Internat. J. Math. & Math. Sci.
VOL. 21 NO. 4 (1998) 823-827

A COUNTER EXAMPLE ON COMMON PERIODIC POINTS OF FUNCTIONS

ALIASGHAR ALIKHANI-KOOPAEI

Mathematics Department
University of Isfahan
Isfahan, Iran

(Received July 19, 1996)

ABSTRACT. By a counter example we show that two continuous functions defined on a compact metric space satisfying a certain semi metric need not have a common periodic point.

KEY WORDS AND PHRASES: Fixed points, periodic points.
1992 AMS SUBJECT CLASSIFICATION CODES: 26A16, 47H10.

1. INTRODUCTION

In [1] we defined the notion of a semi-metric and used it in a contractive type inequality to obtain some results regarding common fixed points of two functions. We proved Theorem 1.1 and gave a counter example illustrating that we cannot replace the contractive coefficient a with 1. However, it is natural to ask (see [2]) if it is possible to prove a version of Theorem 1.1 with (1.1) amended to read strict inequality, a replaced by 1, and with the additional requirement that $x \neq y$, for the situation in which the functions are defined on a compact metric space X . Theorem 1.2 provides a partial answer to this question . Here we show that in general we can not expect to prove such a result. We begin with Theorem 1.1 and Theorem 1.2 as well as some preliminaries from [1].

THEOREM 1.1. *Let f and g be selfmaps of the unit interval and let $h : I \times I \to [0,\infty)$ be a function having property P_1. Suppose g is continuous on I and A is a nonempty closed $g-$invariant subset of $F(f)$. If there exists a real number a, $0 \leq a < 1$ such that for all x and y in $F(f)$, f and g satisfy the following inequality:*

$$h(fx,fy) \leq a \cdot \max\{h(gx,gy), h(gx,fx), h(gy,fy), h(gy,fx), h(fx,gy)\}, \tag{1.1}$$

then f and g have a unique common fixed point.

THEOREM 1.2. *Suppose f and g are two selfmaps of a compact metric space X with g continuous, and let $h : X \times X \to [0, \infty)$ be a function having property P_1. If for all $x \neq y$ in X, f and g satisfy the following inequality:*

$$h(fx, fy) < \max\{h(gx, gy), h(gx, fx), h(gy, fy), h(gy, fx), h(fx, gy)\}, \tag{1.2}$$

then one of the following holds:

(i) either f and g have a common fixed point.

(ii) or every nonempty closed $g-$invariant subset of $F(f)$ contains a perfect minimal set B such that the functions $\phi_1(x) = h(gx, x)$ and $\phi_2(x) = h(x, gx)$, do not attain their minimum or maximum on B.

Throughout g^n denotes the n fold composition of g with itself and X is a compact metric space. The orbit of x under the homeomorphism g (a one to one function g), $O(g, x)$ is the set $\{g^k(x) : -\infty < k < \infty\}$. A subset Y of X is called invariant under g if $g(Y) \subseteq Y$. A closed, invariant, nonempty subset of X is called minimal if it contains no proper subset that is also closed, invariant and nonempty. The sets $P(f)$ and $F(f)$ are the sets of periodic points and the fixed points of f, respectively. The space $\Sigma_2 = \{s = (s_0 s_1 s_2 ...) : s_j = 0 \text{ or } 1\}$ is called the sequence space on the two symbols 0 and 1. For two sequences $s = (s_0 s_1 s_2 ...)$ and $t = (t_0 t_1 t_2 ...)$, their distance is defined by $d[s, t] = \sum_{i=0}^{\infty} | s_i - t_i | /2^i$. It is clear that (Σ_2, d) is a compact metric space.

Let C be the Cantor Middle-Third set obtained as follows. Let $A_0 = (1/3, 2/3)$ be the middle third of the unit interval I and $I_0 = I - A_0$. Let $A_1 = (1/9, 2/9) \cup (7/9, 8/9)$ be the middle third of the two intervals in I_0 and $I_1 = I_0 - A_1$. Inductively, let A_n denote the middle third of the intervals in I_{n-1} and let $I_n = I_{n-1} - A_n$ and $C = \cap_{n \geq 0} I_n$. For each $x \in C$, we attach an infinite sequence of 0's and 1's, $S(x) = (s_0 s_1 s_2 ...)$, according to the rule: $s_0 = 1$ if x belongs to the left component of I_0; $s_0 = 0$ if x belongs to the right component of I_0. Since x belongs to some component of I_{n-1}, and I_n is obtained by removing the middle third of this interval. Therefore we may set $s_n = 1$ if x belongs to the left hand interval and $s_n = 0$ otherwise. By this way we can think of elements of the Cantor set C as elements of Σ_2 and vice versa. Define $A : \Sigma_2 \to \Sigma_2$ by $A(s_0 s_1 s_2 ...) = (s_0 s_1 s_2 ...) + (100...)$ mod 2, i.e., A is obtained by adding 1 mod 2 to s_0 and carrying the result to the right. For example $A(0000...) = (1000...), A(1000...) = (0100...), A(0100...) = (1100...), A(110\overline{110}...) = (001\overline{110}...), A(111...) = (000...)$. The map A is known as the adding machine (see [4]).

2. RESULTS

We first show that A is a homeomorphism on Σ_2 (or in another word C) and the orbit of every point of C under A is dense in $C = \Sigma_2$. Since Σ_2 does not have a nonempty proper closed invariant subset under A, it is a perfect minimal set.

LEMMA 2.1 *A is a homeomorphism from C to itself.*

PROOF. To see this we show that A is continuous, one one, onto on C with A^{-1} also continuous.

To see that A is continuous, let x be an arbitrary point of C and $\epsilon > 0$. Let N be a positive integer such that $1/2^N < \epsilon$. Choose $\delta = 1/2^N$. If $d(y,x) < \delta$, the sequences x and y have identical first N elements, hence $A(x)$ and $A(y)$ have also identical first N terms. Thus $d(A(x), A(y)) < 1/2^N < \epsilon$, implying the continuity of A at x.

To see that A is one one on C, let $x = (x_0x_1...), y = (y_0y_1...)$ be two points of C with $x \neq y$, then there exists a least nonnegative integer N such that $x_N \neq y_N$. Obviously the corresponding elements of the sequences $A(x)$ and $A(y)$ are different, hence A is one one.

To see that A is onto, let $y = (y_0y_1y_2...)$ and N be the smallest nonnegative integer such that $y_N = 1$. Then for $x = (11...0y_{N+1}y_{N+2}...)$ we have $A(x) = y$.

Since (Σ_2, d) is a compact metric space and A is continuous on $C = \Sigma_2$, the image of every closed subset of C under A is a closed set, implying the continuity of A^{-1}.

LEMMA 2.2. *The orbit of every point of C under A is dense in C.*

PROOF. Let $x = (x_0x_1x_2...)$ and $y = (y_0y_1y_2...)$ be two arbitrary points of $C = \Sigma_2$. For $\epsilon > 0$, choose a positive integer N so that $1/2^N < \epsilon$. Let N_1 be the least positive integer such that the sequences x and y have identical first $N_1 - 1$ elements. Then for $k_1 = 2^{N_1-1}$ the two sequences $A^{k_1}(x)$ and y have at least N_1 identical first elements. Similarly suppose N_2 is the least positive integer such that the first $N_2 - 1$ elements of the two sequences $A^{k_1}(x)$ and y are identical. Then $N_2 \geq N_1 + 1$ and for $k_2 = 2^{N_2-1}$ the two sequences $A^{k_1+k_2}(x)$ and y have identical first N_2 elements. By repeating this process we obtain a positive integer $m = k_1 + k_2 + ... + k_l$ such that $A^m(x)$ and y have identical first N elements, implying $d(A^m(x), y) < 1/2^N < \epsilon$. Since x and y were arbitrary we may interchange the role of x with y. Thus the result is established.

DEFINITION 2.1. Let X be a compact metric space. The function $h : X \times X \to [0, \infty)$ is said to have property P_1 if it satisfies the following conditions:

(i): $h(x,y) = 0$ if and only if $x = y$,

(ii): if $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} y_n = y_0$, and $\lim_{n\to\infty} h(x_n, y_n) = 0$, then $x_0 = y_0$.

The following theorem is based on an example which illustrates that assertion (ii) of Theorem 1.2 may occur.

THEOREM 2.1. *There exist two continuous functions f and g selfmaps of a compact metric space (X,d), a $g-$minimal perfect set $B \subseteq F(f)$ and a function $h : X \times X \to [0,\infty)$ having property P_1 such that for all $x \neq y$ both in B, f and g satisfy the following:*

$$\begin{aligned} h(fx, fy) \ < \ \max\{&h(gx,gy), h(gx,fx), h(gy,fy), \\ &h(gy,fx), h(fx,gy)\}, \end{aligned} \tag{2.1}$$

yet f and g do not have a common periodic point.

PROOF. Consider the compact metric space (Σ_2, d). For each $x \in \Sigma_2$, let $f(x) = x$ and

$g(x) = A(x)$, where A is the adding machine. It is clear that $B = \Sigma_2$ is a g-invariant perfect subset of $F(f)$. Suppose $H = \{O(g,x) : x \in B\}$. By the axiom of choice there is a set E such that E has exactly one element from each element of H. Choose an arbitrary point $x_0 \in E$. Define the function $h : B \times B \to [0,\infty)$ as follows:

(a): $h(t,s) = 0$ if $s = t$.

(b): Suppose $M = O(g,x_0) \times O(g,x_0)$. We define h on M as

(i): for $n,m \geq 0, n \neq m$, $h(g^n x_0, g^m x_0) = h(g^m x_0, g^n x_0) = 7 - 1/2^{(m+n)}$.

(ii): For $m < 0, n = 0$, $h(g^n x_0, g^m x_0) = h(g^m x_0, g^n x_0) = 3 - 1/2^{-m}$.

(iii): For $m < 0 < n$, $h(g^n x_0, g^m x_0) = h(g^m x_0, g^n x_0) = 5 - 1/2^{(-m+n)}$.

(iv): For $m \neq n, m < 0, n < 0$, $h(g^n x_0, g^m x_0) = h(g^m x_0, g^n x_0) = 1 + 1/2^{-(m+n)}$.

(c): Let $z \in E$ and $z \neq x_0$,

(i): for each integers m and n, define $h(g^n z, g^m z) = h(g^m z, g^n z) = h(g^n x_0, g^m x_0) = h(g^m x_0, g^n x_0)$

(ii): For each integers m and n, $m \neq n$ and each $t \in E$, $z \in E$, $t \neq z$, define $h(g^n z, g^m t) = h(g^m t, g^n z) = h(g^n x_0, g^m x_0)$.

(iii): For each integers $m \neq n$, define $h(g^n z, g^m x_0) = h(g^m x_0, g^n z) = h(g^n x_0, g^m x_0)$.

(iv): For each integer n and each $t \neq z$ both different from x_0, define $h(g^n(z), g^n(t)) = h(g^n(t), g^n(z))$:

$$h(g^n z, g^n x_0) = h(g^n x_0, g^n z) = \begin{cases} 4 - 1/2^{(n+2)} & 0 \leq n, \\ 3 + 1/2^{-n} & n < 0. \end{cases}$$

Since g is one to one on B, the function h is well defined on $B \times B$. The function h satisfies the property P_1 since, for $x = y$, $h(x,y) = 0$ and for each $(x,y) \in B \times B$, $h(x,y) \geq 1$. It remains to show that for every $t \neq s$ in B the inequality (2.1) is satisfied. To show this let $t, s \in B$, and $t \neq s$. We distinguish several different cases.

Case 1: There exists $x_0 \in E$ such that $t = g^n(x_0)$, $s = g^m(x_0)$ for some integers m and n.

(i): If $m = n$ then $h(t,s) = 0$, but $h(gt,s) > 0$.

(ii) If $n, m \geq 0$ and $m \neq n$, then $h(t,s) = h(g^n x_0, g^m x_0) = 7 - 1/2^{(m+n)}$ and $h(gt,gs) = h(g^{(n+1)} x_0, g^{(m+1)} x_0) = 7 - 1/2^{(m+n+2)}$. Hence $h(t,s) < h(gt,gs)$.

(iii) If $m < 0$, $n = 0$, $h(t,s) = h(x_0, g^m x_0) = 3 - 1/2^{-m}$. Suppose $m < -1$. Then $h(gt,gs) = h(gx_0, g^{(m+1)} x_0) = 5 - 1/2^{-m}$. If $m = -1$, then $h(t,s) = 5/2$, but $h(gt,gs) = h(gx_0, g^{(m+1)} x_0) = h(gx_0, x_0) = 13/2$. Hence in this case we have $h(t,s) < h(gt,gs)$.

(iv) If $m < 0 < n$, then $h(t,s) = h(g^n x_0, g^m x_0) = 5 - 1/2^{(-m+n)}$. If $m = -1$, we have $h(gt,gs) = h(g^{(n+1)} x_0, g^{(m+1)} x_0) = h(g^{(n+1)} x_0, x_0) = 7 - 1/2^{(n+1)}$. If $m < -1$, then $h(gt,s) = h(g^{(n+1)} x_0, g^m x_0) = 5 - 1/2^{(-m+n+1)}$ and $h(gt,t) = h(g^{(n+1)} x_0, g^n x_0) = 7 - 1/2^{(n+1+n)} = 7 - 1/2^{(2n+1)}$. Hence we have $h(t,s) < \max\{h(gt,gs), h(gt,s)\}$ and $h(t,s) < \max\{h(gt,gs), h(gt,t)\}$.

(v) If $m \neq n, m < 0, n < 0$, then $h(t,s) = h(g^n x_0, g^m x_0) = 1 + 1/2^{-(m+n)}$. Suppose either $m = -1$ or $n = -1$. Without loss of generality we may assume that $m = -1$. Then we have $h(t,s) = 1 + 1/2^{(1-n)}$ and $h(gt,gs) = h(g^{(n+1)} x_0, g^{(m+1)} x_0) = h(g^{(n+1)} x_0, x_0) = 3 - 1/2^{-(n+1)}$. For $m < -1$ and $n < -1$ we also have $h(t,s) = 1 + 1/2^{-(m+n)}$ and $h(gt,gs) = h(g^{(n+1)} x_0, g^{(m+1)} x_0) = 1 + 1/2^{-(m+n+2)}$, implying $h(t,s) < h(gt,gs)$. Thus again we have $h(t,s) < \max\{h(gt,gs), h(gt,s)\}$

and $h(t,s) < \max\{h(gt,gs),h(gt,t)\}$.

Case 2: There exist $x_1 \in E$, $x_2 \in E$ such that $t = g^n(x_1)$, $s = g^m(x_2)$, for some integers m and n. If $m \neq n$, then $h(t,s) = h(g^n x_1, g^m x_2) = h(g^n x_0, g^m x_0) < \max\{h(gt,gs),h(gt,s)\}$ and $h(t,s) < \max\{h(gt,gs),h(gt,t)\}$. If $n = m \geq 0$, then $h(t,s) = h(g^n x_1, g^n x_2) = 4 - 1/2^{(n+2)}$ and h(gt, gs)= $h(g^{(n+1)}x_1,\ g^{(n+1)}x_2) = 4 - 1/2^{(n+3)}$, thus $h(t,s) < h(gt,gs)$. If $n = m = -1$, then $h(t,s) = h(g^{-1}x_1, g^{-1}x_2) = 7/2$, and $h(gt,gs) = h(x_1,x_2) = 15/4$. If $n = m < -1$, then $h(t,s) = 3 + 1/2^{-n}$ and $h(gt,gs) = 3 + 1/2^{-(n+1)}$. Hence for this case we also have $h(t,s) < \max\{h(gt,gs),h(gt,s)\}$ and $h(t,s) < \max\{h(gt,gs),h(gt,t)\}$.

We see that inequality (2.1) is satisfied for every $t,s \in B$, yet f and g do not have any common periodic point. In fact the set of fixed points of f is identical to B, while g does not have any periodic point in B.

Remark 2.1. We may choose the functions f and g to be continuous selfmaps of the unit interval. For example let $\{(a_n,b_n)\}_{n=1}^{\infty}$ be the complementary intervals of the middle third Cantor set C. Define

$$f(x) = \begin{cases} x & x \in C, \\ a_n & a_n < x \leq (a_n+b_n)/2, \\ 2(x-b_n)+b_n & (a_n+b_n)/2 < x \leq b_n, \end{cases} \tag{2.2}$$

for $n = 1,2,3,...$, g a continuous extension of A, and the semi metric h on C as in the above theorem. It is clear that $F(f) = C$ and the orbit of each point outside C under f (*i.e.*$\{x, f(x), f^2(x), f^3(x), ...\}$) is attracted to a point in C , also f and g satisfy the inequality (2.1) on $F(f)$, yet they do not have any common periodic points.

ACKNOWLEDGMENT. I would like to thank professor B.E. Rhoades for his comments (reference[3]) which led to this work. I Also wish to thank professor A. M. Bruckner for his valuable suggestions.

AUTHOR'S PRESENT ADDRESS. Department of Mathematics, The University of Tennessee at Chattanooga, Chattanooga, TN 37403- 2598.

REFERENCES

1. ALIKHANI-KOOPAEI, A. A., *On common fixed and periodic points of commuting functions*, International Journal of Mathematics and Mathematical Sciences, to appear.

2. RHOADES, B.E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226(1977), 257-290.

3. RHOADES, B.E., Private Communication.

4. DEVANEY, R. L.,*An introduction to chaotic dynamical systems*, second edition, Addition-Wesley Publishing Inc., 1989.