1.2811^+$. Subsequently, Gau [6] considered the general case and proved a new inequality for the weight coefficient $\omega(q,n)$ as

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{\theta_p}{n^{1/p}} \quad \left(q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N\right), \tag{1.8}$$

where $\theta_p = (p-1)$. Recently, Gau [7] replaced $(p-1)$ by $\theta_p = \theta_p(n) > 0$ in (1.8). But the problem is that $\theta_p(n)$ depends on both p and q. Simultaneously, Yang [8] found that $\theta_p = \theta = 0.341295^+$, but the constant $\theta_p = \theta$ is not the best possible value. Finally, Yang and Gau [9] found the best possible value for $\theta_p = \theta = 1 - C = 0.42278433^+$, where C is a Euler constant. They also proved the following new Hardy-Hilbert's inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/p}}\right]a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1-C}{n^{1/q}}\right]b_n^q\right\}^{1/q}. \tag{1.9}$$

It is important to point out that (1.5) and (1.8) are different, and the constant η_p in (1.5) depends on p.

The main objective of this paper is to prove an improved version of (1.5) as

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}} \quad \left(q > 1, \frac{1}{p} + \frac{1}{q} = 1, n \in N\right), \tag{1.10}$$

and then prove a strengthened version of Hardy-Hilbert's inequality as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}}\right]a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q}+n^{-1/p}}\right]b_n^q\right\}^{1/q}. \tag{1.11}$$

For this, we need the following inequality (see Yang [8] Lemma 1): If

$$f(x) > 0, f^{(2r-1)}(x) < 0, f^{(2r)}(x) \geq 0, x \in [1,\infty)(r = 1,2), f^{(r)}(\infty) = 0 (r = 0,1,2,3,4),$$

and $\int_1^{\infty} f(x)dx < \infty$, then

$$\sum_{m=1}^{\infty} f(m) \leq \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1). \tag{1.12}$$

2. SOME LEMMAS

LEMMA 2.1. If $q > 1$, $p^{-1} + q^{-1} = 1$, $n \in N$, then

$$\omega(q,n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}}[f_n(p) + g_n(p)], \tag{2.1}$$

where $\omega(q,n)$ is defined by (1.5), and

$$f_n(p) := p + \frac{1}{12p} + \frac{1}{(1+p)n} + \frac{1}{12pn^2} + \frac{1}{3(1+3p)n^3},$$
$$g_n(p) := \frac{-1}{12pn} - \frac{1}{2(1+2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}.$$

PROOF. Let

$$f(x) = \frac{1}{(x+n)x^{1/q}}, \quad x \in [1,\infty)\,(q > 1,\, n \in N).$$

By (1.12), we obtain that

$$\sum_{m=1}^{\infty} \frac{1}{(m+n)m^{1/q}} \le \int_1^{\infty} \frac{1}{(x+n)x^{1/q}}\,dx + \left(\frac{7}{12} - \frac{1}{12p}\right)\frac{1}{1+n} + \frac{1}{12(1+n)^2}. \tag{2.2}$$

Since

$$\begin{aligned}\int_0^{1/n} \frac{1}{(1+y)y^{1/q}}\,dy &= \int_0^{1/n} \sum_{\nu=0}^{\infty}(-1)^\nu y^{\nu-1/q}dy \\ &= \sum_{\nu=0}^{\infty}(-1)^\nu \int_0^{1/n} y^{\nu-1/q}dy = \frac{p}{n^{1/p}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(1+\nu p)n^\nu} \\ &> \frac{p}{n^{1/p}}\sum_{\nu=0}^{3}\frac{(-1)^\nu}{(1+\nu p)n^\nu} = \frac{1}{n^{1/p}}\left[p + \sum_{\nu=1}^{3}\frac{(-1)^\nu}{\nu n^\nu} - \sum_{\nu=1}^{3}\frac{(-1)^\nu}{\nu(1+\nu p)n^\nu}\right].\end{aligned}$$

Putting $x = ny$, we find that

$$\begin{aligned}\int_1^{\infty} \frac{1}{(x+n)x^{1/q}}\,dx &= \frac{1}{n^{1/q}}\int_{1/n}^{\infty}\frac{1}{(1+y)y^{1/q}}\,dy \\ &= \frac{1}{n^{1/q}}\left[\int_0^{\infty}\frac{1}{(1+y)y^{1/q}}\,dy - \int_0^{1/n}\frac{1}{(1+y)y^{1/q}}\,dy\right] \\ &= \frac{1}{n^{1/q}}\left[\frac{\pi}{\sin(\pi/p)} - \frac{p}{n^{1/p}}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(1+\nu p)n^\nu}\right]' \\ &< \frac{1}{n^{1/q}}\frac{\pi}{\sin(\pi/p)} - \frac{1}{n}\left[p + \sum_{\nu=1}^{3}\frac{(-1)^\nu}{\nu n^\nu} - \sum_{\nu=1}^{3}\frac{(-1)^\nu}{\nu(1+\nu p)n^\nu}\right],\end{aligned}$$

we then find that

$$\frac{1}{1+n} = \frac{1}{n}\left(1 + \frac{1}{n}\right)^{-1} < \frac{1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right),$$

and

$$\frac{1}{(1+n)^2} = \frac{1}{n^2}\left(1 + \frac{1}{n}\right)^{-2} < \frac{1}{n^2}\left(1 - \frac{2}{n} + \frac{3}{n^2}\right).$$

Substituting the above results in (2.2), by (1.5), we have (2.1). This proves the lemma.

LEMMA 2.2. If $p > 1$, $n \in N$, then

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}. \tag{2.3}$$

PROOF. Since

$$\begin{aligned} f_n'(p) &= 1 - \frac{1+n^2}{12n^2p^2} - \frac{1}{(1+p)^2 n} - \frac{1}{(1+3p)^2 n^3} \\ &> 1 - \frac{1+n^2}{12n^2} - \frac{1}{(1+1)^2 n} - \frac{1}{(1+3)^2 n^3} \\ &= \frac{11}{12} - \frac{1}{12n^2} - \frac{1}{4n} - \frac{1}{16n^3} > 0, \end{aligned}$$

and

$$g_n'(p) = \frac{1}{12p^2 n} + \frac{1}{(1+2p)^2 n^2} > 0,$$

then $f_n(p) + g_n(p)$ is strictly increasing for $p \in (1, \infty)$, and

$$f_n(p) + g_n(p) > \lim_{p \to 1}(f_n(p) + g_n(p)) = \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}.$$

Thus the lemma is proved.

LEMMA 2.3. If $q > 1$, $p^{-1} + q^{-1} = 1$, $n \in N$, then inequality (1.10) is valid. So is the following inequality:

$$\omega(p, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + n^{-1/p}}. \tag{2.4}$$

PROOF. Since for $n \geq 3$,

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right)\left(1 + \frac{1}{2n}\right) = \frac{1}{2} + \frac{1}{n}\left(\frac{1}{6} - \frac{1}{24n} - \frac{1}{2n^2} - \frac{1}{4n^3}\right) > \frac{1}{2},$$

then

$$\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3} > \frac{1}{2 + n^{-1}} \quad (n \geq 3).$$

By (2.1) and (2.3), we have

$$\begin{aligned} \omega(q, n) &< \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}}\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right) \\ &< \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} \quad (n \geq 3). \end{aligned}$$

Taking $\theta_p = 1 - C$, by (1.8) (see Yang and Gau [9]), we find that

$$\omega(q, 1) < \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{1} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2 \times 1 + 1}. \tag{2.5}$$

Since $C < 3/5 = 0.6$, then we have

$$\frac{1}{2 \times 2^{1/p} + 2^{-1/q}} < \frac{1 - C}{2^{1/p}},$$

and

$$\omega(q, 2) < \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{2^{1/p}} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2 \times 2^{1/p} + 2^{-1/q}}. \tag{2.6}$$

It follows that for $n = 1, 2$, (1.10) also holds. Then (1.10) is valid for any $n \in N$. Interchanging p, q in (1.10), since $\frac{\pi}{\sin(\pi/p)} = \frac{\pi}{\sin(\pi/q)}$, we have (2.4). The lemma is proved.

3. MAIN RESULTS

THEOREM 3.1. If $p > 1$, $p^{-1} + q^{-1} = 1$, $a_n \geq 0$, $b_n \geq 0$, and $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then inequality (1.11) is valid. We also have

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}}\right]a_n^p. \tag{3.1}$$

When $p = q = 2$, this inequality reduces to the form

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^2 < \pi\sum_{n=1}^{\infty}\left[\pi - \frac{1}{2\sqrt{n}+\sqrt{n^{-1}}}\right]a_n^2. \tag{3.2}$$

PROOF. By Holder's inequality, we have

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{(m+n)^{1/p}}\left(\frac{m}{n}\right)^{1/pq}a_m\right]\left[\frac{1}{(m+n)^{1/q}}\left(\frac{n}{m}\right)^{1/pq}b_n\right] \\
&\leq \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{1/q}a_m^p\right\}^{1/p}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/p}b_n^q\right\}^{1/q} \\
&= \left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}\right]a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/p}\right]b_n^q\right\}^{1/q} \\
&= \left\{\sum_{n=1}^{\infty}\omega(q,n)a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\omega(p,n)b_n^q\right\}^{1/q}.
\end{aligned}$$

Hence, by (1.10) and (2.4), inequality (1.11) holds.

Since by (2.4), $\omega(p,n) < \frac{\pi}{\sin(\pi/p)}$, then by Holder's inequality, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{a_n}{m+n} &= \sum_{n=1}^{\infty}\left[\frac{a_n}{(m+n)^{1/p}}\left(\frac{n}{m}\right)^{1/pq}\right]\left[\frac{1}{(m+n)^{1/q}}\left(\frac{m}{n}\right)^{1/pq}\right] \\
&\leq \left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}\right]a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{1/p}\right\}^{1/q} \\
&= \left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}\right]a_n^p\right\}^{1/p}\{\omega(p,n)\}^{1/q}, \\
&< \left\{\sum_{n=1}^{\infty}\left[\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}\right]a_n^p\right\}^{1/p}\left\{\frac{\pi}{\sin(\pi/p)}\right\}^{1/q}.
\end{aligned}$$

By (1.10), we find

$$\begin{aligned}
\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p &< \left[\frac{\pi}{\sin(\pi/p)}\right]^{p/q}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}a_n^p \\
&= \left[\frac{\pi}{\sin(\pi/p)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{1/q}\right]a_n^p \\
&= \left[\frac{\pi}{\sin(\pi/p)}\right]^{p-1}\sum_{n=1}^{\infty}\omega(q,n)a_n^p \\
&< \left[\frac{\pi}{\sin(\pi/p)}\right]^{p-1}\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p}+n^{-1/q}}\right]a_n^p.
\end{aligned}$$

This proves result (3.1). Thus the proof of Theorem 3.1 is complete.

4. CONCLUDING REMARKS

(a) Inequality (1.11) is a definite improvement over (1.6).

(b) Since, for $n \geq 3$, $C > \left(\frac{n+1}{2n+1}\right)$, then

$$\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}} < \frac{\pi}{\sin(\pi/p)} - \frac{(1-C)}{n^{1/p}}, \quad (n \geq 3). \tag{4.1}$$

In view of (2.5), (2.6) and (3.3), it follows that (1.9) and (1.11) represent two distinct versions of strengthened inequalities. But they are not comparable.

(c) Inequality (3.1) reduces to

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \sum_{n=1}^{\infty} a_n^p, \tag{4.2}$$

This is an equivalent form of Hardy-Hilbert's inequality (1.4) (see Hardy et al. [11], Chapter 9).

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