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LEVEL CROSSINGS AND TURNING POINTS OF RANDOM HYPERBOLIC POLYNOMIALS

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ABSTRACT. In this paper, we show that the asymptotic estimate for the expected number of K-level crossings of a random hyperbolic polynomial $a_1 \sinh x + a_2 \sinh 2x + \cdots + a_n \sinh nx$, where $a_j (j = 1, 2, \ldots, n)$ are independent normally distributed random variables with mean zero and variance one, is $(1/\pi)\log n$. This result is true for all K independent of x, provided $K \equiv K_n = O(\sqrt{n})$. It is also shown that the asymptotic estimate of the expected number of turning points for the random polynomial $a_1 \cosh x + a_2 \cosh 2x + \cdots + a_n \cosh nx$, with $a_j (j = 1, 2, \ldots, n)$ as before, is also $(1/\pi)\log n$.

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1. Introduction. Let

$$T_1(x) = \sum_{j=1}^{n} a_j \cosh jx \tag{1.1}$$

and

$$T_2(x) = \sum_{j=1}^{n} a_j \sinh jx, \tag{1.2}$$

where $a_1, a_2, \ldots, a_n$ is a sequence of independent normally distributed random variables with mean zero and variance one. Let $N_K^{(i)}(\alpha, \beta)$ be the number of real roots of the equation $T_i(x) - K = 0$ and $M^{(i)}(\alpha, \beta)$ be the number of real roots of $T_i'(x)$. In both cases, the interval is (α, β) and $i = 1, 2$. Clearly, $M^{(i)}(\alpha, \beta)$ represents the number of turning points of $T_i(x)$ on the interval (α, β). It proves convenient to denote the expected number of real roots of $T_i(x) - K$ and $T_i'(x)$, on the interval (α, β), by $EN_K^{(i)}(\alpha, \beta)$ and $EM^{(i)}(\alpha, \beta)$, respectively. Bharucha-Reid and Sambandham [1] reported an unpublished result of Das [2], where it is stated that for $K = 0$, $EN_0^{(1)}(-\infty, \infty) \sim (1/\pi)\log n$, the random coefficients being the same as in (1.1). Farahmand [4] obtained the same asymptotic value as Das for $EN_K^{(1)}(-\infty, \infty)$, where $K = O(\sqrt{n})$. Farahmand highlighted the surprising way the hyperbolic polynomial mimics certain characteristics of both the trigonometric and the algebraic polynomials. In a more recent work, Farahmand [5] showed that $EM^{(2)}(-\infty, \infty) \sim (1/\pi)\log n$, where the random coefficients are again those outlined in (1.1) and (1.2). Clearly, there remains two unsolved complementary problems, that is $EM^{(1)}(-\infty, \infty)$ and $EN_K^{(2)}(-\infty, \infty)$. These problems are the subject of this paper. We prove the following theorems.

Theorem 1. *If the coefficients of $T_2(x)$ in (1.2) are independent normally distributed random variables with mean zero and variance one and $K_n \equiv K$ such that $K^2/(n\log n)$ tends to zero as n tends to infinity, then*

$$EN_K^{(2)}(-\infty,\infty) \sim \left(\frac{1}{\pi}\right)\log n. \tag{1.3}$$

Theorem 2. *If the coefficients of $T_1(x)$ in (1.1) are independent normally distributed random variables with mean zero and variance one and n tends to infinity, then*

$$EM^{(1)}(-\infty,\infty) \sim \left(\frac{1}{\pi}\right)\log n. \tag{1.4}$$

These results may not have been derived until now because of the difficulty in applying the conventional Rice formula to both of these cases near zero. In our method, we make the unusual step of determining the expected number of roots for the polynomial's derivative to simplify the analysis. More comprehensive results are known for the random algebraic polynomial $\sum_{j=1}^{n} a_j x^j$. We refer the reader to the pioneering works of Littlewood and Offord [8, 7] and the more recent works of Wilkins [11] and Offord [9].

2. Level crossings and turning points formulae. We employ the extension of a formula obtained by Rice [10] given by Farahmand [3], where $\operatorname{erf}(x) = \int_0^x \exp(-t^2)\,dt$, that is

$$\begin{aligned} EN_k^{(i)}(\alpha,\beta) &= \int_\alpha^\beta \left(\frac{\Delta_i}{\pi A_i^2}\right)\exp\left(\frac{-B_i^2K^2}{2\Delta_i^2}\right)dx \\ &\quad + \int_\alpha^\beta \left(\frac{\sqrt{2}}{\pi}\right)KD_iA_i^{-3}\exp\left(\frac{-K^2}{2A_i^2}\right)\operatorname{erf}\left(\frac{KD_i}{A_i\Delta_i\sqrt{2}}\right)dx \\ &= I_1^{(i)}(\alpha,\beta)+I_2^{(i)}(\alpha,\beta). \end{aligned} \tag{2.1}$$

The Rice formula applied to the roots of $T_i'(x)$ gives

$$EM^{(i)}(\alpha,\beta) = \int_\alpha^\beta \left(\frac{\Lambda_i}{\pi B_i^2}\right)dx. \tag{2.2}$$

Before we give generic definitions for the individual elements in (2.1) and (2.2), we note two fundamental facts that make these definitions easier,

$$\operatorname{var}\{T_i(x)-L\} = \operatorname{var}\{T_i(x)\} \tag{2.3}$$

and

$$\operatorname{cov}\{T_i(x)-L,T_i'(x)\} = \operatorname{cov}\{T_i(x),T_i'(x)\}. \tag{2.4}$$

Using the properties (2.3) and (2.4), we define the variance and covariance elements of (2.1) and (2.2) as

$$\begin{gathered} A_i^2 = \operatorname{var}\{T_i(x)\}, \qquad B_i^2 = \operatorname{var}\{T_i'(x)\}, \qquad C_i^2 = \operatorname{var}\{T_i''(x)\}, \\ D_i = \operatorname{cov}\{T_i(x),T_i'(x)\}, \qquad E_i = \operatorname{cov}\{T_i'(x),T_i''(x)\}, \\ \Delta_i^2 = A_i^2B_i^2-D_i^2, \qquad \text{and} \qquad \Lambda_i^2 = B_i^2C_i^2-E_i^2. \end{gathered} \tag{2.5}$$

To enable us to prove Theorem 2 as simply as possible, we define $Q^{(i)}(\alpha,\beta)$ to be the number of roots of $T_i''(x)$ (the second derivative of $T_i(x)$) on the interval (α,β) and we give the Rice formula for the expected number of roots of $T_i''(x)$.

$$EQ^{(i)}(\alpha,\beta) = \int_\alpha^\beta \left(\frac{\Psi_i}{\pi C_i^2}\right) dx, \tag{2.6}$$

where

$$F_i^2 = \operatorname{var}\{T_i^{(3)}(x)\}, \qquad G_i = \operatorname{cov}\{T_i(x), T_i^{(3)}(x)\}, \tag{2.7}$$

and

$$\Psi_i^2 = C_i^2 F_i^2 - G_i^2. \tag{2.8}$$

3. Evaluation of variances and covariances. Since the coefficients of (1.1) and (1.2) are independent normally distributed random variables with mean zero and variance one,

$$A_2^2 = \sum_{j=1}^n \sinh^2 jx, \tag{3.1}$$

$$B_1^2 = \sum_{j=1}^n j^2 \sinh^2 jx, \tag{3.2}$$

$$B_2^2 = \sum_{j=1}^n j^2 \cosh^2 jx, \tag{3.3}$$

$$C_1^2 = \sum_{j=1}^n j^4 \cosh^2 jx, \tag{3.4}$$

$$C_2^2 = \sum_{j=1}^n j^4 \sinh^2 jx, \tag{3.5}$$

$$D_2 = \sum_{j=1}^n j \cosh jx \sinh jx, \tag{3.6}$$

$$E_1 = E_2 = \sum_{j=1}^n j^3 \cosh jx \sinh jx, \tag{3.7}$$

$$F_1^2 = \sum_{j=1}^n j^6 \sinh^2 jx. \tag{3.8}$$

At this point, we rewrite (3.1), (3.2), (3.3), (3.4), (3.6), and (3.7) in a form we utilize later. We observe that

$$\sinh^2 jx = \frac{(\cosh 2jx - 1)}{2}, \tag{3.9}$$

and use the hyperbolic equivalent to the two formulae given at 1.342.2 in [6, p. 36], that is

$$\begin{aligned}\sum_{j=1}^n \cosh 2jx &= \frac{\cosh(n+1)x \sinh nx + \cosh nx \sinh(n+1)x}{2\sinh x} - \frac{1}{2} \\ &= \frac{\sinh(2n+1)x}{2\sinh x} - \frac{1}{2}.\end{aligned} \tag{3.10}$$

Thus, summing (3.9) and employing (3.10), we find that

$$A_2^2 = \frac{\sinh(2n+1)x}{4\sinh x} - \frac{(2n+1)}{4}. \tag{3.11}$$

By repeated differentiation of (3.11), we find that

$$\begin{aligned} B_i^2 &= (2n^2+2n+1)\frac{\sinh(2n+1)x}{8\sinh x} - (2n+1)\frac{\cosh x\cosh(2n+1)x}{8\sinh^2 x} \\ &\quad + \frac{\sinh(2n+1)x}{8\sinh^3 x} + \frac{(-1)^i n(n+1)(2n+1)}{12}, \quad i=1,2, \end{aligned} \tag{3.12}$$

$$\begin{aligned} C_1^2 &= (2n^4+4n^3+6n^2+4n+1)\frac{\sinh(2n+1)x}{8\sinh x} \\ &\quad - (2n^2+2n+1)(2n+1)\frac{\cosh x\cosh(2n+1)x}{8\sinh^2 x} \\ &\quad + (3n^2+3n+2)\frac{\sinh(2n+1)x}{4\sinh^3 x} - 3(2n+1)\frac{\cosh x\cosh(2n+1)x}{8\sinh^4 x} \\ &\quad + 3\frac{\sinh(2n+1)x}{8\sinh^5 x} + \frac{n(n+1)(2n+1)(3n^2+2n-1)}{60}, \end{aligned} \tag{3.13}$$

$$D_2 = (2n+1)\frac{\cosh(2n+1)x}{8\sinh x} - \frac{\cosh x\sinh(2n+1)x}{8\sinh^2 x}, \tag{3.14}$$

$$\begin{aligned} E_1 = E_2 &= (n^2+n+1)(2n+1)\frac{\cosh(2n+1)x}{8\sinh x} \\ &\quad - (3n^2+3n+1)\frac{\cosh x\sinh(2n+1)x}{8\sinh^2 x} \\ &\quad + (6n+3)\frac{\cosh(2n+1)x}{16\sinh^3 x} - \frac{3\cosh x\sinh(2n+1)x}{16\sinh^4 x}. \end{aligned} \tag{3.15}$$

4. Proof of the theorems. Looking at the properties of the random coefficients in (1.1) and (1.2), it is clear that $EN_k^{(i)}(0,\infty) = EN_K^{(i)}(-\infty,0)$ and $EM^{(i)}(0,\infty) = EM^{(i)}(-\infty,0)$. In the proofs that follow, we consider the interval $(0,\infty)$ only, it proves advantageous to break this interval into the three subintervals $(0,\sqrt{\log n}/n)$, $(\sqrt{\log n}/n,1)$, and $(1,\infty)$. We start by proving Theorem 1 and Theorem 2 on the interval $(0,\sqrt{\log n}/n)$.

From Rolle's theorem, it is evident that, for any differentiable function $P(x)$ with r roots in the interval (a,b), $P'(x)$ (the derivative of $P(x)$) has r' roots on (a,b), where

$$r \le r' + 1. \tag{4.1}$$

From (2.1), we know that

$$EN_K^{(2)}(\alpha,\beta) \le EN_0^{(2)}(\alpha,\beta), \tag{4.2}$$

where $N_K^{(2)}(\alpha,\beta)$ and $N_0^{(2)}(\alpha,\beta)$, are the numbers of K level and zero level crossings of (1.2), respectively, on the interval (α,β). Employing (4.1), (4.2), (2.2), (3.3), and (3.5), we find that

$$\begin{aligned} EN_K^{(2)}\left(0,\frac{\sqrt{\log n}}{n}\right) &\le EM^{(2)}\left(0,\frac{\sqrt{\log n}}{n}\right)+1 \le \left(\frac{1}{\pi}\right)\int_0^{\sqrt{\log n}/n}\left(\sqrt{\frac{C_2^2}{B_2^2}}\right)dx+1 \\ &\le \left(\frac{1}{\pi}\right)\int_0^{\sqrt{\log n}/n} n\,dx+1 = O\left(\sqrt{\log n}\right). \end{aligned} \tag{4.3}$$

Obviously, taking advantage of the same analysis and using (2.6), (3.4), and (3.8), we can show that

$$\begin{aligned} EM^{(1)}\left(0,\frac{\sqrt{\log n}}{n}\right) &\le EQ^{(1)}\left(0,\frac{\sqrt{\log n}}{n}\right)+1 \\ &\le \left(\frac{1}{\pi}\right)\int_0^{\sqrt{\log n}/n} n\,dx+1 \\ &= O\left(\sqrt{\log n}\right). \end{aligned} \tag{4.4}$$

We can show that the second integral on the right-hand side of (2.1) does not contribute to the leading behaviour of $EN_K^{(2)}(0,\infty)$. Since we know that $\operatorname{erf}(x) \le \sqrt{\pi/2}$, for all values of x, and by making the simple substitution $u = K/A_2$, it is clear that

$$\begin{aligned} I_2^{(2)}(0,\infty) &\le \left(\frac{1}{\sqrt{2\pi}}\right)\int_0^\infty KD_2A_2^{-3}\exp\left(\frac{-K^2}{2A_2^2}\right)dx \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)\int_0^\infty \exp\left(\frac{-u^2}{2}\right)du = \frac{1}{2}. \end{aligned} \tag{4.5}$$

Having derived (4.5), it becomes apparent that we can handle the remaining analysis required for both theorems in tandem. At this point, we concentrate on the interval $(\sqrt{\log n}/n,1)$; this is the only interval that contributes to the leading behaviour of $EN_K^{(2)}(0,\infty)$ and $EM^{(1)}(0,\infty)$. To find the dominant terms in (3.11), (3.12), (3.13), (3.14), and (3.15), we observe that, in this interval,

$$\coth x < \frac{e}{x} < \frac{en}{\sqrt{\log n}}. \tag{4.6}$$

Employing (4.6), it becomes a trivial task to show that the derivative of $f_{n,p}(x) = \sinh nx/(\sinh x)^p$ is positive for all $p < e^{-1}\sqrt{\log n}$. Therefore, since $\sinh(x) < 4x$ in $(0,1)$,

$$f_{n,p}(x) \ge \sinh\left\{\sqrt{\log n}\right\}\left[n\left(4\sqrt{\log n}\right)^{-1}\right]^p \tag{4.7}$$

for all sufficiently large n.

Hence,

$$f_{2n+1,p}(x) \ge \left(\frac{n^p}{48}\right)(\log n)^{-p/2}\exp\left\{\sqrt{\log n}\right\} \tag{4.8}$$

for all $p = 1,2,3,4,\ldots$. Now, using (3.11), (3.12), (3.13), (3.14), and (3.15) and since $\sinh x \ge x/4$ for all $x \in (0,1)$, we can show that, for all sufficiently large n,

$$A_2^2 = \frac{\sinh(2n+1)x}{4\sinh x}\left\{1+O\left\{\frac{\sqrt{\log n}}{\exp(\sqrt{\log n})}\right\}\right\}, \tag{4.9}$$

$$B_1^2 = B_2^2 = (2n^2+2n+1)\frac{\sinh(2n+1)x}{8\sinh x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}, \tag{4.10}$$

$$C_1^2 = (2n^4+4n^3+6n^2+4n+1)\frac{\sinh(2n+1)x}{8\sinh x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}, \tag{4.11}$$

$$D_2 = (2n+1)\frac{\cosh(2n+1)x}{8\sinh x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}, \tag{4.12}$$

$$E_1 = E_2 = (n^2+n+1)(2n+1)\frac{\cosh(2n+1)x}{8\sinh x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}. \tag{4.13}$$

Due to cancellation of terms, it is not sufficient to use (4.9), (4.10), and (4.12) to compute Δ_2^2. Instead, we use (3.11), (3.12), and (3.14) to show that

$$\Delta_2^2 = \frac{\sinh^2(2n+1)x}{64\sinh^4 x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}. \tag{4.14}$$

We compute Ψ_1^2 from (4.10), (4.11), and (4.13) directly to give

$$\Psi_1^2 = n^4\frac{\sinh^2(2n+1)x}{64\sinh^4 x}\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}. \tag{4.15}$$

Since we have already determined $I_2^{(2)}(0,\infty)$, we concentrate on $I_1^{(2)}(\sqrt{\log n}/n,1)$. From (2.1), (4.9), (4.10), and (4.12), we have

$$\begin{aligned} I_1^{(2)}\left(\frac{\sqrt{\log n}}{n},1\right) &= (2\pi)^{-1}\int_{\sqrt{\log n}/n}^{1}\operatorname{cosech} x\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}dx \\ &\quad\times\exp\left(-8n^2\frac{\sinh^3 xK^2}{\sinh(2n+1)x}\right)dx \\ &= (2\pi)^{-1}\int_{\sqrt{\log n}/n}^{1}\operatorname{cosech} x\left\{1+O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\}dx \\ &\quad+O\left(K^2n^2\int_{\sqrt{\log n}/n}^{1}\frac{\sinh^2 x}{\sinh(2n+1)x}dx\right). \end{aligned} \tag{4.16}$$

The first integral on the right-hand side of (4.16) can be evaluated directly. To determine the second integral, we use the fact that $x \le \sinh x \le 4x$ on the interval $0 \le x \le 1$. Thus,

$$\begin{aligned} I_1^{(2)}\left(\frac{\sqrt{\log n}}{n},1\right) &= (2\pi)^{-1}\left\{\log\left(\tanh\left(\frac{1}{2}\right)\right)-\log\left(\tanh\left(\frac{\sqrt{\log n}}{2n}\right)\right)\right\} \\ &\quad+O\left\{K^2n\int_{2\sqrt{\log n}}^{2n}u^2\operatorname{cosech}(u)\,du\right\} \\ &= (2\pi)^{-1}\log n+O(\log\log n)+O\left(\frac{K^2}{n}\right). \end{aligned} \tag{4.17}$$

Since $K = o(\sqrt{n\log n})$, it is clear from (4.17) that $I_2^{(2)}(\sqrt{\log n}/n, 1) = (2\pi)^{-1}\log n$. We now evaluate $EM^{(1)}(\sqrt{\log n}/n, 1)$ using (2.2), (4.10), and (4.15).

$$\begin{aligned}
EM^{(1)}\left(\frac{\sqrt{\log n}}{n}, 1\right) &= (2\pi)^{-1}\int_{\sqrt{\log n}/n}^{1} \operatorname{cosech} x\, dx\left\{1 + O\left\{\frac{1}{\sqrt{\log n}}\right\}\right\} \\
&= (2\pi)^{-1}\left\{\log\left(\tanh\left(\frac{1}{2}\right)\right) - \log\left[\tanh\left\{\frac{\sqrt{\log n}}{2n}\right\}\right]\right\} \\
&= (2\pi)^{-1}\log n + O\{\log\log n\}.
\end{aligned} \tag{4.18}$$

All that remains to complete the proofs of the two theorems is to evaluate $EN_K^{(2)}(1,\infty)$ and $EM^{(1)}(1,\infty)$. The same function $f_{n,p}(x)$, as used in the previous section, is obviously strictly increasing on the interval $(1,\infty)$. Since $\sinh nx > \exp(nx)/3$ and $\sinh x \le \exp(x)/2$, it is clear that

$$f_{n,p}(x) \ge 2\frac{\exp(n-p)}{3}. \tag{4.19}$$

Using inequality (4.19), it can be shown that

$$A_2^2 = \frac{\sinh(2n+1)x}{4\sinh x}\left\{1 + O\left\{n\exp(-2n)\right\}\right\}, \tag{4.20}$$

$$B_1^2 = B_2^2 = (2n^2+2n+1)\frac{\sinh(2n+1)x}{8\sinh x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}, \tag{4.21}$$

$$C_1^2 = (2n^4+4n^3+6n^2+4n+1)\frac{\sinh(2n+1)x}{8\sinh x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}, \tag{4.22}$$

$$D_2 = (2n+1)\frac{\cosh(2n+1)x}{8\sinh x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}, \tag{4.23}$$

$$E_1 = E_2 = (n^2+n+1)(2n+1)\frac{\cosh(2n+1)x}{8\sinh x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}, \tag{4.24}$$

$$\Delta_2^2 = \frac{\sinh^2(2n+1)x}{64\sinh^4 x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}, \tag{4.25}$$

$$\Lambda_1^2 = n^4\frac{\sinh^2(2n+1)x}{64\sinh^4 x}\left\{1 + O\left\{\frac{1}{n}\right\}\right\}. \tag{4.26}$$

Employing (2.1), (4.2), (4.20), and (4.25), we find that

$$\begin{aligned}
EN_K^{(2)}(1,\infty) &\le \left(\frac{1}{2\pi}\right)\int_1^{\infty} \operatorname{cosech} x\, dx\left\{1 + O\left\{\frac{1}{n}\right\}\right\} \\
&= \left(\frac{1}{2\pi}\right)\log\left[\tanh\left(\frac{x}{2}\right)\right]\Big|_1^{\infty}\left\{1 + O\left\{\frac{1}{n}\right\}\right\} \\
&= O\{1\}.
\end{aligned} \tag{4.27}$$

Similarly, using (2.2), (4.21), and (4.26), we find that

$$EM^{(1)}(1,\infty) = O\{1\}. \tag{4.28}$$

If we combine (4.3), (4.5), (4.16), and (4.27), it is obvious that Theorem 1 is proved. Bringing together (4.4), (4.18), and (4.28), the proof of the second theorem is also complete.

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