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OSCILLATION OF A HIGHER ORDER NEUTRAL DIFFERENCE EQUATION WITH A FORCING TERM

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Abstract. The authors obtain oscillation results for the even order forced neutral difference equation

$$\Delta^m(y_n+p_ny_{n-k})+q_nf(y_{n-\ell})=h_n. \tag{$*$}$$

Examples illustrating the results are included.

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1. Introduction. In this paper, we consider forced even order nonlinear neutral difference equations of the form

$$\Delta^m(y_n+p_ny_{n-k})+q_nf(y_{n-\ell})=h_n, \tag{1}$$

where $m\ge 2$ is even, $k,\ell\in\mathbb{N}=\{0,1,2,\dots\}$, $\Delta y_n=y_{n+1}-y_n$ is the usual forward difference operator, $\{p_n\},\{q_n\}$, and $\{h_n\}$ are real sequences, and $f:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous with $uf(u)>0$ for $u\ne 0$.

Let $\sigma=\max\{k,\ell\}$ and let $N_0\in\mathbb{N}$ be fixed. By a *solution* of (1), we mean a real sequence $\{y_n\}$ defined for all $n\ge N_0-\sigma$ and satisfying (1) for all $n\ge N_0$. Here, we are concerned only with the nontrivial solutions of (1). Such a solution $\{y_n\}$ of (1) is said to be *oscillatory* if, for any $N\ge N_0$, there exists $n>N$ such that $y_{n+1}y_n\le 0$. Otherwise, the solution is said to be *nonoscillatory*. Throughout the paper, we assume that the following conditions hold:

(C_1) $q_n\ge 0$ for all $n\in\mathbb{N}$, and q_n is not eventually identically zero;

(C_2) f is nondecreasing and there exists $K>0$ such that

$$|f(uv)|\ge K|f(u)||f(v)| \quad \text{for all } u,v\in\mathbb{R}, \tag{2}$$

and

$$\int_0^{\pm c}\frac{ds}{f(s)}<\infty \quad \text{for all } c>0. \tag{3}$$

In recent years, the oscillation of delay difference equations, especially unforced equations, has been studied by a variety of authors. For recent contributions to the literature, see, for example, the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references contained therein. However, relatively few oscillation results are known for forced equations (see [5, 6, 7, 8, 9, 10, 11]). In this paper, we give sufficient conditions which ensure that all solutions of (1) are oscillatory under the influence of certain classes of forcing terms.

In the sequel, we often make use of the following conditions:

(H_1) $0 \le p_n < P_1 < 1$, where P_1 is a constant;
(H_2) there exists a real sequence $\{F_n\}$ such that $\Delta^m F_n = h_n$;
(H_3) $\sum_{n=N_0}^{\infty} q_n f\left(\left(\frac{n-l}{2^{m-1}}\right)^{(m-1)}\right) = \infty$;
(H_4) $\{F_n\}$ is oscillatory and $\lim_{n\to\infty} F_n = 0$;
(H_5) $\{F_n\}$ is k periodic;
(H_6) $\sum_{n=N_0}^{\infty} q_n = \infty$;
(H_7) there exists $\gamma > 0$ such that $f(u)/u \ge \gamma > 0$ for $u \ne 0$.

We also need the following lemmas whose proof can be found in [1].

Lemma 1 ([1, Thm. 1.7.11]). *Let $z_n > 0$ be defined for $n \ge a$ with $\Delta^m z_n$ of constant sign for $n \ge a$ and not identically zero. Then there exists an integer j, $0 \le j \le m$, with $m+j$ odd for $\Delta^m z_n \le 0$ and $m+j$ even for $\Delta^m z_n \ge 0$, such that for $n \ge a$*

$$\begin{aligned} &j \le m-1 \text{ implies } (-1)^{j+i}\Delta^i z_n > 0 \quad \text{for } j \le i \le m-1 \\ &j \ge 1 \text{ implies } \Delta^i z_n > 0 \quad \text{for } 1 \le i \le j-1. \end{aligned} \tag{4}$$

Lemma 2 ([1, Cor. 1.7.12]). *Let $z_n > 0$ be defined for $n \ge a$ with $\Delta^m z_n \le 0$ for $n \ge a$ and not eventually identically zero. Then there exists an integer $N_1 \ge a$ such that*

$$z_n \ge \frac{(n-N_1)^{(m-1)}}{(m-1)!}\Delta^{m-1} z_{2^{m-j-1}n} \tag{5}$$

for $n \ge N_1$, where j is defined in Lemma 1.

Remark 1. Observe that under the hypotheses of Lemma 1, if z_n is increasing, then

$$z_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{(m-1)}\Delta^{m-1} z_n \tag{6}$$

for $n \ge 2^{m-1}N_1$.

2. Main results.

Our first theorem is a new result for unforced equations, but the technique of proof will be used in subsequent theorems for forced equations.

Theorem 1. *Let $h_n \equiv 0$ for all $n \in \mathbb{N}$, and let (H_1) and (H_3) hold. Then all solutions of (1) are oscillatory.*

Proof. Let $\{y_n\}$ be a solution of (1) with $y_n > 0$, $y_{n-k} > 0$, and $y_{n-\ell} > 0$ for $n \ge N_1 \ge N_0$. Setting

$$z_n = y_n + p_n y_{n-k}, \tag{7}$$

we obtain $z_n \ge y_n > 0$ and

$$\Delta^m z_n = -q_n f(y_{n-\ell}) \le 0 \tag{8}$$

for $n \geq N_1$. By Lemma 1, there exists an odd integer j with $0 \leq j \leq m$ such that

$$\Delta^i z_n > 0 \quad \text{for } i = 1, \dots, j-1$$

and

$$(-1)^{j+i}\Delta^i z_n > 0 \quad \text{for } i = j, j+1, \dots, m-1 \tag{9}$$

for $n \geq N_2$ for some $N_2 \geq N_1$.

Since m is even, $\Delta z_n > 0$ and $\Delta^{m-1} z_n > 0$ for $n \geq N_2$. From (7), we have

$$z_n - p_n y_{n-k} = y_n, \tag{10}$$

so $z_n \geq y_n$ and $\{z_n\}$ increasing imply that

$$0 < (1-P_1)z_n \leq (1-p_n)z_n \leq y_n. \tag{11}$$

Again, since z_n is increasing, Remark 1 and (11) imply that there exists $N_3 \geq N_2$ such that

$$y_n \geq (1-P_1)z_n \geq \frac{(1-P_1)}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{(m-1)} \Delta^{m-1} z_n \tag{12}$$

for $n \geq 2^{m-1}N_3$. Applying (C$_2$) to (12) yields

$$\begin{aligned} f(y_{n-\ell}) &\geq K^2 f\left(\frac{(1-P_1)}{(m-1)!}\right) f\left(\left(\frac{n-\ell}{2^{m-1}}\right)^{(m-1)}\right) f(\Delta^{m-1} z_{n-\ell}) \\ &\geq K_1 f\left(\left(\frac{n-\ell}{2^{m-1}}\right)^{(m-1)}\right) f(\Delta^{m-1} z_n) \end{aligned} \tag{13}$$

for $n \geq N_4 \geq 2^{m-1}N_3$, where $K_1 = K^2 f\left(\frac{(1-P_1)}{(m-1)!}\right) > 0$. Combining (8) and (13), we obtain

$$\Delta^m z_n + K_1 q_n f\left(\left(\frac{n-\ell}{2^{m-1}}\right)^{(m-1)}\right) f(\Delta^{m-1} z_n) \leq 0 \tag{14}$$

for $n \geq N_4$ and summing, we get

$$K_1 \sum_{s=N_4}^{n-1} q_s f\left(\left(\frac{s-\ell}{2^{m-1}}\right)^{(m-1)}\right) \leq -\sum_{s=N_4}^{n-1} \frac{\Delta^m z_s}{f(\Delta^{m-1} z_s)} \leq \int_{\Delta^{m-1} z_n}^{\Delta^{m-1} z_{N_4}} \frac{du}{f(u)}. \tag{15}$$

Letting $n \longrightarrow \infty$ and using (C$_2$), we get

$$\sum_{n=N_4}^{\infty} q_n f\left(\left(\frac{n-\ell}{2^{m-1}}\right)^{(m-1)}\right) < \infty, \tag{16}$$

which contradicts (H$_3$). □

Theorem 2. *If (H$_1$) and (H$_2$)–(H$_4$) holds, then all the solutions of (1) are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1) with $y_n > 0$, $y_{n-k} > 0$, and $y_{n-\ell} > 0$ for all $n \geq N_1 \geq N_0$. For $n \geq N_1$, let

$$x_n = y_n + p_n y_{n-k} - F_n. \tag{17}$$

Then from (1) and (H_2),

$$\Delta^m x_n = -q_n f(y_{n-\ell}) \leq 0. \tag{18}$$

Hence, $x_n > 0$ or $x_n < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. But $x_n < 0$ implies that $0 < y_n < F_n$ for $n \geq N_2$ which is impossible since $\{F_n\}$ oscillates. Thus, $x_n > 0$ for $n \geq N_2$. From Lemma 1, it follows that there is an odd integer j with $0 \leq j \leq m$ such that

$$\Delta^i x_n > 0, \quad \text{for } i = 1, \dots, j-1$$

and

$$(-1)^{j+i}\Delta^i x_n > 0, \quad \text{for } i = j, j+1, \dots, m-1 \tag{19}$$

for $n \geq N_3 \geq N_2$.

Clearly, $\Delta x_n > 0$ for $n \geq N_3$. For $0 < \epsilon < (1-P_1)x_{N_3}$, (H_4) implies that there exists an integer $N_4 > N_3$ such that $|F_n| < \epsilon/2$ for $n \geq N_4$. From (17), we have $y_n \leq x_n + F_n$. So

$$x_n - p_n x_{n-k} \leq y_n - F_n + p_n F_{n-k} < y_n + \frac{\epsilon}{2} + \frac{\epsilon}{2} p_n. \tag{20}$$

Hence,

$$0 < (1-P_1)x_{N_3} - \epsilon < (1-P_1)x_n - \epsilon < y_n \tag{21}$$

for $n \geq N_4$. Setting $r_n = (1-P_1)x_n - \epsilon$ for $n \geq N_4$, we get $0 < r_n < y_n$, $\Delta r_n > 0$, and $\Delta^m r_n = -(1-P_1)q_n f(y_{n-\ell}) \leq 0$. Now, proceeding as in the proof of Theorem 1, we again obtain a contradiction. □

We can remove the "oscillatory" part in condition (H_4) and obtain the weaker conclusion that the solutions either oscillate or converge to zero.

Corollary 3. *If (H_1), (H_2), and (H_3) hold and* $\lim_{n\to\infty} F_n = 0$, *then all the solutions of (1) are either oscillatory or converge to zero.*

Proof. Proceeding as in the proof of Theorem 2, we again obtain that $x_n > 0$ or $x_n < 0$ for $n \geq N_2$. If $x_n < 0$, then $0 < y_n < F_n$. So, $\{y_n\} \longrightarrow 0$ as $n \longrightarrow \infty$. The remainder of the proof is the same as proof of Theorem 2. □

Our next result replaces condition (H_4) with a periodicity condition on forcing term.

Theorem 4. *If (H_1)–(H_3), and (H_5) hold, then every solution of (1) is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1) with $y_n > 0$, $y_{n-k} > 0$, and $y_{n-\ell} > 0$ for all $n \geq N_1 \geq N_0$. Defining x_n as in (17), we have that (18) holds and so either $x_n > 0$ or $x_n < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$.

We claim that $\{y_n\}$ is bounded. If not, then $\{y_n\}$ is unbounded and since $0 < y_n < x_n + F_n$ and $\{F_n\}$ is bounded, $\{x_n\}$ must be unbounded and eventually positive. Clearly, $\Delta x_n > 0$ for large n since $\Delta x_n < 0$ implies that $\{x_n\}$ is bounded. From (17), we have

$$x_n - p_n x_{n-k} = y_n - F_n - p_n p_{n-k} y_{n-2k} + p_n F_{n-k}, \tag{22}$$

for $n \geq N_3$ for some $N_3 \geq N_2$. That is,

$$(1-p_n)x_n \leq y_n - (1-p_n)F_n, \tag{23}$$

or

$$0 < (1-P_1)(x_n + F_n) \leq y_n. \tag{24}$$

Since $\{F_n\}$ is periodic, there exist real numbers c_1 and c_2 and two increasing sequences $\{n_i'\}$ and $\{n_i''\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} n_i' = \lim_{i\to\infty} n_i'' = \infty$, $F_{n_i'} = c_1$, $F_{n_i''} = c_2$, and $c_1 \leq F_n \leq c_2$ for all $n \geq N_0$. Hence, for $n \geq n_i'$, $i \geq 1$, we have

$$x_n + c_1 \geq x_{n_i'} + c_1 = x_{n_i'} + F_{n_i'} \geq y_{n_i'} > 0. \tag{25}$$

Thus,

$$0 < (1-P_1)(x_n + c_1) \leq (1-P_1)(x_n + F_n) \leq y_n \tag{26}$$

for $n \geq n_i'$. Setting $r_n = (1-P_1)(x_n + c_1)$ for $n \geq n_i'$, and $i \geq 1$, we obtain $0 < r_n \leq y_n$, $\Delta r_n > 0$, and

$$\Delta^m r_n = -(1-P_1) q_n f(y_{n-\ell}) \leq 0. \tag{27}$$

Now, applying Lemma 1 and proceeding as in the proof of Theorem 1, we arrive at a contradiction. Thus, our claim holds, that is, $\{y_n\}$ is bounded.

The boundedness of $\{y_n\}$ implies that $\{x_n\}$ is bounded. Since m is even, j is odd. So (19) implies that $\Delta x_n > 0$ for $n \geq N_2$. Again, proceeding as the proof of Theorem 1, we arrive at a contradiction. Hence, $\{y_n\}$ is oscillatory. □

Remark 2. With appropriate modifications in condition (C_1), (C_2), and (H_3), Theorems 1, 2, and 4 and Corollary 3 also hold for the more general equation

$$\Delta^m (y_n + p_n y_{n-k}) + \sum_{j=1}^{m} q_{j,n} f_j(y_{n-\ell_j}) = h_n. \tag{28}$$

Our final result, in this paper, is for the case $p_n \equiv 1$.

Theorem 5. *If $p_n \equiv 1$ and the conditions (H_2) and (H_5)–(H_7) hold, then all the solutions of (1) are oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1) with $y_n > 0$, $y_{n-k} > 0$, and $y_{n-\ell} > 0$ for all $n \geq N_1 \geq N_0$. Since $\{F_n\}$ is periodic, there is a real number ω such that the sequence $\{F_n - \omega\}$ is oscillatory. For $n \geq N_1$, let $w_n = y_n + y_{n-k} - (F_n - \omega)$. Then

$$\Delta^m w_n = -q_n f(y_{n-\ell}) \leq 0, \tag{29}$$

and so $\{w_n\}$ is monotonic. If $w_n < 0$ eventually, then $0 < y_n < F_n - \omega$ for large n which is impossible since $\{F_n - \omega\}$ oscillates. Thus, $w_n > 0$ for $n \geq N_2$ for some $N_2 \geq N_1$. By Lemma 1, we have $\Delta^{m-1} w_n > 0$ for $n \geq N_2$. Summing (29) from N_2 to $n-1$ and applying (H_7), we obtain

$$\Delta^{m-1} w_{N_2} = \sum_{s=N_2}^{n-1} q_s f(y_{s-\ell}) + \Delta^{m-1} w_n > \sum_{s=N_2}^{n-1} q_s f(y_{s-\ell}) > \gamma \sum_{s=N_2}^{n-1} q_s y_{s-\ell}, \tag{30}$$

which yields

$$\sum_{s=N_2}^{\infty} q_s y_{s-\ell} < \infty. \tag{31}$$

From Lemma 1, we see that j is odd, and, hence, $\Delta w_n > 0$ for $n \geq N_2$. This means that for $n \geq N_2$,

$$w_n - w_{n-k} = y_n - y_{n-2k} - (F_n - F_{n-k}), \tag{32}$$

which, in view of (H_5), yields

$$w_n - w_{n-k} = y_n - y_{n-2k} > 0, \tag{33}$$

or $y_n > y_{n-2k}$ for $n \geq N_2$. Therefore, $\liminf_{n\to\infty} y_n > 0$ and so $\sum_{s=N_2}^{\infty} q_s < \infty$, which contradicts ($H_6$). □

It should be pointed out that whether results analogous to Theorems 1, 2, 4, and 5 and Corollary 3 hold when m is odd remains an open question. We conclude this paper with some examples of the above theorems.

Example 1. Consider the difference equation

$$\Delta^m \left(y_n + \tfrac{1}{2} y_{n-k}\right) + 3(2)^{m-1} y_{n-\ell}^{\alpha} = 0, \tag{E_1}$$

where $\alpha \in (0,1)$ is a ratio of odd positive integers, k is any positive even integer, and ℓ is any nonnegative integer such that $\alpha\ell$ is an odd integer. It is easy to see that all the conditions of Theorem 1 are satisfied. In fact, $\{y_n\} = \{(-1)^n\}$ is an oscillatory solution of (E_1).

Example 2. In the equation

$$\Delta^m \left(y_n + \tfrac{1}{2} y_{n-k}\right) + \left(3(2)^{m-1} - \frac{3^m}{2^{n+m}}\right) y_{n-\ell}^{\alpha} = \frac{(-1)^n 3^m}{2^{n+m}}, \tag{E_2}$$

let $\alpha \in (0,1)$ be the ratio of odd positive integer, k an even positive integer, and ℓ any nonnegative integer such that $\alpha\ell$ is an odd integer. If we let $\{F_n\} = \{(-1)^n/2^n\}$, then all the conditions of Theorem 2 are satisfied and, in fact, $\{y_n\} = \{(-1)^n\}$ is oscillatory solution of (E_2).

Example 3. Consider the difference equation

$$\Delta^m\left(y_n+\tfrac{1}{4}y_{n-k}\right)+2^{m-2}y_{n-\ell}^{\alpha}=3(2)^{m-1}(-1)^n, \tag{E_3}$$

where $\alpha\in(0,1)$ is a ratio of odd positive integer, k is an even positive integer, and ℓ is any nonnegative integer such that $\alpha\ell$ is an even integer. Here, we take $\{F_n\}=\{3/2(-1)^n\}$. Then all the conditions of Theorem 4 are satisfied and $\{y_n\}=\{(-1)^n\}$ is an oscillatory solution of (E_3).

Example 4. The difference equation

$$\Delta^m\left(y_n+y_{n-k}\right)+2^{m+1}y_{n-\ell}=2^{m+2}(-1)^n=0, \tag{E_4}$$

where k and ℓ are positive even integers and $\{F_n\}=\{4(-1)^n\}$, satisfies all the conditions of Theorem 5. Here, $\{y_n\}=\{(-1)^n\}$ is an oscillatory solution of (E_4).

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