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TAG-MODULES WITH COMPLEMENT SUBMODULES H-PURE

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ABSTRACT

The concept of a QTAG-module M_R was given by Singh[8]. The structure theory of such modules has been developed on similar lines as that of torsion abelian groups. If a module M_R is such that M⊕M is a QTAG-module, it is called a strongly TAG-module. This in turn leads to the concept of a primary TAG-module and its periodicity. In the present paper some decomposition theorems for those primary TAG-modules in which all h-neat submodules are h-pure are proved. Unlike torsion abelian groups, there exist primary TAG-modules of infinite periodicities. Such modules are studied in the last section. The results proved in this paper indicate that the structure theory of primary TAG-modules of infinite periodicity is not very similar to that of torsion abelian groups.

KEY WORDS AND PHRASES: QTAG-modules, complement submodules, h-pure submodules, h-neat submodules, and basic submodules.

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ξ 1 INTRODUCTION

A module M_R satisfying the following two conditions is called a TAG-module [2].

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any homomorphism f: W → V can be extended to a homomorphism g: U → V provided the composition length $d(U/W) \leq d(V/f(W))$.

If a module satisfies condition (I), it is called a QTAG-module [8]. The main purpose of this paper is to prove some decomposition theorems for a module M, such that M⊕M is a QTAG-module and that is to prove some decomposition theorems for a module M, such that M⊕M is a QTAG-module and that every h-neat (complement) submodule of M is h-pure. An example of such an h-reduced primary TAG -

module, which is not decomposable, is given at the end of the paper. However, it follows fron the results in this paper that any torsion reduced module over a bounded (hnp)-ring, with every complement submodule pure, is decomposable. The main results are given in Theorems (5.5), (5.12) and (5.14). In section 3, a necessary and sufficient condition for a QTAG-module to admit only one basic submomdule is given. In section 4 the concept of neat height of a uniform element in a QTAG-module is discussed. The concept of neat height is used to give, in Theorems (4.6) and (4.7), some criterians for a QTAGmodule, such that every h-neat module is l-embedded in the sense of Moore[5]. The results in sections 3 and 4 can be of independent interest.

ξ 2 PRELIMINARIES

A module in which the lattice of its submodules is linearly ordered under inclusion is called a serial module; in addition if it has finite composition length, it is called a uniserial module. Let M_R be a QTAG-module. An $x \in M$ is called a uniform element, if xR is a non-zero uniform (hence uniserial) submodule of M. For any module A_R with a composition series, d(A) denotes its composition length. Let $x \in M$ be uniform. Then $e(x) = d(xR)$ is called the exponent of x. The equation $[x, y] = n$, will give that y is a uniform element of M, such that $x \in yR$ and $d(yR/xR) = n$. For basic definitions of height of an element of M, the submodule $H_k(M)$ for $k \geq 0$, one may refer to [6] or [8]. For any submodule N of M, and any $y \in N$, $h_N(y)$ will denote the height of y in N; however we write h(y) for $h_M(y)$. A submodule N of M is said to be h-pure in M, if $H_k(M) \cap N = H_k(N)$ for every $k \geq 0$. For any module K, soc(K) denotes the socle of K. M_R is said to be decomposable, if it is a direct sum of uniserial modules.

By using [8, Lemma(2.3)], one can prove the following:

Proposition(2.1). A submodule N of a QTAG-module M is h-pure in M if and only if for any uniform $x \in soc(N)$, $h_N(x) = h(x)$.

The following is of frequent use in this paper.

Proposition(2.2) [8, Lemma(3.9)]. Let N be any h-pure submodule of a QTAG-module M. Then for any uniform $x \in M$, there exists a uniform $x' \in M$, such that for $\bar{x} = x + M \in M/N$, $e(\bar{x}) = e(x')$, $\bar{x} = \bar{x'}$ and $M \cap x'R = 0$.

By using the above proposition, we get that if M/N is decomposable for some h-pure submodule N, then $M = T \oplus N$, for some decomposable submodule T of M. Let K_R be any module. For the definitions of K-injective modules and K-projective modules one may refer to [1]. Lemmas (2.2) and (2.4) in [8] give the following:

Proposition(2.3). Let A and B be two uniserial submodules of a QTAG-modules M, such that $A \cap B = 0$.

(i) If $d(A) \leq d(B)$, then B is A-injective.

(ii) If $d(A) \geq d(B)$, then B is A-projective.

(iii) If $d(A) = d(B)$, then $A \cong B$ if and only if either $soc(A) \cong soc(B)$, or $A/H_1(A) \cong B/H_1(B)$.

M is said to be bounded, if for some k, $H_k(M) = 0$. Any h-pure bounded submodule of M is a summand of M [8, Remark(3.8)]. M is said to be h-divisible, if $h(x) = \infty$ for every $x \in M$. If a uniform element $x \in soc(M)$ has finite height, then for any uniform $y \in M$, with $[x, y] = h(x)$, yR being an h-pure submodule of M, is a summand of M. For general properties of rings and modules one may refer to [3].

ξ3 BASIC SUBMODULES

Throughout M_R is a QTAG-module. A submodule B of M is called a basic submodule of M, if B is a decomposable h-pure submodule of M, such that M/B is h-divisible [7]. As pointed out in [8, Remark(3.12)], M has a basic submodule and any two basic submodules of M are isomorphic.

Lemma(3.1). Let $A_1, A_2, \ldots, A_k$ be any finitely many uniserial summands of M, such that $d(A_i) < d(A_{i+1})$ and $N = \sum_{i=1}^{k} A_i = \oplus\sum_{i=1}^{k} A_i$. Then N is an h-pure submodule of M.

Proof. Consider a uniform element $x \in soc(N)$. Then $x = \Sigma x_i$, $x_i \in A_i$. If for any $i < j$, $x_i \neq 0 \neq x_j$, then by the hypothesis $h(x_i) < h(x_j)$. Thus $h(x) = \{h(x_i) : x_i \neq 0\}$. As each A_i is h-pure, $h(x_i) = h_{A_i}(x_i) = h_N(x_i)$. This gives $h(x) = h_N(x)$. Hence N is -pure.

Lemma(3.2). Let M be such that $\cap_k H_k(M) = 0$ and let M have a basic submodule $B \neq M$. Then for some simple submodule S of soc(M), there exists an h-pure submodule $N = \oplus\sum_{i=1}^{\infty} y_i R$ such that every y_iR is uniserial, $d(y_iR) < d(y_{i+1}R)$ and $S \cong soc(y_iR)$. The heights of the (non-zero) elements of the homogeneous components of soc(M), determined by S, do not have an upper bound.

Proof. Let $\bar{M} = M/B$. Consider a uniform $\bar{z}$ in $soc(\bar{M})$. By (2.2) there exists a uniform $z_1 \in soc(M)$ such that $\bar{z} = \bar{z}_1$. As $\cap_k H_k(M) = 0$, $h(z_1)$ is finite. Let $h(z_1) = n_1$. Then there exists $y_1 \in M$, such that $[z_1, y_1] = n_1$. Then y_1R is an h-pure submodule of M and $B \cap y_1R = 0$. However $h(\bar{z}) = \infty$. So there exists a uniform $u_1 \in M$ with $soc(\bar{u}_1) = \bar{z}R$ and $e(\bar{u}_1) > n_1$ By (2.2) we get uniform $z_2 \in soc(M)$ with $\bar{z}_2 = \bar{z}$, $h(z_2) = n_2 > n_1$. We get $y_2 \in M$ such that $[z_2, y_2] = n_2$. By continuing this process, we get an infinite sequence of uniform elements $\{y_i\}_{i \geq 1}$ of M, such that each y_iR is an h-pure uniserial submodule, $soc(y_iR) = z_iR$ for some $z_i \in M$ satisfying $\bar{z} = \bar{z}_i$, $[z_i, y_i] = n_i = h(z_i)$ and $n_i < n_{i+1}$. If $K = \sum_i y_i R$ is not a direct sum, we get a smallest $i \geq 2$, such that $z_i \in \sum_{k=1}^{i-1} z_k R$. Then $N = \sum_{k=1}^{i-1} y_k R = \oplus\sum_{k=1}^{i-1} y_k R$. By (3.1) N is an h-pure submodule of M. For any uniform $v \in N$, if $v = \Sigma v_j$, with $v_j \in y_jR$, then $h(v) = \min\{h(v_j)\}$ This gives $h(z_i) \leq \max\{h(z_k) : 1 \leq k \leq i-1\}$. This is a contradiction,as $h(z_j) < h(z_i)$ for $j < i$. Hence $K = \oplus\Sigma y_iR$. By using (3.1) we get that K is an h-pure submodule. Clearly soc(K) is homogeneous. The last part is obvious.

Lemma(3.3). Let M be a QTAG-modue such that $M = \oplus\sum_{i=1}^{\infty} y_i R$, y_iR uniserial, $soc(y_iR) \cong soc(y_{i+1}R)$ and $d(y_iR) < d(y_{i+1}R)$. Then M has a basic submodule $B \neq M$.

Proof. By (2.3)(i) we get monomorphisms $\sigma_i : y_iR \to y_{i+1}R$. Write $\sigma_i(y_i) = w_i$. Then w_i is uniform and $e(w_i) = e(y_i)$. Consider $B = \sum_{i=1}^{\infty} w_iR$, and $\overline{M} = M/B$. Let $z \in B$. Then $z = \sum_{i=1}^{s}(y_i - \sigma_i(y_i))r_i = y_1R + \sum_{i=2}^{s}(y_ir_i - \sigma_{i-1}(y_{i-1})r_{i-1}) - \sigma_s(y_s)r_s$, for some $r_i \in R$ and a positive integer s. Here $y_ir_i - \sigma_{i-1}(y_{i-1})r_{i-1} \in y_iR$ and $-\sigma_s(y_sr_s) \in y_{s+1}R$.. Using this, it can be easily proved that $B = \oplus\Sigma w_iR$ and $y_1R \cap B = 0$. Now $\overline{y_1} = \overline{\sigma_1(y_1)}$, and $e(\sigma_1(y_1)) = e(\overline{\sigma_1(y_1)})$. So that $\sigma_1(y_1)R \cap B = 0$. As $\sigma_1(y_1)R \subseteq y_2R$, we get $y_2R \cap B = 0$. By continuing this process, we get $y_iR \cap B = 0$. Clearly $\bar{y}_1 R < \bar{y}_2 R <\ldots$, gives $\overline{M}$ is a serial module of infinite length. It only remains to prove that B is h-pure. In view of (3.1) it is enough to prove that each w_iR is h-pure. Now $y_iR \oplus y_{i+1}R$ being a summand of M, is h-pure. But $y_iR \oplus y_{i+1}R = w_iR \oplus y_{i+1}R$. So w_iR is h-pure in M. This completes the proof.

Theorem(3.4). A QTAG-module M_R has no basic submodule other than M if and only if M is h-reduced and for each homogeneous component K of soc(M), there exists an upper bound on the heights of members of K

Proof. Let M be its only basic submodule. Then by definition M is decomposable and h-reduced. For a simple submodule S of M, we get a summand M_S of M, such that $soc(M_S)$ is the homogeneous component of soc(M) determined by S. If heights of members of $soc(M_S)$ do not have an upper bound, we get a summand $N = \oplus\sum_{i=1}^{\infty} y_iR$ of M_S such that each y_iR is uniserial and $d(y_iR) < d(y_{i+1}R)$. By (3.3) N has a basic submodule $B_1 \neq N$. As N is a summand of M, we get a basic submodule B of M of which B_1 is a summand and $B \neq M$. This is a contradiction. Conversely let the given conditions hold. Then $\cap_k H_k(M) = 0$. The rest follows from (3.3) .

§ 4. H-NEAT HEIGHT

Throughout M_R is a QTAG-module. A submodule N of M is called an h-neat submodule of M if $H_1(M) \cap N = H_1(N)$. As observed in [8], any submodule N of M is h-neat if and only if it is a complement submodule of M, any maximal essential extension K' of a submodule K of M, is an h-neat submodule of M. Any such K' is called an h-neat hull of K. For any uniform $x \in M$, the minimum of all $d(K'/xR)$, where K' runs over all h-neat hulls of xR, is called the h-neat height of x : it is denoted by $h'(x)$. If $x \in N \subseteq M$, then $h'_N(x)$ will denote the neat height of x in N. If N is an h-neat submodule of M, then any h-neat submodule of N is h-neat in M, so that for any uniform $x \in N$, $h'(x) \leq h'_N(x)$. We put $h'(0) = \infty$. In an h-divisible QTAG-module M, every uniform element is of infinite h-neat height.

For any two modules A_R and B_R any homomorphism from a submodule of A into B is called a subhomomorphism from A to B; the set of all subhomomorphisms from A to B is denoted by SH(A, B). An $f \in$ SH(A, B) is said to be maximal, if it has no extension in SH(A, B). Now (2.3) gives the following:

Lemma(4.1). Let xR and yR be any two uniserial submodules of M, such that $xR \cap yR = 0$. Then

(a) For any maximal $f \in$ SH(xR, yR), either domain(f) = xR or range(f) = yR.

(b) Let $z \in xR \oplus yR$ be unifrom, $z = x' + y'$, $x' \in xR$, $y' \in yR$ and $d(x'R) \geq d(y'R)$. The following hold:

(i). $zR \cong x'R$.

(ii) Given any $u = v+w$, $v \in xR$, $w \in yR$ such that $z \in uR$,

(α) if $y' \neq 0$, then $[x', v] = [y', w]$;

(β) if $y' = 0$, then $e(w) \leq [x', v]$

Lemma (4.2). Let xR and yR be two uniserial submodules of M such that $xR \cap yR = 0$. Let $z = x'+y'$, $x' \in xR$, $y' \in yR$, be uniform such that $d(y'R) \leq d(x'R)$. For $T = xR \oplus yR$, the following hold:

(i). For $y' \neq 0$, $h'_T(z)$ is the minimum of $[x', x]$ and $[y', y]$.

(ii). For $y' = 0$, let $f \in SH(xR, yR)$ be maximal with $s = d(\ker f))$, minimal under the condition that $x'R \subseteq \ker f$. If domain(f) = uR, then $h'_T(z) = [x', u]$ = minimum of $[x', x]$ and $e(y) + s - e(x')$.

Proof. $g : x'R \to y'R$ such that $g(x'r) = y'r$ is an R-epimorphism. If $w = a+b$, $a \in xR$, $b \in yR$, is uniform and $z \in wR$, then $f : aR \to bR$ such that $f(ar) = br$, is an extension of g; further $[z, w] = [x', a]$. Any extension $h : a'R \to yR$, $a' \in xR$, of g gives uniform $w' = a' + h(a')$ such that $z \in w'R$. Consequently wR is an h-neat hull of zR if and only if f is maximal. In that case by (4.1) either domain(f) = xR or range(f) = yR. Thus for domain(f) = aR, and uR = ker f, e(a) is the minimum of e(x) and e(y)+e(u). To minimize e(a), we need to minimise s = e(u). So that for minimal e(u), $h'_T(z) = [x', a] = e(a) - e(x') = \min\{e(x), e(y)+e(u)\} - e(x') = \min\{[x', x], e(y)+e(u) - e(x')\}$, as $e(x) - e(x') = [x', x]$ If $y' \neq 0$, then $e(x') = e(u)+e(y')$, so that $e(y)+e(u) - e(x') = e(y) - e(y') = [y', y]$. For $y' = 0$, it is obvious that $x'R \subseteq \ker f$. This proves the result.

Lemma(4.3). Let $M = A \oplus B$ and $x \in M$ be uniform. If $x = a+b$, $a \in A$, $b \in B$ and $d(aR) \geq d(bR)$, then the following hold:

(i). For $b \neq 0$, $h'(x) = \min\{h'_A(a), h'_B(b)\}$.

(ii). If $b = 0$, and B is h-divisible, then $h'(x) = h'_A(a)$

Proof. Now $g : aR \to bR$ given by $g(ar) = br$, is an epimorphism. Let π_1 and π_2 be the projections $A \oplus B \to A$, and $A \oplus B \to B$ respectively. Consider an h-neat hull K of xR. Then K is serial. Let $K_i = \pi_i(K)$. As $d(bR) \leq d(aR)$, we get an epimorphism $\sigma : K_1 \to K_2$ such that for any $x_1 \in K_1$, $\sigma(x_1) = x_2$ if and only if $x_1+x_2 \in K$. Further $aR \subseteq K_1$, $bR \subseteq K_2$ and $d(K/xR) = d(K_1/aR)$ By using (2.3) we get that either K_1 is h-neat or K_2 is h-neat in M.

Case I : $b \neq 0$. Then either K_1 is an h-neat hull of aR or K_2 is an h-neat hull of bR. So that $h'(x) \geq \min\{h'_T(a), h'_T(b)\}$. Let $t = \min\{h'_A(a), h'_B(b)\} < h'(x)$. To be definite let $t = h'_A(a)$. Then we get an h-neat hull a_1R of aR with $[a, a_1] = t$, and a uniform b_1 in M with $[b, b_1] \geq t$. By (2.3) g extends to a homomorphism $f: a_1R \to b_1R$. Then $(a_1+f(a_1))R$ is an h-neat hull of xR with $[x, a_1+f(a_1)] < h'(x)$ This is a contradiction. Similar arguments hold if $t = h'_B(b)$ This proves (i).

Case II : $b = 0$ and B is h-divisible. Any h-neat serial submodule of B is either zero or of infinite length. Thus for K to be an h-neat hull of xR it is necessary and sufficient that K_1 is an h-neat hull of aR.

Thus for $x = a$, $h'(x) = h'_A(a)$

Lemma(4.4). Let $K_R = \oplus\sum_{i=1}^{t} x_i R$ be a QTAG-module with each x_iR uniserial. Let $z = \Sigma z_i$, $z_i \in x_iR$, be uniform. Let z_u be such that $e(z) = e(z_u)$. Then $h'(z)$ is the minimum of the following numbers :

(i). All $[z_i, x_i]$. with $z_i \neq 0$.

(ii). The neat heights of z_u in various $x_uR \oplus x_jR$, with $z_j = 0$.

Proof. The hypothesis on z_u gives that for any i, $\sigma_i : z_uR \to z_iR$ such that $\sigma_i(z_ur) = z_ir$ is an epimorphims. Let $y = \Sigma y_i$, $y_i \in x_iR$, be any uniform element in K such that $z \in yR$. Then $\eta_i : y_uR \to y_iR$ given by $\eta_i(y_ur) = y_ir$ is an extension of σ_i. Clearly if a $z_i \neq 0$, then $[z_u, y_u] = [z_i, y_i]$. So that e(y) is not more than s, the minimum of all those $[z_i, x_i]$ for which $z_i \neq 0$. Thus $h'(z) \leq s$. However, if every $z_i \neq 0$, then by (2.3), it is immediate that for yR to be an h-neat hull of zR, it is necessary that $[z, y] = s$, i.e $h'(z) = s$. Suppose that for some j, $z_j = 0$ and that for $T = x_uR \oplus x_jR$, $h'_T(z_u) < s$. We have a maximal $f \in SH(x_uR, x_jR)$ with ker f of smallest length among those containing z_uR. Let w_uR = domain(f), then $s' = h'_T(z_u) = [z_u, w_u]$. By using (2.3), we obtain a uniform $y = \Sigma_i y_i$ with $z \in yR$, $y_u = w_u$ and $y_j = f(w_u)$. Then yR is an h-neat hull of zR such that $[z, y] = s'$. Thus $h'(z) \leq s_0$, the minimum of the numbers listed in (i) and (ii). Suppose $h'(z) < s_0$. We get a uniform $w = \Sigma w_i$, $w_i \in x_iR$ such that wR is an h-neat hull of zR and $[z, w] = h'(z)$. Then for some j, $w_jR = x_jR$. For this j, $z_j = 0$ and $(w_u+w_j)R$ is an h-neat hull of z_uR. Consequently for $T = x_uR \oplus x_uR$, $h'_T(z_u) \leq h'(z)$. This is a contradiction. This completes the proof.

We now give a criterian in terms of h-neat heights, for a QTAG-module , in which every h-neat submodule is h-pure. We shall give a more general result. Analogous to the definition of an l-embedded subgroup of an abelian p-group given by Moore [5], we define an l-embedded submodule of a QTAG-module. Let Z^+ be the set of all non-negative integers and $l : Z^+ \to Z^+$ be any function such that $n \leq l(n)$, $n \in Z^+$. A submodule N of a QTAG-module M is said to be l-embedded if $H_{l(n)}(M) \cap N \subseteq H_n(N)$ for every $n \in Z^+$. Thus if I is the identity map on Z^+, a submodule N of M is h-pure in M if and only if N is I-embedded. Given $l : Z^+ \to Z^+$ satisfying $l(n) \geq n$, we define $l_1 : Z^+ \to Z^+$ such that for any $n \in Z^+$, $l_1(n)$ is the minimum of all $l(k)$, $k \geq n$. Then l_1 is monotonic. Further any submodule N of M is l-embedded if and only if it is l_1-embedded. So without loss of generality we assume that l is monotic. Further define $l(\infty) = \infty$.

Proposition 4.5. Let M be an h-reduced QTAG-module and $l : Z^+ \to Z^+$ be a monotonic function such that $n \leq l(n)$, $n \in Z^+$. Then every h-neat submodule of M is l-embedded if and only if $h(y) \leq l(h'(y)+1)-1$ for every uniform $y \in M$.

Proof. Let every h-neat submodule of M be l-embedded. Consider a uniform $y \in M$. As M is h-reduced, every h-neat hull of yR is of finite length. Let zR be an h-neat hull of yR such that $[y, z] = h'(y) = t$. Then $H_t(zR) = yR$ and $H_{t+1}(zR) < yR$. Then by the hypothesis, $H_{l(t)}(M) \cap zR \subseteq H_t(zR) = yR$, but $H_{l(t+1)}(M) \cap zR < yR$. Consequently $h(y) \leq l(t+1)-1 = l(h'(y)+1)-1$ Conversely let the inequality hold. So every uniform $y \in M$ has finite height. Let there exist an h-neat submodule N of M that is not l-

embedded. We get smallest positive integer n such that $H_{l(n)}(M) \cap N \not\subset H_n(N)$. Then $H_{l(n-1)}(M) \cap N \subset H_{n-1}(N)$. There exists a uniform $y \in H_{l(n)}(M) \cap N$ such that $y \notin H_n(N)$. As $l(n) \geq l(n-1)$, $y \in H_{n-1}(N)$. So that $h_N(y) = n-1$. Consequently $h'(y) \leq n-1$. By the hypothesis $h(y) \leq l(h'(y)+1)-1 \leq l(n)-1$. However as $y \in H_{l(n)}(M)$, $h(y) \geq l(n)$. This is a contradiction. This proves the result.

Theorem(4.6). Let M be any QTAG-module and $l : Z^+ \to Z^+$ be a monotonic function such that $n \leq l(n)$, $n \in Z^+$. Then every h-neat submodule of M is l-embedded if and only if for any uniform $y \in M$, $h(y) \leq l(h'(y)+1)-1$.

Proof. Let every h-neat submodule of M be l-embedded. Write $M = L \oplus D$, where D is the largest h-divisible submodule of M. Now L is h-reduced and every h-neat submodule of L is l-embedded in L. Consider a uniform $y \in M$. Write $y = y_1+y_2$, $y_1 \in L$, $y_2 \in D$. Suppose $y_1 \neq 0$. Then $h(y) = h(y_1)$. By (4.3), $h'(y) = h'_L(y_1)$ By using (4.5), we get $h(y) = h(y_1) \leq l(h'(y)+1)-1$. Suppose $y_1 = 0$. then $y = y_2 \in D$, hence and $h(y) = \infty$. Let K be any h-neat hull of yR. Consider any $n \geq 0$. Then $H_{l(n)}(M) = H_{l(n)}(L) \oplus D$. As $K \cap D \neq 0$, $H_{l(n)}(M) \cap K \subseteq H_n(K)$, we get $H_n(K) \neq 0$. So that $d(K) = \infty$, $h'(y) = \infty = h(y)$. Once again $h(y) = l(h'(y)+1)-1$. Conversely let the given condition be satisfied. By essentially following the arguments in (4.5), we complete the proof.

Theorem(4.7). Let $M = L \oplus D$ be a QTAG-module such that L is h-reduced and D is h-divisible. For a monotonic function $l : Z^+ \to Z^+$ satisfying $n \leq l(n)$, every h-neat submodule of M is l-embedded if and only if

(i) every h-neat submodule of L is l-embedded in L; and

(ii) for any serial submodue W of D, any non-zero homomorphism $f : W \to L$ is a monomorphism.

Proof. Let every h-neat submodule of M be l-embedded. Then obviously (i) hold. Consider a non-zero homomorphism $f : W \to L$ with $\ker f \neq 0$. then $bR = soc(W) \subset \ker f$. Consider $soc(f(W)) = b_1R$. As $h(b_1) < \infty$, by using (2.3) we can choose W to be such that f(W) is h-neat in L. Then $L_1 = \{x+f(x) : x \in W\}$ is an h-neat hull of bR. So that $h'(b) < \infty$. By (4.6) $h'(b) = \infty$. This gives a contradiction.

Conversely, let the conditions be satisfied. Consider a uniform $y \in M$. Let $y = y_1+y_2$, $y_1 \in L$, $y_2 \in D$. Suppose $y_1 \neq 0$. Then by (4.3) $h(y) = h_L(y_1) \leq l(h'_L(y_1)+1)-1$. Suppose $y_1 = 0$. Then $y = y_2 \in D$. Let K be any h-neat hull of yR. Let K_1 and K_2 be projections of K in L and D respectively. Then $K \cong K_2$ and we get an epimorphism $f : K_2 \to K_1$ with $y \in \ker f$. By (ii), $f = 0$. Consequently $K \subseteq D$ and hence $d(K) = \infty$. So once again $h(y) = l(h'(y)+1)-1$. Hence (4.6) completes the proof.

By taking $l = I$, we get the following:

Corollary (4.8). Let $M = L \oplus D$ be a QTAG-module such that L is h-reduced and D is h-divisible Then the following are equivalent:

(i) Every h-neat submodule of M is h-pure in M.

(ii) For any uniform $y \in M$, $h(y) = h'(y)$.

(iii) Evey h-neat submodule of L is h-pure and for any uniserial submodule W of D any non-zero homomorphism $f: W \to L$ is a monomorphism

ξ 5. H-NEAT SUBMODULES

A module M_R is called a strongly TAG-modue, if $M \oplus M$ is a QTAG-module. We start with the following:

Lemma(5.1). Let M_R be a strongly TAG-module, A and B be two uniserial submodules of some homomorphic images of M . Then the following hold:

(i) If $d(A) \leq d(B)$, then B is A-injective.

(ii) If $d(A) \geq d(B)$, then B is A-projective.

(iii) If $d(A) = d(B)$, then $A \cong B$, whenever $soc(A) \cong soc(B)$ or $A/H_1(A) \cong B/H_1(B)$.

(iv) M is a TAG-module.

Proof. Now A and B are submodules of M/K and M/L for some submodules K and L of M. As $N = M/K \oplus M/L$ is a homorphic image of $M \oplus M$, $A\times 0$, $0\times B$ are submodules of N with zero intersection, (i) , (ii), and (iii) follow from (2.3). Finally (iv) follows from (i).

Let M_R be a strongly TAG-module. Let spec(M) be the set of all simple R-modules which occur as composition factors of some finitely generated submodules of M. Let S, $S' \in$ spec(M). Then S' is called an immediate predecessor of S (and S is called an immediate successor of S') if for some uniserial submodule A of M, $A/H_1(A) \cong S'$ and $H_1(A)/H_2(A) \cong S$. By using (5.1) we get that any $S \in$ spec(M) does not have more than one immediate successor and more than one immediate predecessor.(see also [9]). Let S, $S' \in$ spec(M), S' is called a k-th successor of S, if there exists a sequence $S = S_0, S_1, \ldots, S_k = S'$ of k+1 distinct members S_i of spec(M), such that for $i < k$, S_{i+1} is an immediate successor of S_i; in this situation S is called a k-th predecessor of S'. S is called its own 0-th successor(0-th predecessor). $S'$is called a successor of S , if S' is a k-th successor of S for some positive integer k. Define $S \sim S'$ if for some $k \geq 0$, S' is a k-th successor or k-th predecessor of S. This is an equivalence relation. Any equivalence class C determined by this relation is called a primary class. For a torsion abelian group, each such C is a singleton. However for a torsion module over a bounded (hnp)-ring, each C is finite. For any primary class C in spec(M), the submodule M_c of all those $x \in M$ such that every composition factor of xR is in C, is called the C-primary submodule of M. By using (5.1) one can easily see that M is a direct sum of its C-primary submodules. A module M is called a primary TAG-module if $M \oplus M$ is a TAG-module such that spec(M) is a primary class. Consider a primary TAG-modue M. Let spec(M) have k members, then either k is finite or countable. This k is called the periodicity of M. In this section we study primary TAG-modules.

Lemma(5.2). Let M_R be an h-reduced primary TAG-module of finite periodicity. If there exists a function $f: Z^+ \to Z^+$ such that for any uniform $x \in M$, $h(x) \leq f(h'(x))$, then M is bounded.

Proof. Let M be of periodicity k. For any uniform $x \in M$, $h'(x) < \infty$. This gives $h(x) \leq f(h'(x)) < \infty$. Suppose M is not bounded. Then M has uniserial summands of arbitrarily large lengths. So we can write $M = x_1R \oplus x_2R \oplus M'$, with x_iR non-zero uniserial, $z_iR = soc(x_iR)$, $h(z_2) > \max\{f(j) : 1 \leq j \leq k+d(x_1R)\}$ and $e(x_2) > k$. Now $h(z_2) = [z_2, x_2]$. As M is of periodicity k and $e(x_2) > k$, we get $y_2 \in x_2R$ such that $[z_2, y_2] \leq k-1$ and $soc(x_2R/y_2R) \cong soc(x_1R)$. This gives a maximal $g \in SH(x_2R, x_1R)$ with $d(\ker g) \leq k$ and $z_2R \subseteq \ker g$. Consequently $d(domain(g)) \leq k+d(x_1R)$, $h'(z_2) \leq k+d(x_1R)$. As $h(z_2) \leq f(h'(z_2))$, we get $h(z_2) \leq \max\{f(j) : 0 \leq j \leq k+d(x_1R)\}$. This is a contradiction. Hence M is bounded.

Lemma(5.3).Let M_R be any primary TAG-module of finite periodicity. If every h-neat submodule of M is h-pure, then either M is h-divisible or h-reduced.

Proof. Let M be neither h-reduced nor h-divisible. Then $M = xR \oplus A \oplus M_1$ for some uniform element x and a serial module A of infinite length. Let $zR = soc(A)$. Then $h(z) = \infty$. If the periodicity of M is k, then for some u, $1 \leq u \leq k$, we get a submodule of A of length u satisfying $soc(A/yR) \cong soc(xR)$. By (2.3), we get a maximal $f \in SH(A, xR)$ with $d(domain(f)) \leq e(x)+u$. This gives an h-neat hull K of zR length $e(x)+u$. As K is h-pure, we get $h(z) = d(K)-1 < \infty$. This is contradiction.Hence the result follows.

Lemma(5.4). Let M_R be a primary TAG-module of finite periodicity k. Let $T = xR \oplus A$ be a submodul eof M, with xR uniserial, such that every h-neat submodule of T is h-pure in T. Then the following hold:

(i) If $soc(xR) \cong soc(A)$, then $d(A) \leq d(xR)+k$.

(ii). If soc(xR) is the u-th predecessor of soc(A) for some $u \geq 1$, then $d(A) \leq d(xR)+u$.

Proof. Let $soc(A) = zR$. Let $soc(xR) \cong zR$. For a maximal $f \neq 0$ in SH(A, xR) with $zR \subseteq \ker f$ and $d(\ker f)$ minimal, we have $d(\ker f) = k$, $domain(f) = yR \subseteq A$; further $h'(z) = e(y)-1 = [z, y] \leq e(x)+k-1$. However by (4.8), $h'(z) = h(z)$. So $yR = A$. Consequently $e(y) = d(A) \leq d(xR)+k$. Similarly (ii) follows.

We now prove the first decomposition theorem.

Theorem(5.5). Let M_R be a primary TAG-module ofperiodicity $k < \infty$. Then every h-neat submodule of M is h-pure if and only if either M is h-divisible or $M = \oplus\sum_{\alpha\in\Lambda} x_\alpha R$ such that :

(i). each $x_\alpha R$ is uniserial; and

(ii) for any two distinct $\alpha, \beta \in \Lambda$ the following hold :

(a) if $soc(x_\alpha R) \cong soc(x_\beta R)$, then $d(x_\beta R) \leq d(x_\alpha R)+k$,

(b) if $soc(x_\beta R)$ is a u-th predecessor of $soc(x_\alpha R)$, $1 \leq u \leq k-1$, then $d(x_\alpha R) \leq d(x_\beta R)+u$.

Proof. Let every h-neat submodule of M be h-pure. By (5.2) M is either h-divisible or h-reduced Let M be h-reduced. By (5.2) M is bounded. So that $M = \oplus\sum_{\alpha\in\Lambda} x_\alpha R$, for some uniserial submodules $x_\alpha R$ By applying (5.4) we complete the necessity. Conversely let the given conditions be satisfied. If M is h-divisible, then every h-neat submodule N of M being h-divisible, is a summand of M, consequently N is h-pure. Let M be h-reduced. Consider a uniform $z = \sum_{\alpha\in\Lambda} z_\alpha \in M$ with $z_\alpha \in x_\alpha R$. Then $h(z) = \min\{h(z_\alpha) : z_\alpha \neq 0\} = \min\{[z_\alpha, x_\alpha] : z_\alpha \neq 0\}$. Consider $T = x_\alpha R \oplus x_\beta R$ with $z_\alpha \neq 0$, $z_\beta \neq 0$ and $\alpha \neq \beta$. Let $f \in SH(x_\alpha R, x_\beta R)$ be maximal with the property that $z_\alpha R \subseteq \ker f$ and $d(\ker f)$ is minimal. Either $domain(f) = x_\alpha R$ or range(f)

$= x_\beta R$. If $f = 0$, obviously domain(f) $= x_\alpha R$. Let $f \neq 0$. If $soc(x_\alpha R) \cong soc(x_\beta R)$, then d(ker f) $= \lambda k$ for some $\lambda > 0$. If $soc(x_\alpha R) \not\cong soc(x_\beta R)$, then for some $u \geq 1$ $soc(x_\beta R)$ is the u-th predecessor of $soc(x_\alpha R)$ and d(ker f) $= u+\mu k$ for some $\mu \geq 0$. Thus (a) and (b) yield domain(f) $= x_\alpha R$. Cosequently $h'_T(z_\alpha) = [z_\alpha , x_\alpha] = h(z_\alpha)$ By (4.4), $h'(z) = h(z)$. This proves the result.

The periodicity of a torsion abelian p-group is one. We get the following:

Corollary(5.6). Every neat subgroup of an abelian p-group G , p a prime number, is pure subgroup if and only if either G is a divisible group or $G = A \oplus B$, such that for some positive integer n, A is a direct sum of copies of $Z/(p^n)$ and B is a direct sum of copies of $Z/(p^{n+1})$.

We now discuss the case of a primary TAG-module of infinite periodicity. Henceforth M_R will be a primary TAG-module of infinite periodicity.

Lemma(5.7). Let xR and yR be two h-neat uniserial submodules of M such that soc(xR) $\not\cong$ soc(yR) and soc(yR) is a predecessor of soc(xR). Then :

(i) SH(yR, xR) = 0.

(ii) For any h-neat hull K of $xR \oplus yR$ in M, yR is a summand of K; if in addition xR is h-pure in M, then $K = xR \oplus yR$.

(iii) If xR and yR both are h-pure, then $xR \oplus yR$ is h-pure in M.

Proof. As M is of infinite periodicity and soc(yR) is a predecessor of soc(xR), soc(xR) is not a predecessor of soc(yR). Consequently SH(yR, xR) = 0. Let K be an h-neat hull of $xR \oplus yR$. As rank(K) = 2, $K = A_1 \oplus A_2$ with A_i serial. Consider the projections $f_i : A_1 \oplus A_2 \to A_i$. The restrictiion of one of f_i ,say of f_1 to xR ia a monomorphism. Then $soc(xR) \cong soc(A_1)$ and $soc(yR) \cong soc(A_2)$. Further f_2 embeds yR in A_2. By (i) $SH(A_2, A_1) = 0$. This yields $yR \subseteq A_2$. As $yR \subset' A_2$ annd yR is h-neat, we get $yR = A_2$. Let xR be h-pure in M. So that xR is h-pure in K. Consequently xR is a summand of K. As $xR \not\cong yR$, we get $K = xR \oplus yR$. This proves (ii). Finally let both xR and yR be h-pure in M.Then $M = xR \oplus M_1$ for some submodule M_1. Then $K = xR \oplus (K \cap M_1)$. This gives $yR = K \cap M_1$. So $K \cap M_1$ is h-pure in M_1. Thus $K \cap M_1$ is a summand of M_1. Hence K is a summand of M. This gives (iii).

Lemma(5.8). Let K be any submodule of M with soc(K) homogeneous. Then :

(i) Given any two uniserial submodules A and B of M, either $A \cap B = 0$ or they are comparable under inclusion,

(ii) for any uniform $x \in K$, $h_K(x) = h'_K(x)$, and

(iii) any h-neat submodule of K is h-pure in K.

Proof. (i) Let $A \cap B \neq 0$ and $d(A) \geq d(B)$. Then $A+B = A \oplus C$, with C a proper homomorphic image of B. Suppose $C \neq 0$, then soc(A) = soc(B) $\not\cong$ soc(C). This contradicts the hypothesis that soc(K) is homogeneous. Thus C = 0, so $B \subseteq A$. This proves (i). Consider a uniform $x \in K$. Then by (i) xR has unique h-neat hull D in K. Then $h'_K(x) = d(D)-1$. The uniqueness of D gives D is h-pure. Consequently $h_K(x) = d(D)-1$. This proves (ii). The last part is immediate from (ii)

Corollary(5.9). Let xR and yR be two uniserial submodules of M with $xR \cap yR = 0$. Every h-neat submodule of $T = xR \oplus yR$ is h-pure in T if and only if

(i) soc(xR) ≅ soc(yR), or

(ii) soc(xR) ≇ soc(yR), one of them say soc(xR) is a u-th predecessor of soc(yR) for some u ≥ 1 and $d(yR) \leq d(xR)+u$.

Proof. Let every h-neat submodule of T be h-pure. Let soc(xR) ≇ soc(yR), and soc(xR) be a u-th predecessor of soc(yR). Then as in (5.4), we get $d(yR) \leq d(xR) + u$. Conversely, let T satisfy the given conditions. If soc(xR) ≅ soc(yR), then soc(T) is homogeneous, so by (5.8) every h-neat submodule of T is h-pure in T. Let (ii) hold, soc(yR) be a u-th predecessor of soc(xR). As M is of infinite periodicity, soc(xR) is not a predecessor of yR. So SH(yR, xR) = 0, and for any $0 \neq f \in$ SH(xR, yR), d(ker f) = u. Consider a uniform $z = x_1 + y_1$, $x_1 \in xR$, $y_1 \in yR$.. Let $e(x_1) \geq e(y_1)$. If $y_1 \neq 0$, then $e(x_1) = e(y_1)+u$. So $[x_1, x] \leq [y_1, y]$, and hence $h'_T(z) = [x_1, x] = h_T(z)$. Suppose $y_1 = 0$. Consider a maximal $f \in$ SH(xR, yR) with $z \in$ ker f. Then either f = 0 or d(ker f) = u. As $d(xR) \leq d(yR)+u$, domain(f) = xR. Once again $h'_T(z) = [x_1, x] = h_T(z)$. Let $e(y_1) > e(x_1)$, then $x_1 = 0$, as SH(yR, xR) = 0. Then for any uniform $v \in T$, with $z \in z_1R$, we have $z_1 \in yR$. So yR is the only h-neat hull of zR in T. Thus in all cases $h'_T(z) = h_T(z)$. By (4.8), the result follows.

Lemma(5.10). Let N be the submodule of M generated by those uniform elements $x \in M$ such that soc(xR) has no predecessor in soc(M). Then:

(i) Soc(N) is homogeneous.

(ii). N is an h-pure submodule of M.

(iii) Any h-neat submodule of N is h-pure in M.

Proof. Let A be the set of those uniform $x \in M$ such that soc(xR) has no predecessor in soc(M). For any $x, y \in A$, if soc(xR) ≇ soc(yR), then one of them being a successor of the other, contradicts the hypothesis. So that soc(xR) ≅ soc(yR) for all $x, y \in A$. Consider a uniform $z \in$ soc(N). For some $y_i \in A$, $z \in \sum y_iR = \oplus\sum B_j$, B_j's uniserial. For some j, zR ≅ soc(B_j). But B_j is a homomorphic image of some y_iR. As soc(y_iR) has no predecessor in soc(M), we get $y_iR \cong B_j$. Hence soc(N) is homogeneous. It is now immediate that if for any uniform $x \in M$, soc(xR) ⊆ N, then $x \in N$. This fact gives (ii). By using (4.8) we get (iii).

The submodule N of M generated by those uniform elements $x \in M$, such that soc(xR) has no predecessor in soc(M) is called a terminal submodule of M. We denote this submodule by Ter(M).

Proposition(5.11). Let M_R be a primary TAG-module of infinite periodicity and N = Ter(M). Then :

(i) Any submodule K of N has unique h-neat hull in M,

(ii) for any uniform $x \in N$, $h(x) = h'(x)$; and

(iii) for any decomposition $M = \oplus\sum_{i\in\Lambda} A_i$, $N = \sum(A_i \cap N)$.

Proof. By (5.10) soc(N) is homogeneous. So given a uniform $x \in N$, by (5.8) any two uniform submodules of N containing x are comparable under inclusion. Thus there is unique h-neat hull A_x of xR in M, and by (5.10) $A_x \subseteq N$. For K, the sum L of those A_x for which $x \in K$, is the unique h-neat hull of K. (ii) is immediate from (i). Consider any uniform $x \in N$. Then $x = \sum x_i$, $x_i \in A_i$. If some $x_i \neq 0$, and the mapping $xR \to x_iR$, $xr \to x_ir$ is not one-to-one, then soc(xR) will have a predecessor in soc(M). This gives a contradiction. Hence $xR \cong x_iR$, whenever $x_i \neq 0$. Thus $x_i \in N$ and (iii) follows.

Theorem(5.12). Let M_R be an h-reduced primary TAG-module of infinite periodicity. Then every h-neat submodule of M is h-pure if and only if $M = \oplus\sum_{j=-\infty}^{\infty} K_j \oplus \mathrm{Ter}(M)$ satisfying the following conditions :

(i) for each j, K_j is decomposable and soc(K_j) is homogeneous,

(ii) for $j_1 < j_2$, with $K_{j_1} \neq 0 \neq K_{j_2}$, if z_1R and z_2R are uniseserial summands of K_{j_1} and K_{j_2} respectively, then soc(z_2R) is a u-th predecessor of soc(z_1R) for some positive integer u depending upon j_1 and j_2 , and $d(z_1R) \leq d(z_2R) + u$; and

(iii) if t is the length of a smallest length uniserial summand of N, and S is the simple module determining soc(N), then for any $K_j \neq 0$, if S is a v_j-th predecessor of the simple module S_j determining soc(K_j) , we have $d(zR) \leq t + v_j$ for any uniserial summand zR of K_j.

Proof. Let every h-neat submodule of M be h-pure. Now N = Ter(M). As N is h-reduced, it has a uniserial summand xR of smallest length, say t. Consider $\overline{M}$ = M/N. Let S be a simple submodule of $\overline{M}$. Consider any uniform $\overline{y} \in \overline{M}$ such that $S \cong \mathrm{soc}(\overline{y}R)$. By (2.2), we choose y to be uniform such that $yR \cap N = 0$. Then soc(xR) $\not\cong$ soc(yR). As soc(xR) has no predecessor in soc(M), it is a v-th predecessor of soc(yR) = zR for some $v \geq 1$. Now $h(z) = h'(z) < \infty$. We get $y_1 \in M$ such that $[z, y_1] = h(z)$. Then in $xR \oplus y_1R$, both the summands are h-pure in M. By (5.7) $xR \oplus y_1R$ is h-pure. By (5.9), $d(y_1R) \leq d(xR) + v$. So there is an upper bound on the heights of elements of a particular homogeneous component of soc($\overline{M}$). Hence by (3.4) $\overline{M}$ is its only basic submodule, so it is decomposable. As N is h-pure , by the observation following (2.2), we get $M = K \oplus N$, with K its only basic submodule. As M is primary, spec(M) is countable. We get $K = \oplus\sum_{j=-\infty} K_j$ satisfying (i). Finally (ii) and (iii) follow from (5.9).

Conversely, let M satisfy the given conditions.. Then K satisfies conditions analogous to those given in (5.5). So on the simalar lines as in (5.5), every h-neat submodule of K is h-pure. Consider any uniform $y \in M$. Now $y = y_1 + y_2$. $y_1 \in K$, $y_2 \in N$. If $y_1 = 0$, $y \in N$ and by (5.11), yR has unique h-neat hull in M ; obviously then $h(y) = h'(y)$. Let $y_1 \neq 0$. Suppose $y_2 \neq 0$. then by using (4.3) $h'(y) = h(y)$. Suppose $y_2 = 0$ and $h'(y) < h(y)$. We get an h-neat hull zR of yR with $[y, z] = h'(y)$. Let $z = z_1 + z_2$, $z_1 \in K$, $z_2 \in N$. As $h'_K(y) = h(y)$, $z_2 \neq 0$. One of z_1R and z_2R is h-neat. As $yR \subseteq z_1R$ and $[y, z_1] < h(y) = h_K(y)$, z_1R is not h-neat in K and so in M. Consequently z_2R is h-neat in N, and by (5.10) it is h-pure. For some v, soc(z_1R) is a v-th predecessor of soc(z_2R). So that $d(z_1R) = d(z_2R) + v$. Write soc(z_1R) = gR. Then by using condition (iii), we get $[g, z_1] \leq h(g) \leq d(z_2R) + v - 1 = [g, z_1]$. Conseuently $d(z_1R) - 1 = h(g)$. So z_1R is h-pure. This is a contradiction. This completes the proof.

We now discuss the case of M being not necessarily h-reduced. Write $M = M_1 \oplus D$, where D is the largest h-divisible submodule of M.

Lemma(5.13). If every h-neat submodule of M is h-pure and $D \neq 0$, then Ter(M) $\subseteq$ D ; further Ter(M) is h-divisible.

Proof. Suppose $N = Ter(M) \not\subset D$. We get a uniform $x \in soc(N)$, such that $h(x) < \infty$. Then $x = y + z$. Now $0 \neq y \in M_1$, $z \in D$. By (5.11) $y \in N$. Consider any simple submodule S of D. By the definition of S, it is not a predecessor of soc(yR). So that soc(yR) is a predecessor of S. As D is h-divisible, there exists a uniserial submodule A of D and a homomorphism $f : A \to M_1$ with range(f) = yR and $S \subseteq \ker f$. This contradicts (4.8). Hence $N \subseteq D$. As N is h-pure, it must be h-divisible.

Theorem(5.14) Let M_R be a primary TAG-module of infinite periodicity such that M is not h-divisible and let N = Ter(M). Then every h-neat submodule of M is h-pure if and only if the following hold:

(a) N is h-divisible ,and

(b) $M = N \oplus \sum_{j=-\infty}^{j=\infty} K_j$, where K_j satisfy the following conditions:

(i) if some $K_j \neq 0$, then $soc(K_j)$ is a homogeneous component of soc(M),

(ii) each K_j is a direct sum of serial modules,

(iii) if for some $i < j$, $K_i \neq 0 \neq K_j$ and K_i is not h-divisible , then the simple submodule S_i determining $soc(K_i)$ is a v-th predecessor of the simple submodule determining $soc(K_j)$ for some positive integer v depending on i and j , and for any uniserial submodule A of K_j, $d(A) \leq t + v$, where t is the length of the smallest length uniserial summand of K_i, and

(iv) if for some j, $K_j \neq 0$ and is not h-reduced, then for any $i < j$, K_i is h-divisible.

Proof. Let D be the largest h-divisible submodule of M. Then $D \neq 0$. Let every h-neat submodule of M be h-pure. By (5.13) N is h-divisible. Thus $M = N \oplus M_1 \oplus M_2$ such that $D = N \oplus M_1$ and M_2 is h-reduced. By applying (5.12) to M_2 and using the fact that M_1 is a direct sum of serial modules, we get $M_1 \oplus M_2 = \oplus \sum_{j=-\infty}^{j=\infty} K_j$ satisfying (i), (ii), and (iii). Finally (iv) is an immediate consequence of (iii).

Conversely let the given conditions be satisfied. By comparing these conditions with those in (5.12), we get $M = D \oplus L$ such that $N \subseteq D$. Then SH(N, L) = 0. Consider a uniserial submodule W of D and let $f : W \to L$ be a non-zero homomorphism. Then $W \not\subset N$. For some j, W is isomorphic to a submodule of K_j. This K_j is not h-reduced, $f(W) \subseteq L$, and for some t, f(W) is isomorphic to a submodule of K_t . If j = t, obviously f is a monomorphism. Suppose $t < j$ Then K_t is h-divisible. Let xR = soc(f(W)). As $xR \subseteq L$, $h(x) < \infty$. On the other hand $x \in soc(K_t)$ yields $h(x) = \infty$. This is a contradiction. Hence $j < t$. So $SH(K_j, K_t) = 0$. This once again contradicts the fact that $f \neq 0$. Thus j = t. Hence f is a monomorphism. So by (4.8) the result follows.

We end this paper by giving an example of an h-reduced primary TAG-module M of which every h-neat submodule is h-pure, but it is not decomposable. Such a module has to be of infinite periodicity.

Example. Let F be a Galois field and R be the ring of infinite lower triangular matrices $[a_{ij}]$ over F, where i, j are indexed over the set P of all positive integers. Let $\{e_{ij} : i, j \in P\}$ be the usual set of matrix units in R. Then $M_{kk} = e_{kk}R$ is a uniserial R-module with $d(M_k) = k$; it is annihilated by the ideal A_k of R consisting of those $[a_{ij}] \in R$, such that $a_{ij} = 0$ for $i \leq k$. Observe that each R/A_k is isomorphic to the ring of k×k lower triangular matrices over F. So that any R/A_k-module is a TAG-module. Each M_k embeds in

M_{k+1} under the mapping $e_{kk}r \rightarrow e_{k+1,k}r$, $r \in R$. Let $T = \Pi_k M_k$. Then $M_R = \{x \in T : xA_k = 0 \text{ for some } k\}$ is a primary TAG-module of infinite periodicity. Its socle is homogeneous. By (5 8) every h-neat submodule of M is h-pure. M is h-reduced. Consider a uniform $x \in soc(M)$. then $x = (x_k)$, $x_k \in M_k$. Let u be the smallest integer such that $x_u \neq 0$. Then $xR \cong x_uR$. As $d(M_u) = u$, by using (2.3) it can be easily seen that $h(x) = u-1$. So that for any $i > 1$, $soc(H_{i-1}(M))/soc(H_i) \cong soc(M_i)$. Suppose that M is decomposable, Then $M = \oplus \sum_{j=1}^{\infty} N_j$, where N_j is a direct sum of uniserial modules of length j. Then $soc((H_{i-1}(M))/soc(H_i(M)) \cong soc(N_i)$. Thus $soc(N_i) \cong soc(M_i)$, a simple module. Consequently each N_i is a uniserial module. As F is finite, N_i is a finite set. Consequently M is countable. But by construction M is uncountable This is a contradiction. Hence M is not decomposable.

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