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GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITIES

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ABSTRACT. This paper deals with some new generalizations of Hardy's integral inequalities. Some cases concerning whether the constant factors involved in these inequalities are best possible are discussed in some detail.

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1. Introduction. Hardy et al. ([2, Ch. 9]) proved the following integral inequalities: if

$$p > 1, \qquad \frac{1}{p}+\frac{1}{q}=1, \qquad f(x) \ge 0, \qquad \text{and} \qquad 0 < \int_0^\infty f^p(x)\,dx < \infty, \tag{1.1}$$

then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx < q^p \int_0^\infty f^p(t)\,dt, \tag{1.2}$$

where the constant q^p in (1.2) is best possible.

The dual form of (1.2) is as follows: if

$$0 < \int_0^\infty (xf(x))^p\,dx < \infty, \tag{1.3}$$

then

$$\int_0^\infty \left(\int_x^\infty f(t)\,dt\right)^p dx < p^p \int_0^\infty (tf(t))^p\,dt, \tag{1.4}$$

where the constant p^p in (1.4) is still best possible.

Both inequalities (1.2) and (1.4) are known as Hardy's integral inequalities. They play an important role in mathematical analysis and its applications. Recently, Yang et al. [1] gave some new generalizations of (1.2) which can be stated as follows:

For any a and b, $(0 < a < b < \infty)$, the following inequalities hold:

$$\int_a^b \left(\frac{1}{x}\int_a^x f(t)\,dt\right)^p dx < q^p \left[1-\left(\frac{a}{b}\right)^{1/q}\right]^p \int_a^b f^p(t)\,dt; \tag{1.5}$$

$$\int_a^\infty \left(\frac{1}{x}\int_a^x f(t)\,dt\right)^p dx < q^p \int_a^\infty [1-\theta_p(t)] f^p(t)\,dt \,(0 < \theta_p(t) < 1); \tag{1.6}$$

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx < q^p \int_0^b \left[1-\left(\frac{t}{b}\right)^{1/q}\right] f^p(t)\,dt. \tag{1.7}$$

The main objective of this paper is to consider some cases whether the constant factors involved in (1.5), (1.6), and (1.7) are best possible. This is followed by some more new generalizations of (1.4), (1.5), (1.6), and (1.7).

2. Best possible constant factors. This section deals with calculations of the best possible constant factors involved in (1.5), (1.6), and (1.7).

Theorem 2.1. *If* $b > a > 0$, $p > 1$, $(1/p) + (1/q) = 1$, $f(x) \geq 0$, *and* $0 < \int_a^b f^p(x)dx < \infty$, *then*

$$\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx < q^p \eta_p(a,b) \int_a^b f^p(t)dt, \tag{2.1}$$

where the constant

$$\eta_p(a,b) = \max_{a \leq t \leq b} \left\{ \frac{1}{q} t^{1/q} \int_t^b x^{-1-1/q}\left[1-\left(\frac{a}{x}\right)^{1/q}\right]^{p-1} dx \right\} \tag{2.2}$$

and

$$\frac{1}{p}\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^p < \eta_p(a,b) < \left[1-\left(\frac{a}{b}\right)^{1/q}\right]^p. \tag{2.3}$$

Proof. In view of the proof given in [1, Thm. 2.1], we have

$$\begin{aligned}\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx &\leq q^{p-1}\int_a^b \left\{\int_t^b x^{-1-1/q}\left[1-\left(\frac{a}{x}\right)^{1/q}\right]^{p-1} dx\right\} t^{1/q} f^p(t)dt \\ &= q^p \int_a^b g_p(t) f^p(t)dt,\end{aligned} \tag{2.4}$$

where the weight function $g_p(t)$ is defined by

$$g_p(t) := \frac{1}{q} t^{1/q}\int_t^b x^{-1-1/q}\left[1-\left(\frac{a}{x}\right)^{1/q}\right]^{p-1} dx, \quad t \in [a,b]. \tag{2.5}$$

Setting $\eta_p(a,b) := \max_{a \leq t \leq b} g_p(t)$, since $g_p(t)$ is a nonconstant continuous function, then by (2.4), we have (2.1). Since $g_p(b) = 0$, and for any $t \in [a,b)$,

$$\begin{aligned} g_p(t) &< \frac{1}{q} t^{1/q}\int_t^b x^{-1-1/q}\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^{p-1} dx \\ &= -t^{1/q}\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^{p-1}\left[b^{-1/q}-t^{-1/q}\right] \\ &= \left[1-\left(\frac{a}{b}\right)^{1/q}\right]^{p-1}\left[1-\left(\frac{t}{b}\right)^{1/q}\right] \leq \left[1-\left(\frac{a}{b}\right)^{1/q}\right]^{p},\end{aligned} \tag{2.6}$$

then $\eta_p(a,b) < [1-(a/b)^{1/q}]^p$. Since

$$\begin{aligned} g_p(t) &= \left(\frac{t}{a}\right)^{1/q}\int_t^b \left[1-\left(\frac{a}{x}\right)^{1/q}\right]^{p-1} d\left[1-\left(\frac{a}{x}\right)^{1/q}\right] \\ &= \frac{1}{p}\left(\frac{t}{a}\right)^{1/q}\left\{\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^{p} - \left[1-\left(\frac{a}{t}\right)^{1/q}\right]^{p}\right\},\end{aligned} \tag{2.7}$$

and $g_p'(a) > 0$, then we have $\eta_p(a,b) > g_p(a) = 1/p[1-(a/b)^{1/q}]^p$. This completes the proof. □

Remark. This theorem implies that the constant factor $q^p[1-(a/b)^{1/q}]^p$ in (1.5) is not best possible, and the best value of $k_p(a,b)$ for which (1.5) exists is bounded. More precisely,

$$0 < k_p(a,b) \le q^p \eta_p(a,b) < q^p\left[1-\left(\frac{a}{b}\right)^{1/q}\right]^p. \tag{2.8}$$

Theorem 2.2. *For any $a,b>0$, the same constant factor q^p in (1.6) and (1.7) is best possible.*

Proof. If the constant q^p in (1.6) is not best possible, then there exists $K(0<K<q^p)$, such that

$$\int_a^\infty\left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx < K\int_a^\infty [1-\theta_p(t)]f^p(t)dt < K\int_a^\infty f^p(t)dt. \tag{2.9}$$

Since $\{q/[1-\epsilon(q-1)]\}^p(1-\epsilon pq/(1+\epsilon)) \to q^p(\epsilon\to 0^+)$, then there exists a small number $\epsilon(0<\epsilon<p-1)$, such that $\{q/[1-\epsilon(q-1)]\}^p[1-(\epsilon pq/(1+\epsilon))]>K$. Setting $f_\epsilon(t)=t^{-(1+\epsilon)/p}$, $t\in[a,\infty)$, we obtain

$$\int_a^\infty f_\epsilon^p(t)dt = \frac{1}{\epsilon}a^{-\epsilon}, \tag{2.10}$$

and by Bernoulli's inequality (see [4, Ch. 2.4]),

$$\begin{aligned}\int_a^\infty\left(\frac{1}{x}\int_a^x f_\epsilon(t)dt\right)^p dx &= \left(1-\frac{1+\epsilon}{p}\right)^{-p}\int_a^\infty x^{-(1+\epsilon)}\left[1-\left(\frac{a}{x}\right)^{1-(1+\epsilon)/p}\right]^p dx\\ &> \left(1-\frac{1+\epsilon}{p}\right)^{-p}\int_a^\infty x^{-(1+\epsilon)}\left[1-p\left(\frac{a}{x}\right)^{1-(1+\epsilon)/p}\right]dx\\ &= \left[\frac{q}{1-\epsilon(q-1)}\right]^p\left(1-\frac{\epsilon pq}{1+\epsilon}\right)\frac{1}{\epsilon}a^{-\epsilon} > K\int_a^\infty f_\epsilon^p(t)dt.\end{aligned} \tag{2.11}$$

This is a contradiction, and hence the constant factor q^p in (1.6) is best possible.

If the constant factor q^p in (1.7) is not best possible, then there exists $K(0<K<q^p)$, such that

$$\int_0^b\left(\frac{1}{x}\int_0^x f_\epsilon(t)dt\right)^p dx < K\int_0^b f^p(t)dt. \tag{2.12}$$

There exists a number $\epsilon(>0)$, such that $[q/(1+\epsilon q-\epsilon)]^p>K$. Setting $f_\epsilon(t)=t^{-(1-\epsilon)/p}$, $t\in(0,b]$, then we obtain $\int_0^b f_\epsilon^p(t)dt=(1/\epsilon)b^\epsilon$, and

$$\int_0^b\left(\frac{1}{x}\int_0^x f_\epsilon(t)dt\right)^p dx = \left(\frac{q}{1+\epsilon q-\epsilon}\right)^p\frac{1}{\epsilon}b^\epsilon > K\int_0^b f_\epsilon^p(t)dt. \tag{2.13}$$

This is a contradiction, and the constant factor q^p in (1.7) is best possible. The theorem is proved. □

3. Some new general inequalities.

We first prove two lemmas.

Lemma 3.1. *Let $b>0$, $p>1$, $(1/p)+(1/q)=1$, $f(x)\ge 0$, and $0<\int_0^b(xf(x))^p dx<\infty$. Then there exists a number $x_0\in(0,b)$, such that for any $x\in(0,x_0)$, the following* *inequality is true:*

$$\left(\int_x^b f(t)dt\right)^p < p^{(p-1)}\left(x^{-1/p}-b^{-1/p}\right)^{p-1}\int_x^b t^{p-1/p}f^p(t)dt. \tag{3.1}$$

Proof. For any $x \in (0,b)$, by Holder's inequality, we have

$$\begin{aligned}\left(\int_x^p f(t)dt\right)^p &= \left[\int_x^b t^{(p+1)/pq}f(t)t^{-(p+1)/pq}dt\right]^p \\ &\le \int_x^b t^{(p+1)/q}f^p(t)dt\left(\int_x^b t^{-(p+1)/p}dt\right)^{p/q} \\ &= p^{p-1}\left(x^{-1/p}-b^{-1/p}\right)^{p-1}\int_x^b t^{p-1/p}f^p(t)dt.\end{aligned} \tag{3.2}$$

We have to show that there exists $x_0 \in (0,b)$ such that for any x in $0<x<x_0$, the equality in (3.2) does not hold. Otherwise, there exists $x = x_n \in (0,b)$, $n = 1,2,3,\dots$ and the sequence $\{x_n\}$ decreases to zero such that (3.2) becomes an equality. Moreover, there exist c_n and d_n which are not always zero such that (see [3, p. 29])

$$c_n[t^{(p+1)/pq}f(t)]^p = d_n(t^{-(p+1)/pq})^q, \quad \text{a.e. in } [x_n,b]. \tag{3.3}$$

But $f(t) \ne 0$, a.e. in $[0,b]$, then there exists an integer N, such that for any $n > N$, $f(t) \ne 0$, a.e. in $[x_n,b]$. Hence both $c_n = c \ne 0$, and $d_n = d \ne 0$, for any $n > N$, and then

$$\int_0^b (tf(t))^p dt = \lim_{n\to\infty}\int_{x_n}^b (tf(t))^p dt = \frac{d}{c}\lim_{n\to\infty}\int_{x_n}^b t^{-1}dt = \infty. \tag{3.4}$$

This contradicts the fact that $\int_0^b (xf(x))^p dx < \infty$. Thus, (3.1) is valid. The lemma is proved. □

Lemma 3.2. *Let $a>0$, $p>1$, $(1/p)+(1/q)=1$, $f(x) \ge 0$, and $0 < \int_a^\infty (xf(x))^p dx < \infty$. Then there exists $x_0 \in (a,\infty)$, such that for any $x \in (a,x_0)$,*

$$\left(\int_x^\infty f(t)dt\right)^p < p^{p-1}x^{-1/q}\int_x^\infty t^{p-1/p}f^p(t)dt. \tag{3.5}$$

Proof. We have, by Holder's inequality as in Lemma 3.1, and for any $x \in (a,\infty)$,

$$\left(\int_x^\infty f(t)dt\right)^p \le p^{p-1}x^{-1/q}\int_x^\infty t^{p-1/p}f^p(t)dt. \tag{3.6}$$

We show that there exists $x_0 \in (a,\infty)$, such that (3.6) does not assume equality for any $x \in (a,x_0)$. Otherwise, there exists $x = x_n \in (a,\infty)(n = 1,2,\dots)$, $x_n \downarrow a$, such that (3.6) becomes an equality. By the same argument as in Lemma 3.1, there exists $c>0$ and N, such that for any $n>N$, $[t^{(p+1)/pq}f(t)]^p = c(t^{-(p+1)/pq)})^q$, a.e. in $[x_n,\infty)$, and hence $\int_a^\infty (tf(t))^p dt = c\lim_{n\to\infty}\int_{x_n}^\infty (1/t)\,dt = \infty$. This is a contradiction. Inequality (3.5) is true. The lemma is proved. □

Theorem 3.1. *Let $b>a>0$, $p>1$, $(1/p)+(1/q)=1$, $f(x)\ge 0$, and $0<\int_a^b (xf(x))^p dx<\infty$.*
Then

$$\int_a^b\left(\int_x^b f(t)dt\right)^p dx < p^p\mu_p(a,b)\int_a^b (tf(t))^p dt, \tag{3.7}$$

where

$$\mu_p(a,b) = \max_{a\le t\le b}\left\{\frac{1}{p}t^{-1/p}\int_a^t \left(x^{-1/p}-b^{-1/p}\right)^{p-1}dx\right\} \tag{3.8}$$

and

$$\frac{1}{p}\left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p \le \mu_p(a,b) < \left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p. \tag{3.9}$$

Proof. Using inequality (3.2), we obtain

$$\begin{aligned}\int_a^b\left(\int_x^b f(t)dt\right)^p dx &\le p^{p-1}\int_a^b \left(x^{-1/p}-b^{-1/p}\right)^{p-1}\int_x^b t^{p-1/p}f^p(t)dt\,dx\\ &= p^{p-1}\int_a^b\left\{t^{-1/p}\int_a^t\left(x^{-1/p}-b^{-1/p}\right)^{p-1}dx\right\}(tf(t))^p dt\\ &= p^p\int_a^b h_p(t)(tf(t))^p dt,\end{aligned} \tag{3.10}$$

where the weight function $h_p(t)$ is defined by

$$h_p(t) := \frac{1}{p}t^{-1/p}\int_a^t\left(x^{-1/p}-b^{-1/p}\right)^{p-1}dx, \quad t\in[a,b]. \tag{3.11}$$

Setting $\mu_p(a,b) := \max_{a\le t\le b} h_p(t)$, by (3.10), we have (3.7). Since $h_p(a) = 0$, and for any $t\in(a,b]$,

$$\begin{aligned}h_p(t) &= \frac{1}{p}t^{-1/p}\int_a^t x^{-1+1/p}\left[1-\left(\frac{x}{b}\right)^{1/p}\right]^{p-1}dx\\ &< \frac{1}{p}t^{-1/p}\int_a^t x^{-1+1/p}\left[1-\left(\frac{a}{b}\right)^{1/p}\right]^{p-1}dx\\ &= \left[1-\left(\frac{a}{b}\right)^{1/p}\right]^{p-1}\left[1-\left(\frac{a}{t}\right)^{1/p}\right] \le \left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p,\end{aligned} \tag{3.12}$$

then we have $\mu_p(a,b) < [1-(a/b)^{1/p}]^p$. Since

$$\begin{aligned}h_p(t) &= -\left(\frac{t}{b}\right)^{1/p}\int_a^t\left[1-\left(\frac{x}{b}\right)^{1/p}\right]^{p-1}d\left[1-\left(\frac{x}{b}\right)^{1/p}\right]\\ &= \frac{1}{p}\left(\frac{t}{b}\right)^{1/p}\left\{\left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p-\left[1-\left(\frac{t}{b}\right)^{1/p}\right]^p\right\},\end{aligned} \tag{3.13}$$

then we have

$$\mu_p(a,b) = \max_{a\le t\le b}h_p(t) \ge h_p(b) = \frac{1}{p}\left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p. \tag{3.14}$$

This completes the proof. □

Remark. It follows from this theorem that the best value $\lambda_p(a,b)$ for which inequality (3.7) exists is bounded, that is,

$$0<\lambda_p(a,b)\le p^p\mu_p(a,b) < p^p\left[1-\left(\frac{a}{b}\right)^{1/p}\right]^p. \tag{3.15}$$

Theorem 3.2. *Let $a>0$, $p>1$, $(1/p)+(1/q)=1$, $f(x)\geq 0$, and $0<\int_a^\infty (xf(x))^p dx<\infty$. Then*

$$\int_a^\infty \left(\int_x^\infty f(t)dt\right)^p dx < p^p \int_a^\infty \left[1-\left(\frac{a}{t}\right)^{1/p}\right](tf(t))^p dt, \tag{3.16}$$

where the constant factor p^p in inequality (3.16) is best possible.

Proof. By inequality (3.5), we have

$$\begin{aligned}\int_a^\infty \left(\int_x^\infty f(t)dt\right)^p dx &< p^{p-1}\int_a^\infty x^{-1/q}\int_x^\infty t^{p-1/p}f^p(t)dt\,dx \\ &= p^{p-1}\int_a^\infty \left(\int_a^t x^{1/q}dx\right)t^{p-1/p}f^p(t)dt \\ &= p^p\int_a^\infty \left[1-\left(\frac{a}{t}\right)^{1/p}\right](tf(t))^p dt.\end{aligned} \tag{3.17}$$

Inequality (3.16) is true. If p^p in (3.16) is not possible, then there exists $K(0<K<p^p)$, such that

$$\int_a^\infty \left(\int_x^\infty f(t)dt\right)^p dx < K\int_a^\infty \left[1-\left(\frac{a}{t}\right)^{1/p}\right](tf(t))^p dt < K\int_a^\infty (tf(t))^p dt. \tag{3.18}$$

There exists a small number $\epsilon>0$, such that $p^p[1/(1+\epsilon)^p]>K$. Setting $f_\epsilon(t)=t^{-1-(1+\epsilon)/p}$, $t\in[a,\infty)$, then we have $\int_a^\infty (tf_\epsilon(t))^p dt=(1/\epsilon)a^{-\epsilon}$, and

$$\int_a^\infty \left(\int_x^\infty f_\epsilon(t)dt\right)^p dx = p^p\frac{1}{(1+\epsilon)^p}\cdot\frac{1}{\epsilon}a^{-\epsilon} > K\int_a^\infty (tf_\epsilon(t))^p dt. \tag{3.19}$$

This is a contradiction, and the constant factor p^p in (3.16) is best possible. This proves the theorem. □

Theorem 3.3. *Let $b>0$, $p>1$, $(1/p)+(1/q)=1$, $f(x)\geq 0$, and $0<\int_0^b (xf(x))^p dx<\infty$. Then*

$$\int_0^b \left(\int_x^b f(t)dt\right)^p dx < p^p\int_0^b \mu_p(t)(tf(t))^p dt, \tag{3.20}$$

where the weight function $\mu_p(t)=(1/p)\{1-[1-(t/b)^{1/p}]^p\}(b/t)^{1/p}$, $t\in(0,b]$, and $0<\mu_p(t)<1$; the constant p^p in inequality (3.20) is best possible. When $p=2$, inequality reduces to the form

$$\int_0^b \left(\int_x^b f(t)dt\right)^2 dx < 4\int_0^b \left(1-\frac{1}{2}\sqrt{\frac{t}{b}}\right)(tf(t))^2 dt. \tag{3.21}$$

Proof. In view of (3.1), we find

$$\begin{aligned}\int_0^b \left(\int_x^b f(t)dt\right)^p dx &< p^{p-1}\int_0^b (x^{-1/p}-b^{-1/p})^{p-1}\int_x^b t^{p-1/p}f^p(t)dt\,dx \\ &= p^{p-1}\int_0^b \left[\int_0^t (x^{-1/p}-b^{-1/p})^{p-1}dx\right]t^{p-1/p}f^p(t)dt \\ &= p^{p-1}\int_0^b \left\{\int_0^t x^{-1+1/p}\left[1-\left(\frac{x}{b}\right)^{1/p}\right]^{p-1}dx\right\}\left(\frac{1}{t}\right)^{1/p}(tf(t))^p dt \\ &= p^p\int_0^b \mu_p(t)(tf(t))^p dt,\end{aligned} \tag{3.22}$$

where

$$\mu_p(t) := \frac{1}{p}\left\{1-\left[1-\left(\frac{t}{b}\right)^{1/p}\right]^p\right\}\left(\frac{b}{t}\right)^{1/p}, \quad t \in (0,b]. \tag{3.23}$$

By Bernoulli's inequality, we have

$$0 < \mu_p(t) < \frac{1}{p}\left\{1-\left[1-p\left(\frac{t}{b}\right)^{1/p}\right]\right\}\left(\frac{b}{t}\right)^{1/p} = 1. \tag{3.24}$$

Inequality (3.20) is true. Since $\mu_2(t) = 1-(1/2)\sqrt{t/b}$, inequality (3.21) is also true.

If the constant factor p^p in (3.20) is not best possible, then there exists $K(0 < K < p^p)$, such that

$$\int_0^b \left(\int_x^b f(t)dt\right)^p dx < K\int_0^b \mu_p(t)(tf(t))^p dt < K\int_0^b (tf(t))^p dt. \tag{3.25}$$

There exists a small number $\epsilon, (0 < \epsilon < 1)$, such that $p^p(1/(1-\epsilon)^p)\{1-\epsilon p/[\epsilon+(1-\epsilon)/p]\} > K$. Setting $f_\epsilon(t) = t^{-1-(1-\epsilon)/p}$, $t \in (0,b]$, we obtain $\int_0^b (tf(t))^p dt = (1/\epsilon)b^\epsilon$, and by Bernoulli's inequality,

$$\begin{aligned}
\int_0^b \left(\int_x^b f_\epsilon(t)dt\right)^p dx &= p^p\frac{1}{(1-\epsilon)^p}\int_0^b x^{-1+\epsilon}\left[1-\left(\frac{x}{b}\right)^{(1-\epsilon)/p}\right]^p dx \\
&> p^p\frac{1}{(1-\epsilon)^p}\int_0^b x^{-1+\epsilon}\left[1-p\left(\frac{x}{b}\right)^{(1-\epsilon)/p}\right]dx \\
&= p^p\frac{1}{(1-\epsilon)^p}\left[1-\frac{\epsilon p}{\epsilon+\frac{(1-\epsilon)}{p}}\right]\frac{1}{\epsilon}b^\epsilon \\
&> K\int_0^b (tf_\epsilon(t))^p dt.
\end{aligned} \tag{3.26}$$

This is a contradiction, and the constant factor p^p in (3.20) is best possible. This completes the proof. □

Remark. (1) In the limits $a \to 0$, $b \to \infty$, inequalities (3.7), (3.10), (3.16), and (3.20) reduce to (1.4).

(2) Inequalities (3.7), (3.10), (3.16), and (3.20) are new generalizations of (1.4).

(3) Inequalities (1.4), (3.7), (3.10), (3.16), and (3.20) represent a class, whereas (1.2), (1.6), (1.7), and (2.1) form another class. The constant factors involved in these two classes of inequalities are shown to be best possible except (2.1) and (3.7).

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