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THE RELATIVE DIHEDRAL HOMOLOGY OF INVOLUTIVE ALGEBRAS

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ABSTRACT. Let $f : A \to B$ be a homomorphism of involutive algebras A, B. The purpose of this paper is to define a free involutive algebra resolution of algebra B over f and use it to define and study the relative dihedral homology.

Keywords and phrases. Dihedral homology, free involutive algebra, relative dihedral homology.

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1. Introduction. Let A, B be involutive algebras (an involution $*$ is an anti-automorphism of degree zero and order 2) and let $f : A \to B$ be a homomorphism. Our first aim is to find a free involutive algebra resolution R of algebra B over the homomorphism $f : A \xrightarrow{i} R \xrightarrow{\pi} B$, where i is an inclusion and π is a quasi-isomorphism. The second aim is to define the relative dihedral homology as

$$^{\epsilon}\mathcal{HD}_{\bullet}(A \xrightarrow{f} B) = \mathcal{H}_{\bullet}\left(\frac{R}{(A + [R,R] + \operatorname{Im}(1 - r^{\epsilon})}\right), \tag{1.1}$$

where $[R,R]$ is the commutant of algebra R, r^{ϵ} is the involution on R, and study its main properties.

First, we recall some definitions and facts from [4, 5]. Let A be an associative algebra over a field $k(k = \mathbb{R}$ or $\mathbb{C})$. Define the complex $C(A) = (C_{\bullet}(A), \mathscr{C}_{\bullet})$, where $C_n(A) = A \otimes \cdots \otimes A$ is the tensor product of algebra $A(n+1$ times) and $\mathscr{C}_n : C_n(A) \to C_{n-1}(A)$ is the boundary operator

$$\mathscr{C}_n(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n+1}. \tag{1.2}$$

It is well known that $\mathscr{C}_{n-1}\mathscr{C}_n = 0$, that is, the complex $C(A)$ is a chain complex. This complex is called the Hochschild (simplicial) complex and its homology is called the Hochschild homology $(\mathcal{HH}_{\bullet}(A))$. If A is a unital involutive algebra, then on the complex $C(A)$, one acts by the operators $t_n, r_n : C_n(A) \to C_n(A)$ by means of

$$\begin{aligned} t_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}, \\ r_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^{n(n+1)/2} \epsilon a_0^* \otimes a_n^* \otimes \cdots \otimes a_1^*, \quad \epsilon = \pm 1. \end{aligned} \tag{1.3}$$

Consider the quotient complex $C\mathcal{D}_n(A) = C_n(A)/\operatorname{Im}(1 - t_n) + \operatorname{Im}(1 - r_n)$ of a complex $C_n(A)$. Following [3] the dihedral homology of algebra, A is the homology of the complex $C\mathcal{D}_{\bullet}(A)$.

2. Free involutive algebra resolution. In this part, we discuss the existence of the free involutive algebra resolution. Let $E=\sum_{n=0}^{\infty}E_n$ be a graded involutive vector space over a field K. Suppose that R is a differential graded (in short DG) involutive K-algebra and let $R\langle E\rangle = R * T_k(E)$ be the free product of algebras, where $T_k(E)=\sum_{j\geq 0}E^{\otimes j}$ is the tensor algebra over K. We define an involution on algebra $R\langle E\rangle$ to be the unique anti-automorphism on $R\langle E\rangle$ which restricts to the given involution on R and E (this is enough thanks to the universal property of the tensor algebra and the free product). The product in $R\langle E\rangle$ is given by

$$\begin{gathered}(r_1e_1\cdots r_ne_nr_{n+1})\cdot(\hat{r}_1\hat{e}_1\cdots\hat{r}_k\hat{e}_k\hat{r}_{k+1}) = (r_1e_1\cdots r_ne_n(r_{n+1}\hat{r}_1)\hat{e}_1\cdots\hat{r}_k\hat{e}_k\hat{r}_{k+1}),\\ r_i,\hat{r}_j\in R,\quad e_i,\hat{e}_j\in T_k(E),\quad (r^*)^*=(e^*)^*=e.\end{gathered}\tag{2.1}$$

Definition 2.1. Let $f:R_1\to R_2$ be a homomorphism of involutive differential graded K-algebras. An algebra R_2 is a free algebra over the homomorphism f if there exists an isomorphism $\alpha: R_1\langle E\rangle\simeq R_2$, where E is an involutive differential graded vector space with the following commutative diagram:

$$\begin{array}{ccc} R_1 & \xrightarrow{f} & R_2 \\ & \searrow^{i} & \wr\| \,\alpha \\ & & R_1\langle E\rangle, \end{array}\tag{2.2}$$

where i is the inclusion map.

Lemma 2.2. *Let $f:A\to B$ be a morphism of involutive K-algebra. Then there exists an involutive differential graded algebra $R=\sum_{i=0}R_i$ with the following properties*

(i) *π is surjection and the following diagram is commutative*

$$\begin{array}{ccc} & & R \\ & \nearrow^{i} & \downarrow \pi \\ A & \xrightarrow[f]{} & B, \end{array}\tag{2.3}$$

where i is an inclusion map.

There is an amorphism $j:R\to A$ such that $j\circ i = 1_A$.

(ii) *π is quasi-isomorphism, i.e., $\pi_{\bullet}:\mathcal{H}_{\bullet}(R)\to\mathcal{H}_{\bullet}(B)=B$, where B is a differential graded algebra,*

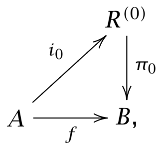
$$(B)_i=\begin{cases}B, & i=0,\\ 0, & i>0, \text{ and the differential } \partial^B=0.\end{cases}\tag{2.4}$$

(iii) *The involutive* DG *algebra R is free over the homomorphism $i:A\to R$.*

Definition 2.3. The involutive DG algebra which satisfies the conditions (i), (ii), and (iii) of Lemma 2.2 is called a free involutive algebra resolution of algebra B over f.

Proof of Lemma 2.2.

First step. We construct a commutative diagram of involutive algebra



(2.5)

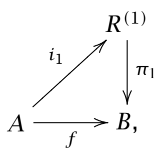
where $R^{(0)}$ is free over the homomorphism $i_0 : A \to R^{(0)}$, π_0 an involutive surjection. Define $A\langle t_i\rangle = A\langle E(t_i)\rangle$, where $E(t_i)$ is an involutive vector space generated by $\{t_i\}$, or generated by the family $\{t_i, t_i^*\}$. The automorphism $* : E(t_i) \to E(t_i)$ is given as follows: $*(t_i) = (t_i^*)$, $*(t_i^*) = t_i$. We choose a system $\{\mathcal{C}_i^{(0)}\}$ of generators in algebra B. This family is assumed to be closed under an involutive on B.

Now, let $R^{(0)} = A\langle t_i^{(0)}\rangle$, where $t_i^{(0)}$ is equivalent to the generator $\{\mathcal{C}_i^{(0)}\}$ of algebra B, and suppose that $\beta_i^{(0)} = t_i^{(0)}$ or $(t_i^{(0)})^*$. We may define π_0 using the universal property of $R^{(0)}$. Let π_0 be the unique homomorphism of involutive algebras $R^{(0)} \to B$ which restricts to f on A and sends $t_i^{(0)}$ to $\mathcal{C}_i^{(0)}$.

Since $i_0 : A \to A\langle t_i^{(0)}\rangle$ is an inclusion map, $i_0(a) = a$, i_0 is an involutive algebra homomorphism and $\pi_0 i_0(a) = \pi_0(a) = f(a)$. Hence, diagram (2.5) is commutative and π_0 is surjective.

Let $j_0 : R^{(0)} \to A$ be the unique homomorphism involutive algebra restricting to the identity on A and mapping the $t_n^{(0)}$ to zero. $R^{(0)}$ is a DG involutive k-algebra: $(R^{(0)})_0 = R^{(0)}$, $i = 0$, $(R^{(0)})_i = 0$, $i > 0$, $\partial^{R^{(0)}} B_{i_j}^{(0)} = 0$. The algebra $R^{(0)}$ is free over the homomorphism $i_0 : A \to R^{(0)}$ since $R^{(0)} = A\langle t^{(0)}\rangle$.

Second step. We construct the second commutative diagram



(2.6)

where $R^{(1)}$ is a free algebra over i_1 and π_1 is an involutive surjection. Choose a system $\mathcal{C}_j^{(1)}$ of generators of $\ker \pi_0$ which is closed under the involution. Let $t_j^{(1)}$ be indeterminates which are in bijection with the $\mathcal{C}_j^{(1)}$. Define $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)}\rangle$, where $t_i^{(0)}$ is defined above. Suppose that $\beta_j^{(1)}$ denotes $t_j^{(1)}$ or $(t_j^{(1)})^*$. The homomorphism π_1 is defined to be the unique homomorphism of involutive algebra $R^{(1)} \to B$ restricting to π_0 on $R^{(0)}$ and sending $t_j^{(1)}$ to 0. We can see, from the above discussion, that the homomorphism π_1 can be defined as π_0 and that π_1 is surjective since $\pi_1(\beta_1^{(0)}) = \mathcal{C}_i$, $\pi_1(\beta_j^{(1)}) = 0$. The homomorphism $i_1 : A \to A\langle t_i^{(0)}, t_j^{(1)}\rangle$ is inclusion. The diagram (2.6) is commutative since $(\pi_1 i_1)(a) = \pi_1(a) = f(a)$. The homomorphism j_1 is defined to be unique homomorphisms: $R^{(1)} \to A$, of involutive algebras restricting to identity on A and mapping $t_i^{(1)}$ to zero. The algebra $R^{(1)} = A\langle t_i^{(0)}, t_j^{(1)}\rangle$ is free over i_1. Finally, we have a differential graded algebra

$$R^{(1)} = (R^{(1)})_0 \oplus (R^{(1)})_0 \oplus \cdots, \qquad \deg \beta_i^{(1)} = 0, \qquad \deg \beta_j^{(1)} = 1. \tag{2.7}$$

Note that the algebra $R^{(1)}$ also has a universal property with respect to derivations (not only homomorphism). This property should be used to define the differential. The differential $\partial^{R^{(1)}}$ of $R^{(1)}$ is the unique derivation on $R^{(1)}$ satisfying the graded Leibniz rule and commuting with the involution which restricts to zero on $R^{(1)}$ and sends $t_j^{(1)}$ to $\mathscr{C}_j^{(1)}$. So, $\partial^{R^{(1)}}\beta_i^{(0)} = 0$, $\partial^{R^{(1)}}\beta_i^{(0)} = \mathscr{C}_j^{(1)} \in \ker\pi_0$, $\partial_i^{R^{(1)}} = 0$, $i > 1$.

In the same manner, we can construct the commutative diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & \overset{i_2}{\nearrow} & \downarrow{\scriptstyle \pi_2} \\ A & \xrightarrow[f]{} & B, \end{array} \tag{2.8}$$

where $R^{(2)} = A\langle t_i^{(0)}, t_j^{(1)}, t_k^{(2)}\rangle$ is a DG algebra, free over i_2, $R^{(2)} = (R^{(2)})_0 \oplus (R^{(2)})_1 \oplus (R^{(2)})_2 \oplus \cdots$, $\deg\beta_i^{(0)} = 0$, $\deg\beta_j^{(1)} = 1$, $\deg\beta_k^{(2)} = 2$, the differential of algebra $R^{(2)}$ is also defined by using a universal property and, hence, $\partial_0^{R^{(2)}}\beta_i^{(0)} = 0$, $\partial_1^{R^{(2)}}\beta_j^{(0)} = \mathscr{C}_j^{(1)}$, $\partial_1^{R^{(2)}}\beta_j^{(2)} = \mathscr{C}_k^{(2)}$, $\partial_i^{R^{(2)}} = 0$, $i > 2$.

Consequently, we can construct an involutive algebra $R^{(i)}$, $i \geq 0$ with the following commutative diagram:

$$\begin{array}{ccccccccccc} & & R^{(0)} & \xrightarrow{P_0} & R^{(1)} & \xrightarrow{P_1} & \cdots & \xrightarrow{P_{n-1}} & R^{(n)} & \xrightarrow{P_0} & \cdots \\ & \overset{i_0}{\nearrow} & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_i} & & & & \downarrow{\scriptstyle \pi_n} & & \\ A & \longrightarrow & B & = & B & = & \cdots & = & B & = & \cdots, \end{array} \tag{2.9}$$

where π_i is an involutive surjection, $i \geq 0$, $i_n = P_{n-1} \circ \cdots \circ P_0 \circ i_0$ is an inclusion map from A to $R^{(n)}$, P_i is also an inclusion map from

$$P_i : A\left\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)}\right\rangle \quad \text{to} \quad A\left\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)}, t_{m_i+1}^{(i+1)}\right\rangle. \tag{2.10}$$

Define $i_n = q_n \circ \cdots \circ q_i \circ j_0$, where q_n is the projection of the map from $A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)}\rangle$ on $A\langle t_{m_0}^{(0)}, t_{m_1}^{(1)}, \ldots, t_{m_i}^{(i)}, t_{m_{i+1}}^{(i+1)}\rangle$. The diagram (2.9) is commutative since $i_{n+1}(\beta_i^{(n)}) = \pi_n(\beta_i^{(n)}) = 0$, $n \geq 0$. Define $R = \lim R_n$, $\pi = \lim \pi_n$, $i = \lim i_n$, $j = \lim j_n$. Then the DG algebra R satisfies the items of Lemma 2.2 since

(1) $\pi = \lim i_n$ is an involutive surjection, the diagram

$$\begin{array}{ccc} & & R^{(2)} \\ & \overset{i_2}{\nearrow} & \downarrow{\scriptstyle \pi_2} \\ A & \xrightarrow[f]{} & B \end{array} \tag{2.11}$$

is commutative since $i(a) = a$, $\pi(a) = f(a)$.

(2) π is a quasi-isomorphism of DG algebras

$$\begin{array}{ccccccccc} (R)_0 & \xleftarrow{\partial_0^R} & (R)_1 & \xleftarrow{\partial_1^R} & \cdots & \xleftarrow{\partial_n^R} & (R)_n & \xleftarrow{\partial_{n+1}^R} & \cdots \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_n} & & \\ B & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & \cdots, \end{array} \tag{2.12}$$

where $\partial_i^R = \lim \partial_i^R$, $(R)_0 = \ker(\pi)_0 = B$, $\operatorname{Im}\partial^R = \ker\partial_0^R$, i.e., $\mathcal{H}_0(R) = B$, $\mathcal{H}_i(R) = 0$.

(3) The DG involutive algebra R is free over the homomorphism $i : A \to R$ since $R = \langle E\rangle$, E is an involutive vector space generated by the system

$$\left\{t_{i_0}^{(0)}, t_{i_1}^{(1)}, \ldots, t_{i_n}^{(n)}, \ldots\right\}. \tag{2.13}$$

□

3. The relative dihedral homology. In this part, we define the relative dihedral homology and study its properties. Let f be a morphism of involutive algebras A and B over a field K with characteristic zero. Let R_f^B be a free involutive algebra resolution of algebra B over f and, for $r_1, r_2 \in R_f^B$, let $[r_1, r_2] = r_1r_2 - (-1)^{|r_1||r_2|}r_2r_1$, where $|r_i| = \deg r_i$, $i = 1,2$. Let $\mathcal{C} = [R_f^B, R_f^B]$ be the linear space generated by $[r_1, r_2]$, r_1, $r_2 \in R_f^B$. It is clear that, $\mathcal{C} = [R_f^B, R_f^B]$ is a K-submodule of a K-module R_f^B. We construct the complex $([R_f^B, R_f^B] + \operatorname{Im}(1 - r^\epsilon))$, where $r^\epsilon(P) = \epsilon(-1)^{|P|(|P|-1)/2}P^*$, $*$ is an involutive on R_f^B, $\epsilon = \pm 1$. It is clear, from the definition of R_f^B, that $\operatorname{Im}(1 - r^\epsilon)$ is a subcomplex of R_f^B. We have

$$\begin{aligned}
\partial[r_1r_2] &= r_1r_2 - (-1)^{|r_1||r_2|}r_2r_1 \\
&= \partial r_1r_2 + (-1)^{|r_1|}r_1\partial r_2 - (-1)^{|r_1||r_2|}\left(\partial r_2r_1 + (-1)^{|r_2|}r_2\partial r_1\right) \\
&= \partial r_1r_2 - (-1)^{|r_2|(|r_1|+1)}r_2\partial r_1 + (-1)^{|r_1|}\left(r_1\partial r_2 - (-1)^{|r_1|(|r_2|+1)}\partial r_2r_1\right) \\
&= [\partial r_1, r_2] + (-1)^{|r_1|}[r_1, \partial r_2], |\partial r_i| = |r_i| - 1, \quad i = 1,2.
\end{aligned} \tag{3.1}$$

Then $[R_f^B, R_f^B]$ is a subcomplex in R_r^B. Therefore, the chain complex of the K-module $[R_f^B, R_f^B] + \operatorname{Im}(1 - r^\epsilon)$ is a subcomplex of R_f^B.

Definition 3.1. Let $f : A \to B$ be an involutive K-algebra (char $K = 0$) homomorphism, R_f^B be a free involutive algebra resolution of algebra B over f. Then the relative dihedral homology is defined as follows:

$${}^\epsilon\mathcal{HD}_i\left(A \xrightarrow{f} B\right) = \mathcal{H}_i\left(\frac{R_f^B}{\left[R_f^B, R_f^B\right] + \operatorname{Im}(1 - r^\epsilon)}\right). \tag{3.2}$$

The main properties of the relative dihedral homology are submitted in Theorems 3.2, 3.6, and 3.7.

Theorem 3.2. *Let A be an involutive algebra. Then ${}^\epsilon\mathcal{HD}_i(A \to 0) = {}^\epsilon\mathcal{HD}_{i-1}(A)$, where ${}^\epsilon\mathcal{HD}_i(A)$ is the dihedral homology of k-algebra A (*char$(k) = 0$*).*

Proof. To do this, we need the following definition and lemmas.

Definition 3.3. The K-algebra $A\langle t\rangle$, generated by the elements $a_0ta_1t\cdots ta_n$, $n \geq 0$, can be considered as an involutive DG algebra by requiring that the morphism $A \to A\langle t\rangle$ is a morphism of involutive differential graded algebras (A is viewed as a DG algebra concentrated in degree 0) and that $\deg t = 1$, $\partial t = 0$, and $t^* = t$.

Lemma 3.4. *The algebra $A\langle t\rangle$ is splitable and is a free involutive algebra resolution of the algebra $B = 0$ over the homomorphism $A \to 0$.*

Proof. Define the following chain complex

$$A \xleftarrow{\partial} AtA \xleftarrow{\partial} AtAtA \xleftarrow{\partial} \cdots \xleftarrow{\partial} At\cdots tA \xleftarrow{\partial} \cdots, \tag{3.3}$$

where $At\cdots tA$ (t-n-times) is a K-module and the boundary operator ∂ is given by

$$\begin{aligned}\partial(a_0ta_1t\cdots ta_{n-1}ta_n) &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t\cdots ta_i(\partial t)a_{i+1}t\cdots ta_n \\ &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t\cdots t(a_ia_{i+1})t\cdots ta_n.\end{aligned} \tag{3.4}$$

Note that the differential ∂ in $a\langle t\rangle$ is equivalent to the operator $\mathscr{C}'_n : C_n(A) \to C_{n-1}(A)$ (see [4]), defined by

$$\mathscr{C}'_n(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n. \tag{3.5}$$

Following [4], the complex $(C_n(A),\mathscr{C}'_n)$ is splitable and so the complex $A\langle t\rangle$ is also splitable, that is, $\mathscr{H}_\bullet(A\langle t\rangle) = 0$. Therefore, the algebra $A\langle t\rangle$ is a free involutive algebra resolution of the algebra $B = 0$ over the homomorphism $A \to 0$. □

Lemma 3.5. *The complex $A\langle t\rangle/[A,A\langle t\rangle]$ is the standard simplicial (Hochschild) complex.*

Proof. Consider the factor complex $A\langle t\rangle/[A,A\langle t\rangle]$. The complex $A\langle t\rangle/[A,A\langle t\rangle]$ is generated by the elements $a_0ta_1t\cdots a_{n-1}t$, since $a_0ta_1t\cdots ta_{n-1}ta_n = a_na_0 \times ta_1t\cdots ta_{n-1}t(\text{mod}[A,A\langle t\rangle])$. The action of the differential ∂ on the complex $A\langle t\rangle/[A,A\langle t\rangle]$ is given by

$$\begin{aligned}\partial(a_0ta_1t\cdots ta_{n-1}ta_n) &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t\cdots t(a_ia_{i+1})t\cdots a_{n-1}ta_nt \\ &\quad + (-1)^n a_0ta_1t\cdots a_{n-1}ta_n \\ &= \sum_{i=0}^{n-1}(-1)^i a_0ta_1t\cdots t(a_ia_{i+1})t\cdots a_{n-1}ta_nt \\ &\quad + (-1)^n a_na_0ta_1t\cdots a_{n-1}t.\end{aligned} \tag{3.6}$$

Consider the complex

$$A \xleftarrow{\text{id}} A \xleftarrow{\mathscr{C}} A^{\otimes 2} \xleftarrow{\mathscr{C}} \cdots \xleftarrow{\mathscr{C}} A^{\otimes n} \xleftarrow{\mathscr{C}} \cdots, \tag{3.7}$$

where $\mathscr{C}$ is the differential in the standard Hochschild complex (see [4]). Since the space $(A\langle t\rangle/[A,A\langle t\rangle])_{n+1}$ identifies with the space

$$A^{\otimes n+1} : a_0ta_1\cdots ta_nt \longrightarrow a_0\otimes a_1\otimes\cdots\otimes a_n, \tag{3.8}$$

and the differential in $A\langle t\rangle/[A,A\langle t\rangle]$ identifies with the differential in the standard Hochschild complex, $A\langle t\rangle/A+[A,A\langle t\rangle]$ is the Hochschild (simplicial) complex of algebra A. □

Now, we prove Theorem 3.2. Consider the factor complex: $A\langle t\rangle/[A\langle t\rangle, A\langle t\rangle]+\mathrm{Im}\,(1-r^{\epsilon})$, such that

$$\begin{aligned} a_0ta_1t\cdots ta_{n-1}t &= (-1)^n a_nta_0ta_1t\cdots a_{n-1}t, \\ a_0ta_1t\cdots ta_{n-1}t &= (-1)^{n(n+1)/2}\epsilon ta_n^* ta_{n-1}^*\cdots ta_1^* ta_0^* \\ &= (-1)^{n(n+1)/2}\epsilon ta_0^* ta_n^* t\cdots ta_1^* t, \end{aligned} \tag{3.9}$$

where $\epsilon=\pm 1$, $\deg a_0ta_1t\cdots ta_{n-1}t = n$, $\deg(a_nt)=1$, $\deg(a_n^*)=0$, $\deg a_0t\cdots a_nt = n+1$. The dihedral homology of $\langle t\rangle$ is the homology of the factor complex $A\langle t\rangle/[A\langle t\rangle, A\langle t\rangle]+\mathrm{Im}\,(1-r^{\epsilon})$. By factoring $A\langle t\rangle$, first by the subcomplex $A\leftarrow 0\leftarrow 0\leftarrow\cdots$ and then by the subcomplex $[A\langle t\rangle, A\langle t\rangle]+\mathrm{Im}\,(1-r^{\epsilon})$, we get a homomorphism ${}^{\epsilon}C\mathcal{D}_{\bullet}(A\to 0)\to {}^{\epsilon}C\mathcal{D}_{\bullet-1}(A)$, which induces an isomorphism in the dihedral homology groups ${}^{\epsilon}\mathcal{HD}_i(A\to 0)\to {}^{\epsilon}\mathcal{HD}_{-1}(A)$. □

Theorem 3.6. *Let $f: A\to B$ be a homomorphism of involutive algebras over a field K (char $K=0$). Then the relative dihedral homology ${}^{\epsilon}\mathcal{HD}_i(A\xrightarrow{f}B)$ does not depend on the choice of the resolution.*

Proof. The homomorphism f induces a homomorphism of chain complexes

$$f_{\bullet}: {}^{\epsilon}C\mathcal{D}_{\bullet}(A)\longrightarrow {}^{\epsilon}C\mathcal{D}_{\bullet}(B), \tag{3.10}$$

where ${}^{\epsilon}C\mathcal{D}_{\bullet}(A)$ is the dihedral complex. Consider the diagram

$$\begin{array}{ccc} & & R_f^B \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[f]{} & B, \end{array} \tag{3.11}$$

where R_f^B is defined above, i is the inclusion map. The idea of the proof is to show that the cone of the map i is quasi-isomorphic to an arbitrary category (see [2]), to the complex: $R_f^B/[R_f^B, R_f^B]+\mathrm{Im}\,(1-r^{\epsilon})$. Since

$$\mathcal{H}_i(R_f^B)=\begin{cases} B, & i=0, \\ 0, & i>0, \end{cases} \tag{3.12}$$

then the isomorphism $\pi_{\bullet}: {}^{\epsilon}C\mathcal{D}_{\bullet}(R_f^B)\to {}^{\epsilon}C\mathcal{D}_{\bullet}(B)$ induces an isomorphism of the homology of these complexes. Since $i_{\bullet}: {}^{\epsilon}C\mathcal{D}_{\bullet}(A)\to {}^{\epsilon}C\mathcal{D}_{\bullet}(R_f^B)$ is an inclusion, then $M(i_{\bullet})\approx {}^{\epsilon}C\mathcal{D}_{\bullet}(R_f^B)/{}^{\epsilon}C\mathcal{D}(A)$, where $M(i_{\bullet})$ is the cone of the map i (see [6]).

Note that the symbol $\approx$ always denotes a quasi-isomorphism. It is clear, from the above discussion, that the following diagram is commutative

$$\begin{array}{ccc} & & {}^{\epsilon}C\mathcal{D}_{\bullet}(R_f^B) \\ & \overset{i_{\bullet}}{\nearrow} & \downarrow{\scriptstyle \pi_{\bullet}} \\ {}^{\epsilon}CD_{\bullet}(A) & \xrightarrow[f_{\bullet}]{} & {}^{\epsilon}C\mathcal{D}_{\bullet}(B), \end{array} \tag{3.13}$$

and, hence, $M(f_\bullet) \approx {}^\epsilon C\mathfrak{D}_\bullet(R_f^B)/{}^\epsilon C\mathfrak{D}_\bullet(A)$. Following [1], we have $CC_\bullet(R_f^B)/CC_\bullet(A) \approx R_f^B/A + [R_f^B, R_f^B]$ and by using the spectral sequence $E_{ij}^2 = {}^\epsilon\mathcal{H}_\bullet(\mathbb{Z}/2, \mathcal{H}C_j(R_f^B)) = {}^\epsilon\mathcal{H}D_{i+j}(R_f^B)$, we have

$$\frac{{}^\epsilon C\mathfrak{D}_\bullet\left(R_f^B\right)}{{}^\epsilon C\mathfrak{D}_\bullet(A)} \approx \frac{R_f^B}{A + \left[R_f^B, R_f^B\right] + \operatorname{Im}(1 - r^\epsilon)}. \tag{3.14}$$

So, $M(f_\bullet) \approx R_f^B/A + [R_f^B, R_f^B] + \operatorname{Im}(1 - r^\epsilon)$. Then ${}^\epsilon\mathcal{H}\mathfrak{D}_i(A \xrightarrow{f} B)$ does not depend on the choice of R_f^B. □

Theorem 3.7. *Let A, B, and C be involutive algebras. Then the following sequence $A \xrightarrow{f} B \xrightarrow{g} C$ induces the long exact sequence of the relative dihedral homology*

$$\cdots \longrightarrow {}^\epsilon\mathcal{H}\mathfrak{D}_i\left(A \xrightarrow{f} B\right) \longrightarrow {}^\epsilon\mathcal{H}\mathfrak{D}_i\left(A \xrightarrow{g\circ f} C\right) \longrightarrow {}^\epsilon\mathcal{H}\mathfrak{D}_i\left(B \xrightarrow{g} C\right) \longrightarrow {}^\epsilon\mathcal{H}\mathfrak{D}_{i-1}\left(A \xrightarrow{f} B\right) \longrightarrow \cdots. \tag{3.15}$$

Proof. In Theorem 3.6, it has been proved that any homomorphism $f : A \to B$ of involutive algebras in an arbitrary category is equivalent to an inclusion $i : A \to R_f^B$

$$\begin{array}{ccc} & & R_f^B \\ & \overset{i}{\nearrow} & \downarrow \wr \\ A & \xrightarrow{f} & B. \end{array} \tag{3.16}$$

Then, for an arbitrary sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of involutive algebras, we have the following complex

$$\begin{array}{ccccccc} A & \xrightarrow{i} & R_f^B & = & B & \xrightarrow{i'} & R_g^C \\ & \underset{f}{\searrow} & \| & & & \underset{g}{\searrow} & \| \\ & & B & & & & C. \end{array} \tag{3.17}$$

Consider the following sequence of mapping cones

$$0 \longrightarrow M(i_\bullet) \longrightarrow M(i'_\bullet) \longrightarrow M(i'_\bullet 0 i_\bullet) \longrightarrow 0. \tag{3.18}$$

In general, the sequence (3.18) is not exact. In fact, the composition of two morphisms will be non zero. However, the cone over the morphism $M(i_\bullet) \to M(i'_\bullet)$ is canonically homotopy equivalent to $M(i'_\bullet 0 i_\bullet)$. So, we get the following long exact sequence of the relative dihedral homology

$$\begin{aligned} \cdots \longrightarrow {}^\epsilon\,\mathcal{H}\mathfrak{D}_i\left(A \xrightarrow{f} B\right) &\longrightarrow {}^\epsilon\,\mathcal{H}\mathfrak{D}_i\left(A \xrightarrow{g\circ f} C\right) \\ \longrightarrow {}^\epsilon\,\mathcal{H}\mathfrak{D}_i\left(B \xrightarrow{g} C\right) &\longrightarrow {}^\epsilon\,\mathcal{H}\mathfrak{D}_{i-1}\left(A \xrightarrow{f} B\right) \longrightarrow \cdots. \end{aligned} \tag{3.19}$$

□

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