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STABILITY FOR RANDOMLY WEIGHTED SUMS OF RANDOM ELEMENTS

TIEN-CHUNG HU and HEN-CHAO CHANG

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ABSTRACT. Let $\{X_n : n = 1,2,3,\ldots\}$ be a sequence of i.i.d. random elements taking values in a separable Banach space of type p and let $\{A_{n,i} : i = 1,2,3,\ldots; n = 1,2,3,\ldots\}$ be an array of random variables. In this paper, under various assumptions of $\{A_{n,i}\}$, the necessary and sufficient conditions for $\sum_{i=1}^{\infty} A_{n,i} X_i \to 0$ a.s. are obtained. Also, the necessity of the assumptions of $\{A_{n,i}\}$ is discussed.

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1. Introduction. Let $\{X_n : n = 1,2,3,\ldots\}$ be a sequence of independent identically distributed (i.i.d.) random variables and let $\{a_{n,i} : i = 1,2,\ldots,n; n = 1,2,3,\ldots\}$ be a triangular array of constants. Many papers were devoted to extending various types of convergence modes to weighted sums $W_n = \sum_{i=1}^{n} a_{n,i} X_i$ in the literature. However, we are only interested in the work of almost sure convergence. The sequence $\{\sum_{i=1}^{n} a_{n,i}\}$ converging to 0 at a certain rate as $n \to \infty$ is a traditional assumption. For example, under the assumption $\sum_{i=1}^{n} a_{n,i}^2 = O(n^{-2/r})$, $W_n \to 0$ a.s. if $E|X_1|^r < \infty$ and $EX_1 = 0$ (See Chow and Lia [3] and Choi and Sung [2]). On the other hand, Padgett and Taylor [4] extended the usual convergence theorems to weighted sums of random elements in a separable Banach space. It would be interesting to extend the results with random weights.

Let $\{A_{n,i} : i = 1,2,\ldots,n; n = 1,2,3,\ldots\}$ be a triangular array of random variables such that $\sum_{i=1}^{n} A_{n,i}^2 = O(n^{-2/r})$ a.s., Ahmad [1] obtained $W_n = \sum_{i=1}^{n} A_{n,i} X_i \to 0$ a.s. if $E\|X_1\|^r < \infty$ and $EX_1 = 0$. We note that, for the Marcinkiewicz-Zygmund law of large numbers, we take the uniform weight $a_{n,i} = n^{-1/r}$ but the condition $\sum_{i=1}^{n} a_{n,i}^2 = O(n^{-2/r})$ cannot be satisfied. The purpose of this paper is to extend the randomly weighted sums of a triangular array of random variables to that of an infinite array of random elements such that the Marcinkiewicz-Zygmund law of large numbers can be obtained as a corollary.

In Section 2, we establish the Marcinkiewicz-Zygmund law of large numbers in a separable Banach space of Type p. In Section 3, we consider an infinite array of random variables $\{A_{n,i} : n, i = 1,2,3,\ldots\}$ as the weight under various assumptions of $\{A_{n,i}\}$, we obtain that $W_n = \sum_{i=1}^{\infty} A_{n,i} X_i \to 0$ a.s. if and only if $EX_1 = 0$ (when it exists) and $E\|X_1\|^r < \infty$.

2. The Marcinkiewicz law in a space of type p. Let (Ω, F, P) be a probability space and $\mathbf{B}$ be a real separable Banach space with norm $\|\bullet\|$. A random element is defined to be an F-measurable mapping of Ω into $\mathbf{B}$ with the Borel σ-field. The concept of independent random elements is a direct extension of the concept of independent random variables. A detailed account of basic properties of random elements in real Banach spaces can be found in Taylor [6].

In this section, we prove the Marcinkiewicz-Zygmund law of large numbers in a space of type p. First, we introduce a space of type p.

Definition 1. Let $1 \le p \le 2$ and $\{r_i : i = 1,2,3,\dots\}$ be a sequence of independent random variables with $\Pr(r_i = \pm 1) = 1/2$. A separable Banach space $\mathbf{B}$ is said to be of type p if there exists a constant C such that

$$E\left\|\sum_{i=1}^{n} r_i x_i\right\| \le C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p} \tag{2.1}$$

for every $n \in N$ and all $x_1,\dots,x_n \in \mathbf{B}$.

Woyczyński [7] proved the equivalent condition of a space of type p.

Lemma 1 (Woyczyński [7]). *Let $1 \le p \le 2$ and $q \ge 1$. The following properties of $\mathbf{B}$ are equivalent :*

(i) *The separable Banach space $\mathbf{B}$ is of type p.*

(ii) *There exists C such that, for every $n \in N$ and for any sequence $\{X_i : i = 1,2,\dots,n\}$ of independent random elements in $\mathbf{B}$ with $EX_i = 0$, $i = 1,2,\dots,n$,*

$$E\left(\left\|\sum_{i=1}^{n} X_i\right\|^q\right) \le CE\left(\left(\sum_{i=1}^{n}\|X_i\|^p\right)^{q/p}\right). \tag{2.2}$$

Using Lemma 1, some elementary properties of spaces of type p can be easily proved. Every separable Hilbert space and finite-dimensional Banach space are of type 2. Every separable Banach space is at least of type 1, and the ℓ^p and L^p are of type $\min\{2,p\}$ for $p \ge 1$. If $\mathbf{B}$ is a space of type p and $1 \le q \le p$, then $\mathbf{B}$ is a space of type q. Before considering the Marcinkiewicz-Zygmund law of large numbers in a space of type p, we need the following definition and lemmas.

Definition 2. Let $\mathbf{B}$ be a separable Banach space, $\mathbf{B}^*$ the dual space of $\mathbf{B}$, and $\mathbf{B}'$ the unit ball in $\mathbf{B}^*$. X is a random element in $\mathbf{B}$. The directionally maximum median of X is defined by

$$\rho(X) \equiv \sup_{f \in \mathbf{B}'} |\mu(f(X))|, \tag{2.3}$$

where $\mu(Y)$ denotes the minimum median in absolute value of the random variable Y.

Lemma 2 (Sakhanenko [5]). *Let $X_1,\dots,X_n$ be independent random elements in $\mathbf{B}$ and $S_k = \sum_{i=1}^{k} X_i$, then, for every $t > 0$,*

$$\Pr\left(\max_{1\le k\le n}\|S_k\| > t\right) \le 2\Pr\left(\|S_n\| > t - \max_{1\le k\le n}\rho(S_n - S_k)\right). \tag{2.4}$$

Lemma 3. *Let $\{X_n : n = 1,2,3,\dots\}$ be a sequence of independent random elements in a separable Banach space. If $S_n = \sum_{i=1}^{n} X_i$ converges to a random element S in probability, then S_n converges to S a.s.*

Proof. Since S_n converges to S in probability, take ϵ such that $0 < \epsilon < 1/2$, then there exists an integer n_0 such that if $m > n \geq n_0$,

$$\Pr\left(||S_m - S|| > \frac{\epsilon}{2}\right) < \frac{\epsilon}{2} \qquad \text{and} \qquad \Pr\left(||S_n - S|| > \frac{\epsilon}{2}\right) < \frac{\epsilon}{2}. \tag{2.5}$$

So,

$$\Pr\left(||S_m - S_n|| > \epsilon\right) < \Pr\left(||S_m - S|| > \frac{\epsilon}{2}\right) + \Pr\left(||S_n - S|| > \frac{\epsilon}{2}\right) < \epsilon < \frac{1}{2}. \tag{2.6}$$

We have $\mu(\|S_m - S_n\|) < \epsilon$ for any $m > n \leq n_0$, where $\mu(Y)$ is minimum median in absolute value of the random variable Y.

By Lemma 2, if $m_1 > n > n_0$,

$$\begin{aligned}\Pr\left(\max_{n<m<m_1} ||S_m - S_n|| > 2\epsilon\right) &\leq 2\Pr\left(||S_{m_1} - S_n|| > 2\epsilon - \max_{n<m<m_1} \rho(S_{m_1} - S_m)\right) \\ &\leq 2\Pr\left(||S_{m_1} - S_n|| > 2\epsilon - \max_{n<m<m_1} \mu\left(||S_{m_1} - S_m||\right)\right) \\ &\leq 2\Pr\left(||S_{m_1} - S_n|| > \epsilon\right) < 2\epsilon.\end{aligned} \tag{2.7}$$

Let $m_1 \to \infty$, then if $m > n > n_0$, we have $\Pr(\max_{n<m} \|S_m - S_n\| > 2\epsilon) < 2\epsilon$.

We obtain S_n converges to some random element a.s., and S_n converges to S in probability. Hence, S_n converges to S a.s. □

Now, we prove the Marcinkiewicz-Zygmund law of large numbers in a space of type p.

Theorem 1. *Let* **B** *be a separable Banach space of type P and $\{X_n : n = 1,2,3,\dots\}$ be a sequence of independent and identically distributed random elements in* **B**. *Then, for any $0 < r < p$,*

$$\frac{S_n - nc}{n^{1/r}} \longrightarrow 0 \quad a.s. \tag{2.8}$$

for some constant c if and only if $E\|X_1\|^r < \infty$.

Moreover, if $r \geq 1$, $c = EX_1$ and $0 < r < 1$, c is arbitrary.

Proof.

Necessary part. Since (2.8) holds,

$$\frac{X_n}{n^{1/r}} = \frac{S_n - nc}{n^{1/r}} - \left(\frac{n-1}{n}\right)^{1/r} \frac{S_{n-1} - nc}{(n-1)^{1/r}} \longrightarrow 0 \quad \text{a.s.,} \tag{2.9}$$

whence, by the Borel-Cantelli lemma, $\sum_{n=1}^{\infty} \Pr(\|X_1\| > n^{1/r}) < \infty$. Thus, $E\|X_1\|^r < \infty$.

Sufficient part. Since $E\|X_1\|^r < \infty$, define $Y_n = n^{-1/r} X_n I(\|X_n\| \leq n^{1/r})$ and $A_j = \{(j-1)^{1/r} < \|X_1\| \leq j^{1/r}\}$. Choose a positive number α such that $r < \alpha \leq p$ and $\alpha \geq 1$. We have

$$\sum_{n=1}^{\infty} E\|Y_n\|^{\alpha} = \sum_{n=1}^{\infty}\sum_{j=1}^{n} n^{-\alpha/r} \int_{A_j} \|X_1\|^{\alpha} dp = \sum_{j=1}^{\infty}\sum_{n=j}^{\infty} n^{-\alpha/r} \int_{A_j} \|X_1\|^{\alpha} dp$$

$$\begin{aligned}&\le c_1\sum_{j=1}^{\infty} j^{1-\alpha/r}\int_{A_j}\|X_1\|^{\alpha}dp \le c_1\sum_{j=1}^{\infty}\int_{A_j}\|X_1\|^{r}dp\\&= c_1E\|X_1\|^{r}<\infty,\quad\text{for some constant } c_1.\end{aligned}\tag{2.10}$$

From Lemma 1,

$$\begin{aligned}E\left\|\sum_{i=n}^{m}(Y_i-EY_i)\right\|^{\alpha} &\le c_2E\left(\sum_{i=n}^{m}\|Y_i-EY_i\|^{p}\right)^{\alpha/p}\\&\le c_2\sum_{i=n}^{m}E\|Y_i-EY_i\|^{\alpha}\le 2^{\alpha}c_2\sum_{i=n}^{m}E\|Y_i\|^{\alpha}.\end{aligned}\tag{2.11}$$

We have $\sum_{i=1}^{n}(Y_i-EY_i)$ converges to some random element Y_0 in L^{α}. Therefore, $\sum_{i=1}^{n}(Y_i-EY_i)\to Y_0$ in probability. By Lemma 3, $\sum_{i=1}^{n}(Y_i-EY_i)\to Y_0$ a.s. Since

$$\sum_{n=1}^{\infty}\Pr\left(\frac{X_n}{n^{1/r}}\neq Y_n\right)=\sum_{n=1}^{\infty}\Pr\left(\|X_1\|>n^{1/r}\right)\le E\|X_1\|^{r}<\infty,\tag{2.12}$$

so,

$$\sum_{n=1}^{\infty}\frac{X_n-E(X_nI(\|X_n\|\le n^{1/r}))}{n^{1/r}}=\sum_{n=1}^{\infty}\left(\frac{X_n}{n^{1/r}}-EY_n\right)\quad\text{converges a.s.}\tag{2.13}$$

If $0<r<1$, we can choose $\alpha=1$. Then $\sum_{n=1}^{\infty}E\|Y_n\|<\infty$. We have $\sum_{n=1}^{\infty}(X_n)/(n^{1/r})$ converges a.s. By Kronecker lemma, $(S_n-nc)/(n^{1/r})$ converges a.s. for any constant c.

If $r=1$, by Kronecker lemma, $(S_n/n)-(1/n)\sum_{i=1}^{n}E(X_iI(\|X_i\|\le n))$ converges a.s. and $E(X_nI(\|X_n\|\le n))\to EX_1$, we have (2.8) holds.

If $r>1$, we can show that

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{-1/r}E\|X_nI(\|X_n\|>n^{1/r})\|\\&\quad\le\sum_{n=1}^{\infty}n^{-1/r}E(\|X_1\|I(\|X_1\|>n^{1/r}))=\sum_{n=1}^{\infty}n^{-1/r}\sum_{j=n+1}^{\infty}\int_{A_j}\|X_1\|dp\\&\quad=\sum_{j=2}^{\infty}\sum_{n=1}^{j-1}n^{-1/r}\int_{A_j}\|X_1\|dp\le\frac{r}{r-1}\sum_{j=1}^{\infty}(j-1)^{(r-1)/r}\int_{A_j}\|X_1\|dp\\&\quad\le\frac{r}{r-1}\sum_{j=1}^{\infty}\int_{A_j}\|X_1\|^{r}dp=\frac{r}{r-1}E\|X_1\|^{r}<\infty.\end{aligned}\tag{2.14}$$

Therefore, $\sum_{n=1}^{\infty}(X_n-EX_1)/n^{1/r}$ converges a.s. We have (2.8) holds by Kronecker lemma. □

3. The convergence of the weighted sums. Throughout this section, we deal with the almost sure convergence of randomly weighted sums $\sum_{i=1}^{\infty}A_{n,i}X_i$, where $\{X_n : n=1,2,3,\dots\}$ is a sequence of independent and identically distributed random elements in a space of type p and $\{A_{n,i} : n,i=1,2,3,\dots\}$ is an array of random variables satisfying some conditions.

Theorem 2. *Let* $\mathbf{B}$ *be a separable Banach space of type* p*. Let* $\{X_n : n = 1,2,3,\dots\}$ *be a sequence of independent and identically distributed random elements in* $\mathbf{B}$ *such that* $E\|X_1\|^r < \infty$ *and* $r < p$*. Moreover, we assume that* $EX_1 = 0$ *when* $r \geq 1$*. Let* $\{A_{n,i} : n, i = 1,2,3,\dots\}$ *be an array of random variables such that* $\{A_{n,i}\}$ *and* $\{X_i\}$ *are independent and satisfying*

$$A_{n,i} = O(i^{-1/r}) \quad \text{a.s. for every } n, \tag{3.1}$$

$$\lim_{n\to\infty} A_{n,i} = 0 \qquad \text{a.s. for every } i, \tag{3.2}$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{for every } n, \tag{3.3}$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i} - A_{n,i+1}| < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \tag{3.4}$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i}X_i = 0 \quad \text{a.s.} \tag{3.5}$$

Conversely, if

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i}X_i = 0 \quad \text{a.s.} \tag{3.6}$$

for all arrays $\{A_{n,i}\}$ *satisfying the above conditions, then* $E\|X_1\|^r < \infty$*.*

Proof. Since $\{A_{n,i}\}$ and $\{X_i\}$ are independent, if $r > 1$, we choose $p = q = r$ in Lemma 1, then

$$E\left(\sum_{i=1}^{\infty} \|A_{n,i}X_i\|\right)^r \leq \sum_{i=1}^{\infty} E|A_{n,i}|^r E\|X_1\|^r < \infty. \tag{3.7}$$

If $r \leq 1$, it is obvious that

$$E\left(\sum_{i=1}^{\infty} \|A_{n,i}X_i\|\right)^r \leq \sum_{i=1}^{\infty} E|A_{n,i}|^r E\|X_1\|^r < \infty. \tag{3.8}$$

Therefore, $\sum_{i=1}^{\infty} A_{n,i}X_i$ converges a.s.

Since $A_{n,i} = A_{n,i}^+ - A_{n,i}^-$, without loss of generality, we can assume that $A_{n,i} \geq 0$. Let $S_k = \sum_{i=1}^{k} X_i$, $Y_k = (S_k/k^{1/r})$ for every $k \geq 1$ and $S_0 = 0$. By Theorem 1, we have $\lim_{k\to\infty} Y_k = 0$ a.s.

$$\sum_{i=1}^{\infty} A_{n,i}X_i = \sum_{i=1}^{\infty} A_{n,i}(S_i - S_{i-1}) = \lim_{N\to\infty}\left(\sum_{i-1}^{N-1} (A_{n,i} - A_{n,i+1})S_i + A_{n,N}S_N\right). \tag{3.9}$$

Since $\{i^{1/r}A_{n,i}\}$ is bounded a.s. for every n and r,

$$\lim_{N\to\infty} A_{n,N}S_N = \lim_{N\to\infty} (N^{1/r}A_{n,N})Y_N = 0 \quad \text{a.s.} \tag{3.10}$$

We have

$$\sum_{i=1}^{\infty} A_{n,i}X_i = \sum_{i=1}^{\infty} i^{1/r}(A_{n,i} - A_{n,i+1})Y_i \quad \text{a.s.} \tag{3.11}$$

Let $B_{n,i} = i^{1/r}(A_{n,i} - A_{n,i+1})$. Hence, $\sum_{i=1}^{\infty} |B_{n,i}| \le M$ a.s. for every n and $\lim_{n\to\infty} B_{n,i} = 0$ a.s. for every i. Define $D^0 = \{w : \lim_{i\to\infty} Y_i(w) = 0\}$ and $D_n^1 = \{w : \sum_{i=1}^{\infty} |B_{n,i}(w)| \le M\}$ for each n and $D_i^2 = \{w : \lim_{n\to\infty} B_{n,i}(w) = 0\}$ for each i. For every $w \in D^0 \cap \bigcap_{i=1}^{\infty}(D_i^1 \cap D_i^2)$ and every $\epsilon > 0$, we can choose A such that $\|Y_i(w)\| < \epsilon$ for $i \ge A$,

$$\begin{aligned}\sum_{i=1}^{\infty} \|B_{n,i}(w) Y_i(w)\| &\le \sum_{i=1}^{A-1} |B_{n,i}(w)| \|Y_i(w)\| + \sum_{i=A}^{\infty} |B_{n,i}(w)| \|Y_i(w)\| \\ &\le \max_{i \le A-1} \|Y_i(w)\| \sum_{i=1}^{A-1} |B_{n,i}(w)| + M\epsilon \longrightarrow M\epsilon \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.12}$$

Since $\Pr(D^0 \cap \bigcap_{i=1}^{\infty}(D_i^1 \cap D_i^2)) = 1$, $\lim_{n\to\infty} \sum_{i=1}^{\infty} \|B_{n,i} Y_i\| = 0$ a.s. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i} X_i = \lim_{n\to\infty} \sum_{i=1}^{\infty} B_{n,i} Y_i = 0 \quad \text{a.s.} \tag{3.13}$$

If $\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0$ a.s. for all arrays $\{A_{n,i}\}$ satisfying the above conditions, we can choose

$$A_{n,i} = \begin{cases} n^{-1/r} & \text{if } i \le n, \\ 0 & \text{if } i > n. \end{cases} \tag{3.14}$$

Then (2.8) holds. By Theorem 1, we have $E\|X_1\|^r < \infty$. □

Remark 1. The following example claims that condition (3.4) cannot be omitted. Consider the real number space R as a space of type 2. Choose a sequence $\{X_n : n = 1, 2, 3, \dots\}$ of independent and identically distributed random variables with $EX_1^2 < \infty$ and $EX_1 = 0$. Define

$$A_{n,i} = \begin{cases} n^{-1/2} & \text{if } i \le n, \\ 0 & \text{if } i > n. \end{cases} \tag{3.15}$$

Choose any $r < 2$ so that condition (3.4) does not hold. By the Central Limit Theorem, $\sum_{i=1}^{\infty} A_{n,i} X_i$ cannot converge to 0 a.s.

Choi and Sung [2] considered the almost sure convergence of $\sum_{i=1}^{\infty} a_{n,i} X_i$ for triangular array of constants. Their Theorem 3 can be regarded as a corollary of Theorem 2.

Corollary 1 (Choi and Sung [2, Theorem 3]). *Let $\{X_n : n = 1, 2, 3, \dots\}$ be independent and identically distributed random variables with $EX_1 = 0$ and $E|X_1|^r < \infty$ for some $1 \le r < 2$. Let $\{a_{n,i} : i = 1, 2, \dots, n; n = 1, 2, 3, \dots\}$ be a triangular array of constants satisfying $\sum_{i=1}^{n} |a_{n,i} - a_{n,i+1}| = O(n^{-1/r})$, where $a_{n,n+1} = 0$. Then*

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_{n,i} X_i = 0 \quad \textit{a.s.} \tag{3.16}$$

Proof. By Theorem 2, we must show that there is a constant $M > 0$ such that

$$\lim_{n\to\infty} a_{n,i} = 0 \quad \text{for every } i \tag{3.17}$$

and

$$\sum_{i=1}^{\infty} i^{1/r}|a_{n,i}-a_{n,i+1}| < M \quad \text{for every } n. \tag{3.18}$$

There is a constant $C > 0$ such that $\sum_{i=1}^{n} |a_{n,i}-a_{n,i+1}| \le Cn^{-1/r}$. We have $|a_{n,i}| \le Cn^{-1/r}$ for every i. So, $\lim_{n\to\infty} a_{n,i} = 0$ for every i.

Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} i^{1/r}|a_{n,i}-a_{n,i+1}| &= \sum_{i=1}^{n} i^{1/r}|a_{n,i}-a_{n,i+1}| \\ &\le n^{1/r}\sum_{i=1}^{n}|a_{n,i}-a_{n,i+1}| \le C.\end{aligned} \tag{3.19}$$

So, the proof is complete. □

The assumptions of $\{A_{n,i}\}$ in Theorem 2 can be simplified as in Theorem 3 for $r < 1$ and Theorem 4 for $r \ge 1$.

Lemma 4. *Let $\{b_n : n = 1,2,3,\dots\}$ be a sequence of positive numbers. If $\sum_{i=1}^{\infty} i|b_i - b_{i+1}| < \infty$ and $\sum_{i=1}^{\infty} b_i < \infty$, then there exists $C > 0$ such that $ib_i < C$ for all i.*

Proof. Since $\sum_{i=1}^{\infty} i|b_i-b_{i+1}| < \infty$, there exists $N > 0$ such that $\sum_{k=N}^{\infty} i|b_i-b_{i+1}| < 1$. If the result of this Lemma is false, then any $n,l > 0$, there exists $i > l$ such that $b_i > n/i$. We define

$$n_i \equiv \inf\left\{i : i > 2n_{j-1} \text{ and } b_i > \frac{J}{i}\right\} \quad \text{if } j \ge 2. \tag{3.20}$$

And

$$n_0 \equiv 0, \qquad n_1 \equiv \inf\left\{i : i > N \text{ and } b_i > \frac{1}{i}\right\}. \tag{3.21}$$

We see that

$$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty}\sum_{k=n_{i-1}+1}^{n_i} b_k = \sum_{i=1}^{\infty}\sum_{k=n_{i-1}+1}^{n_i}\left(b_{n_i} + (b_k - b_{n_i})\right). \tag{3.22}$$

If $m > n \ge N$, then

$$\sum_{i=n}^{m} b_i - b_m \le \sum_{i=n}^{m}\sum_{k=i}^{m-1}|b_k-b_{k+1}| = \sum_{k=n}^{m-1}\sum_{i=n}^{k}|b_k-b_{k+1}| \le \sum_{k=n}^{m-1} K|b_k-b_{k+1}| < 1. \tag{3.23}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} b_i &= \sum_{i=1}^{\infty}\sum_{k=n_{i-1}+i}^{n_i}\left(b_{n_i}+(b_k-b_{n_i})\right) \\ &\ge \sum_{i=2}^{\infty}\left(\frac{i}{n_i}(n_i-n_{i-1})-1\right) \ge \sum_{i=2}^{\infty}\left(\frac{i}{n_i}\times\frac{n_i}{2}-1\right) = \infty.\end{aligned} \tag{3.24}$$

But $\sum_{i=1}^{\infty} b_i < \infty$ and the proof is complete. □

When $r < 1$, Theorem 2 can be rewritten as follows.

Theorem 3. *Let* $\mathbf{B}$ *be a separable Banach space of type* p. *Let* $\{X_n : n = 1,2,3,\dots\}$ *be a sequence of independent and identically distributed random elements in* $\mathbf{B}$ *such that* $E\|X_1\|^r < \infty$ *and* $r < 1$. *Let* $\{A_{n,i} : n,i = 1,2,3,\dots\}$ *be an array of random variables such that* $\{A_{n,i}\}$ *and* $\{X_i\}$ *are independent, and satisfying*

$$\lim_{n\to\infty} A_{n,i} = 0 \quad \text{a.s. for every } i, \tag{3.25}$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{a.s. for every } n, \tag{3.26}$$

and

$$\sum_{i=1}^{\infty} i|A_{n,i} - A_{n,i+1}|^r < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \tag{3.27}$$

Then $\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i}X_i = 0$ *a.s.*

Conversely, if $\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i}X_i = 0$ *a.s. for all arrays* $\{A_{n,i}\}$ *satisfying the above conditions, then* $E\|X_1\|^r < \infty$.

Proof. Since $A_{n,i} = A_{n,i}^+ - A_{n,i}^-$, without loss of generality, we can assume that $A_{n,i} \ge 0$. We consider $A_{n,i}^r = b_i$ in Lemma 4. Since $\sum_{i=1}^{\infty} i|A_{n,i}^r - A_{n,i+1}^r| \le \sum_{i=1}^{\infty} i|A_{n,i} - A_{n,i+1}|^r$, for $r < 1$, we have $A_{n,i} = O(i^{-1/r})$ a.s. for every n. From the proof of Theorem 2, we have $\lim_{i\to\infty} Y_i = 0$ a.s. and $\sum_{i=1}^{\infty} A_{n,i}X_i = \sum_{i=1}^{\infty} i^{1/r}(A_{n,i} - A_{n,i+1})Y_i$ a.s., where $Y_i = (1/i^{1/r})\sum_{j=1}^{i} X_j$.

Define $B_{n,i} = i^{1/r}(A_{n,i} - A_{n,i+1})$. Hence, $\sum_{i=1}^{\infty} |B_{n,i}|^r \le M$ a.s. and $\lim_{n\to\infty} B_{n,i} = 0$ a.s. Let $D = D^0 \cap \bigcap_{i=1}^{\infty}(D_i^1 \cap D_i^2)$, where the definitions of D^0, D_i^1, and D_i^2 are the same as in Theorem 2. For every $w \in D$ and every $\epsilon > 0$, we can choose A such that $\|Y_i(w)\| < \epsilon$ for $i \ge A$,

$$\begin{aligned}\sum_{i=1}^{\infty} \|B_{n,i}Y_i(w)\|^r &\le \sum_{i=1}^{A-1} |B_{n,i}|^r \|Y_i(w)\|^r + \sum_{i=A}^{\infty} |B_{n,i}|^r \|Y_i(w)\|^r \\ &\le \max_{i\le A-1} \|Y_i(w)\|^r \sum_{i=1}^{A-1} |B_{n,i}|^r + M\epsilon \longrightarrow M\epsilon \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.28}$$

Since $\Pr(D) = 1$, $\lim_{n\to\infty} \sum_{i=1}^{\infty} \|B_{n,i}Y_i\|^r = 0$ a.s. Therefore,

$$\lim_{n\to\infty} A_{n,i}X_i = \lim_{n\to\infty} \sum_{i=1}^{\infty} B_{n,i}Y_i = 0 \quad \text{a.s.} \tag{3.29}$$

The proof of the converse part is the same as the proof of Theorem 2. So the proof is complete. □

When $r \ge 1$, we can obtain the following theorem:

Theorem 4. *Let* $\mathbf{B}$ *be a separable Banach space of type* p. *Let* $\{X_n : n = 1,2,3,\dots\}$ *be a sequence of independent and identically distributed random elements in* $\mathbf{B}$ *such that* $EX_1 = 0$ *and* $E\|X_1\|^r < \infty$ *for* $1 \le r < p$. *Let* $\{A_{n,i} : n,i = 1,2,3,\dots\}$ *be an array of* *random variables such that* $\{A_{n,i}\}$ *and* $\{X_i\}$ *are independent, and satisfying*

$$\lim_{n\to\infty} A_{n,i} = 0 \quad \text{a.s. for every } i, \tag{3.30}$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{for every } n, \tag{3.31}$$

and

$$\sum_{i=1}^{\infty} i^{1/r} \left| A_{n,i}^r - A_{n,i+1}^r \right|^{1/r} < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \tag{3.32}$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \tag{3.33}$$

Conversely, if

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \tag{3.34}$$

for all arrays $\{A_{n,i}\}$ *satisfying the above conditions, then* $E\|X_1\|^r < \infty$.

Proof. We see that

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i} - A_{n,i+1}| < \sum_{i=1}^{\infty} i^{1/r} \left| A_{n,i}^r - A_{n,i+1}^r \right|^{1/r} < M \tag{3.35}$$

and

$$\sum_{i=1}^{\infty} i^{1/r} \left| A_{n,i}^r - A_{n,i+1}^r \right|^{1/r} < \infty \Longrightarrow \sum_{i=1}^{\infty} i \left| A_{n,i}^r - A_{n,i+1}^r \right| < \infty \quad (\text{since } r \geq 1). \tag{3.36}$$

So, from the proofs of Theorem 2 and Theorem 3, we can obtain this theorem. □

Now, we consider a very special case of $\{A_{n,i}\}$. Let $A_{n,i} = n^{-1/r}$ for $i = 1,2,\ldots,n$ and $A_{n,i} = 0$ for $i > n$. The assumptions of $\{A_{n,i}\}$ in Theorem 4 can be easily verified. Therefore, the Marcinkiewicz-Zygmund law of large numbers in a space of type p can be obtained as the following corollary.

Corollary 2. *Let* **B** *be a separable Banach space of type* p *and* $\{X_n : n = 1,2,3,\ldots\}$ *be a sequence of independent and identically distributed random elements in* **B** *with zero means. For any* $1 \leq r < p$, *we have if* $E\|X_1\|^r < \infty$, *then*

$$\left(\frac{1}{n}\right)^{1/r} \sum_{i=1}^{n} X_i \longrightarrow 0 \quad \text{a.s.} \tag{3.37}$$

Proof. Let $A_{n,i} = n^{-1/r}$ for $i = 1,2,\ldots,n$ and $A_{n,i} = 0$ for $i > n$. Since

$$\lim_{n\to\infty} A_{n,i} = \lim_{n\to\infty} n^{-1/r} = 0, \tag{3.38}$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r = \sum_{i=1}^{n} \frac{1}{n} = 1, \tag{3.39}$$

and

$$\sum_{i=1}^{\infty} i^{1/r} \left| A_{n,i}^r - A_{n,i+1}^r \right|^{1/r} = n^{1/r} \cdot n^{-1/r} = 1, \tag{3.40}$$

the proof is complete by Theorem 4. □

References

[1] I. A. Ahmad, *An almost sure convergence theorem for weighted sum of random elements in separable Banach spaces with random weights*, Sankhyā Ser. A **44** (1982), no. 2, 262–268. MR 84c:60010. Zbl 586.60008.

[2] B. D. Choi and S. H. Sung, *Almost sure convergence theorems of weighted sums of random variables*, Stochastic Anal. Appl. **5** (1987), no. 4, 365–377. MR 88m:60079. Zbl 633.60049.

[3] Y. S. Chow and T. L. Lai, *Limiting behavior of weighted sums of independent random variables*, Ann. Probab. **1** (1973), 810–824. MR 50 5909. Zbl 303.60025.

[4] W. J. Padgett and R. L. Taylor, *Convergence of weighted sums of random elements in Banach spaces and Fréchet spaces*, Bull. Inst. Math. Acad. Sinica **2** (1974), 389–400, Collection of articles in celebration of the sixtieth birthday of Ky Fan. MR 50 11360. Zbl 301.60004.

[5] A. I. Sakhanenko, *On Lévy-Kolmogorov Inequalities for Banach-space-valued Random Variables*, Theory Probab. Appl. **29** (1984), 830–836.

[6] R. L. Taylor, *Stochastic convergence of weighted sums of random elements in linear spaces*, Lecture Notes in Mathematics, vol. 672, Springer-Verlag, Berlin, 1978. MR 80g:60006. Zbl 443.60004.

[7] W. A. Woyczyński, *On Marcinkiewicz-Zygmund laws of large numbers in Banach spaces and related rates of convergence*, Probab. Math. Statist. **1** (1980), no. 2, 117–131. MR 82i:60026. Zbl 502.60006.

Hu: Department of Mathematics, National Tsing Hua University, Hsinchu, 30043, Taiwan

Chang: Institute of Mathematics, Academic Sinica, Taipei, 11529, Taiwan