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BOUNDED SETS IN THE RANGE OF AN X^{**}-VALUED MEASURE WITH BOUNDED VARIATION

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ABSTRACT. Let X be a Banach space and $A \subset X$ an absolutely convex, closed, and bounded set. We give some sufficient and necessary conditions in order that A lies in the range of a measure valued in the bidual space X^{**} and having bounded variation. Among other results, we prove that X^* is a G. T.-space if and only if A lies inside the range of some X^{**}-valued measure with bounded variation whenever X_A is isomorphic to a Hilbert space.

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1. Introduction. If X is an infinite dimensional Banach space, it is well known that its unit ball cannot lie inside the range of an X^{**}-valued measure with bounded variation. This paper is devoted to a study of bounded sets in a Banach space having that property. Since the convex and closed hull of the range of a measure is the range of another one (see Diestel and Uhl [3]), we can reduce to consider bounded subsets which are closed and absolutely convex. Such a set is called a Banach disc. If $A \subset X$ is a Banach disc, recall that $\cup_{i=1}^{\infty} nA$ is a vector space and that it can be endowed with the norm

$$\rho_A(x) = \inf\{\lambda > 0 : x \in \lambda A\} \quad \text{for all } x \in \bigcup_{i=1}^{\infty} nA. \tag{1.1}$$

It is easy to prove that $(\cup_{i=1}^{\infty} nA, \rho_A)$is a Banach space which will be denoted by X_A and that A is the unit closed ball of this space. j_A will denote the canonical map from X_A into X. Since $j_A(A) = A$, it follows that j_A is a bounded linear map (see Junek [4]).

First of all, we obtain characterizations of linear operators $T : X \to Y$ that take the unit ball of X into a subset of Y lying in the range of some Y-valued (respectively, Y^{**}-valued) measure with bounded variation. Concretely, we prove that operators belonging to $\mathcal{W} \circ \Pi_1^d(X,Y)$ take the unit ball B_X into a set lying in the range of some Y-valued measure with bounded variation. Nevertheless, we show that there exist operators satisfying that property but they do not belong to the class $\mathcal{W} \circ \Pi_1^d$.

Next we give some sufficient and necessary conditions in order that a Banach disc A in X lies in the range of some X^{**}-valued measure with bounded variation. Under certain conditions about A we prove that X_A must be isomorphic to a Hilbert space. When X_A is isomorphic to a Hilbert space we say that A is a Hilbert disc. Among other results, we show that the Banach spaces whose dual spaces are G. T.-spaces are the only Banach spaces for which every Hilbert disc lies inside the range of some X^{**}-valued measure with bounded variation.

Finally, we consider a class of subsets of $\Pi_1(X,Y)$ which we call (μ,c)-uniformly dominated. Such a set is a subset M of $\Pi_1(X,Y)$ with the property

$$\|Tx\| \le c\int_{B_{X^*}} |\langle x,x^*\rangle|\,d\mu(x^*) \quad \text{for all } x\in X,\ T\in M, \tag{1.2}$$

where c is a positive constant and μ is a probability measure on $(B_{X^*}, \star\text{-weak})$. Questions regarding the finer structure of these sets have found interest since Grothendieck-Pietsch's discovery of their famous Domination theorem. We prove that M is uniformly dominated if and only if there exists some X^*-valued measure m with bounded variation such that

$$T^*(B_{Y^*}) \subset \operatorname{rg}(m) \quad \text{for all } T\in M. \tag{1.3}$$

We use the classical notation in Banach spaces theory. We consider all Banach spaces over real numbers. If X is a Banach space, X^* will denote its dual space and B_X its closed unit ball. For a subset K of X, $\overline{\operatorname{aco}}(K)$ will be the absolutely convex and closed hull of K.

We consider only countably additive measures defined on σ-algebras. If Σ is a σ-algebra of subsets of a set Ω, X is a Banach space and $m:\Sigma\to X$ is such a measure, we denote by $|m|$ the total variation of m. If $|m|$ is finite, we say that m has bounded variation. The range of m is denoted by $\operatorname{rg}(m)$, that is, $\operatorname{rg}(m)=\{m(A): A\in\Sigma\}$.

2. Banach operators taking the unit ball inside the range of some vector measure. Following Piñeiro [6], we denote by $\mathcal{R}_{bv}(X,Y)$ the vector space of linear operators taking compact subsets of X into sets that lie inside the range of a Y-valued measure with bounded variation. The author proved that an operator T belongs to $\mathcal{R}_{bv}$ if and only if the adjoint map $T^*: Y^*\to X^*$ is 1-summing. Next we prove that, in fact, these operators take the unit ball of X inside the range of some Y^{**}-valued measure with bounded variation.

Theorem 2.1. *Let X and Y be Banach spaces. Suppose that $T: X\to Y$ is a bounded operator. T maps the unit ball B_X in a set lying in the range of a Y^{**}-valued measure with bounded variation if and only if the adjoint map T^* is 1-summing.*

Proof. $(\Rightarrow)$ Let $T: X\to Y$ be an operator such that $T(B_X)$ lies inside the range of a Y^{**}-valued measure with bounded variation. Obviously, the map $i_Y\circ T$ belongs to $R_{bv}(X,Y^{**})$ and by Piñeiro [6] the map $(i_Y\circ T)^*$ is 1-summing. Then $T^*: Y^*\to X^*$ is 1-summing too.

$(\Leftarrow)$ Let $T: X\to Y$ be an operator such that its adjoint is 1-summing. In light of Grothendieck-Pietsch's Domination theorem (see Diestel et al. [2]) there is a regular Borel probability measure μ on $(B_{Y^{**}}, \text{weak}^*)$ such that $T^*: Y^*\to X^*$ factors through a subspace H of $L^1(\mu)$ in the way

$$\begin{array}{ccc} Y^* & \xrightarrow{\quad T^* \quad} & X^* \\ & {}_{C}\searrow \quad \nearrow_{D} & \\ & H, & \end{array} \tag{2.1}$$

where C and D are bounded linear maps and $\tilde{C}: Y^* \to L^1(\mu)$ is integral. By duality we have

$$\begin{array}{ccccc} X & \xrightarrow{D_0^*} & H^* & \xrightarrow{C^*} & Y^{**} \\ & & \phi \uparrow & \nearrow_{\tilde{C}^*} & \\ & & L^\infty(\mu), & & \end{array} \tag{2.2}$$

where D_0^* denotes the restriction map of D^* to X and ϕ is the quotient map. $\tilde{C}^*$ is integral and ω^*-ω continuous. So, if we define a Y^{**}-valued measure by $m(E) = \tilde{C}^*(x_E)$ for all Borel set E in $(B_{Y^{**}}, \text{weak}^*)$, m is a countably additive Y^{**}-valued measure having bounded variation (see Diestel-Uhl [3]). By Piñeiro [6], there is another Y^{**}-valued measure $\tilde{m}$ such that $\tilde{C}^*(B_{L^\infty(\mu)}) = \operatorname{rg}(\tilde{m})$ and $|\tilde{m}| \le 2|m|$. Finally, we have

$$\begin{aligned} T(B_X) &= C^* \circ D_0^*(B_X) \subset \|D_0^*\| C^*(B_{H^*}) \\ &= \|D_0^*\| C^*\left(\overline{\phi(B_{L^\infty(\mu)})}\right) \subset \|D_0^*\| \overline{C^*(\phi(B_{L^\infty(\mu)}))} \\ &= \|D_0^*\| \overline{\tilde{C}^*(B_{L^\infty(\mu)})} = \|D_0^*\| \tilde{C}^*(B_{L^\infty(\mu)}) = \|D_0^*\| \operatorname{rg}(\tilde{m}). \end{aligned} \tag{2.3}$$

Now, we give a characterization of operators $T: X \to Y$ taking the unit ball of X inside the range of some Y-valued measure with bounded variation. □

Theorem 2.2. *Let X and Y be Banach spaces and $T: X \to Y$ an operator. $T(B_X)$ lies inside the range of some Y-valued measure with bounded variation if and only if T^* factors through a subspace H of an $L^1(\mu)$-space*

$$\begin{array}{ccc} Y^* & \xrightarrow{\quad T^* \quad} & X^* \\ & {}_{A}\searrow \quad \nearrow_{B} & \\ & H \subset L^1(\mu), & \end{array} \tag{2.4}$$

where μ is a positive and finite measure, $A: Y^* \to L^1(\mu)$ is integral and $\star$-weak-weak continuous, and $B: H \to X^*$ is a bounded operator.

Proof. We only prove the "if part" because the "only if part" is similar to the above theorem. Let $T: X \to Y$ be an operator such that $T(B_X) \subset \operatorname{rg}(m)$, $m: \Sigma \to Y$ is a vector measure with bounded variation. We denote by μ the variation measure of m. The integration operator $I: L^\infty(\mu) \to Y$ defined by $I(f) = \int f\, dm$ for all $f \in L^\infty(\mu)$ is 1-summing and, therefore, integral (see Diestel-Uhl [3]). So $I^*: Y^* \to L^\infty(\mu)^*$ is integral, but

$$I^*(y^*) = \frac{d(y^* \circ m)}{d\mu} \in L^1(\mu) \quad \text{for all } y^* \in Y^*. \tag{2.5}$$

Then the rank of I^* is contained in $L^1(\mu)$ and the map

$$A : y^* \in Y^* \longrightarrow \frac{d(y^* \circ m)}{d\mu} \in L^1(\mu) \tag{2.6}$$

is $\star$-weak-weak continuous and integral. Finally, let

$$H = \overline{\{Ay^* : y^* \in Y^*\}}^{L^1(\mu)}, \tag{2.7}$$

and define $B : H \to X^*$ by $B(Ay^*) = T^*y^*$. Since $T(B_X) \subset \mathrm{rg}(m)$, it is easy to prove that B is well defined and continuous. □

Remark 2.3. (a) In [5], it is showed that there exist operators taking the unit ball in the range of some Y^{**}-valued measure with bounded variation, but they do not take it in the range of some Y-valued measure of bounded variation.

(b) Suppose that $T_1 : X \to Y$ is an operator whose adjoint map T_1^* is 1-summing and $T_2 : Y \to Z$ is a weakly compact operator. Then, Theorem 2.2 tells us that the map $T_2 \circ T_1$ takes the unit ball of X into a subset of Z lying in the range of some Z-valued measure with bounded variation. Now, we prove that the converse is not true, i.e., there exist operators taking the unit ball inside the range of some vector measure with bounded variation but they do not belong to the class $\mathcal{W} \circ \mathcal{R}_{bv}$ (here $\mathcal{W}$ denotes the operator ideal of all weakly compact operators). To see this, we need the following result.

Lemma 2.1. *Let $T = T_2 \circ T_1$ where $T_1 \in \mathcal{R}_{bv}(X,Y)$ and $T_2 \in \mathcal{W}(Y,Z)$. Then the sequence (Tx_n) lies inside a countable sum of segments whenever (x_n) is a bounded sequence in X.*

Recall that a countable sum of segments in a Banach space X is a set of the form

$$\sum_{n=1}^{\infty} [-\omega_n, \omega_n] = \left\{ \sum_{n=1}^{\infty} \alpha_n \omega_n : (\alpha_n) \in B_{\ell_\infty} \right\}, \tag{2.8}$$

where $\sum \omega_n$ is an absolutely convergent series in X. Such a set is obviously the range of a vector measure with bounded variation. Piñeiro [7] proved that a sequence (x_n) lies inside a countable sum of segments if and only if the operator

$$(\alpha_n) \in \ell_1 \longrightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X \tag{2.9}$$

is nuclear.

Proof. Given a bounded sequence (x_n) in X, we define the operator

$$S : (\alpha_n) \in \ell_1 \longrightarrow \sum_{n=1}^{\infty} \alpha_n T x_n \in Z, \tag{2.10}$$

where S is the composition of $T_0 : (\alpha_n) \in \ell_1 \to \Sigma \alpha_n T_1 x_n \in \mathbb{N}Y$ and $T_2 : Y \to Z$. It follows from Theorem 2.1 that the sequence $(T_1 x_n)$ lies inside the range of some Y^{**}-valued measure with bounded variation. By [7, Lem 2] T_0 is integral. Now by Grothendieck's theorem (see Diestel and Uhl [3]) the composition $T_2 \circ T_0$ is nuclear.

Now we give the counterexample. Anantharaman and Diestel [1] showed that the unit ball of ℓ_2 is the range of a c_0-valued measure of bounded variation. Nevertheless, the canonical basis (e_n) does not lie in a countable sum of segments in c_0 because the identity operator from ℓ_1 into c_0 is not nuclear. □

3. Banach discs lying in the range of some X^{**}-valued measure. Piñeiro and Rodriguez-Piazza [8] proved that only finite dimensional Banach spaces X have the property that every compact subset is contained in the range of some X-valued measure with bounded variation. Then a natural question arises: given an infinite dimensional Banach space X, which bounded subsets of X have the following property (P)?

(P) *"Every compact subset of A is contained in the range of an X-valued measure with bounded variation"*

We have obtained the following results:

Theorem 3.1. *Let X be a Banach space. If $A \subset X$ is a Banach disc with property (P), then A is contained in the range of some X^{**}-valued measure with bounded variation.*

Proof. If A is a Banach disc having property (P), it is obvious that the operator $j_A : X_A \to X$ maps every compact $K \subset A$ (compact in X_A) into a set lying inside the range of a X-valued measure with bounded variation. Again the theorem of Piñeiro [6] tells us that $(j_A)^* : X^* \to (X_A)^*$ is 1-summing. Theorem 2.1 concludes the proof. □

Remark. By Theorem 2.1, a set $A \subset X$ lies inside the range of some X^{**}-valued measure with bounded variation if and only if $(j_A)^* : X^* \to (X_A)^*$ is 1-summing. According to Grothendieck-Pietsch's theorem, there exist a regular Borel positive measure μ on $(B_{X^{**}}, \star\text{-weak})$ such that

$$\|(j_A)^*(x^*)\| \le \int_{B_{X^{**}}} |\langle x^*, x^{**}\rangle| \, d\mu \quad \text{for all } x^* \in X^*. \tag{3.1}$$

On the other hand, we have

$$\|(j_A)^*(x^*)\| = \sup_{a\in A} |\langle a, (j_A)^*(x^*)\rangle| = \sup_{a\in A} |\langle a, x^*\rangle|. \tag{3.2}$$

So, (3.1) can be written in the form

$$\sup_{a\in A} |\langle a, x^*\rangle| \le \int_{B_{X^{**}}} |\langle x^*, x^{**}\rangle| d\mu \quad \text{for all } x^* \in X^*. \tag{3.3}$$

Note that (3.3) implies that the operator

$$x^* \in X^* \longrightarrow (\langle a, x^*\rangle)_{a\in A} \in \ell^\infty(A) \tag{3.4}$$

is integral. So we have obtained the following result.

Theorem 3.2. *Let X be a Banach space and $A \subset X$ a Banach disc. The following statements are equivalent:*

(i) *The adjoint operator of $j_A : X_A \to X$ is 1-summing.*
(ii) *The operator $(\psi_a) \in \ell^1(A) \to \Sigma_{a\in A}\psi_a a \in X$ is integral.*
(iii) *A lies inside the range of some X^{**}-valued measure with bounded variation.*

As usual "$\circ$" denotes the polar in the duality $\langle X, X^*\rangle$, "$\bullet$" the polar in the duality $\langle X^*, X^{**}\rangle$ and $X^*_{A^\circ}$ the vector space $X^*_{/p^{-1}_{A^\circ}\{0\}}$ endowed with the norm $\tilde{P}_{A^\circ}(x^* + P^{-1}_{A^\circ}\{0\}) = P_{A^\circ}(x^*)$, for all $x^* \in X^*$ (see Junek [4]). Recall that $P_{A^\circ}(x^*) = \sup_{x\in A}|\langle x, x^*\rangle|$ for all $x^* \in X^*$. In general, $X^*_{A^\circ}$ is not complete.

Theorem 3.3. *Let X be a Banach space and $A \subset X$ a Banach disc such that $X^*_{A^\circ}$ is a Banach space. If A is contained in the range of some X^{**}-valued measure with bounded variation, then the Banach space X_A is isomorphic to a Hilbert space.*

Proof. We first prove that $(X^{**})_{A^{\circ\bullet}}$ is isomorphic to a Hilbert space. As we have mentioned earlier, there is a regular Borel positive measure μ on $(B_{X^{**}}, \text{weak}^*)$ satisfying (3.1). We may define on X^* a scalar product $(\cdot/\cdot)$ by letting

$$(x^*/y^*) = \int_{B_{X^{**}}} \langle x^*, x^{**}\rangle\langle y^*, y^{**}\rangle d\mu \quad \text{for all } x^*, y^* \in X^*. \tag{3.5}$$

We denote by $p(\cdot)$ the associate seminorm defined by

$$p(x^*) = \left(\int_{B_{X^{**}}} |\langle x^*, x^{**}\rangle|^2 d\mu\right)^{1/2} \quad \text{for all } x^* \in X^*. \tag{3.6}$$

By Hölder's inequality we have

$$\int_{B_{X^{**}}} |\langle x^*, x^{**}\rangle| d\mu \le \mu(B_{X^{**}})^{1/2} p(x^*) \quad \text{for all } x^* \in X^*. \tag{3.7}$$

Therefore, (3.3) and (3.7) yields

$$\sup_{a\in A} |\langle a, x^*\rangle| \le \mu(B_{X^{**}})^{1/2} p(x^*) \le \mu(B_{X^{**}})\|x^*\| \quad \text{for all } x^* \in X^*. \tag{3.8}$$

Now we consider the vector space $X^*_{/p^{-1}\{0\}}$ endowed with the scalar product

$$\big((x^* + p^{-1}\{0\})/(y^* + p^{-1}\{0\})\big) = (x^*/y^*) \quad \text{for all } x^*, y^* \in X^*. \tag{3.9}$$

We can define a linear map $J : X^*_{/p^{-1}\{0\}} \to X^*_{A^\circ}$, $J(x^* + p^{-1}\{0\}) = x^* + p^{-1}_{A^\circ}\{0\}$ for all $x^* \in X^*$. From (3.8) it follows that J is well defined and continuous. Obviously, it is a surjection. Then $X^*_{A^\circ}$ is isomorphic to a quotient of the prehilbertian space $X^*_{/p^{-1}\{0\}}$. This implies that $(X^*_{A^\circ})^*$ is isomorphic to a subspace of the Hilbert space $(X^*_{/p^{-1}\{0\}})^*$. As $(X^*_{A^\circ})^*$ and $X^{**}_{A^{\circ\bullet}}$ are isometric (see Junek [4]), we have proved that $X^{**}_{A^{\circ\bullet}}$ is isomorphic to a Hilbert space. Finally, we show that X_A is isomorphic to a subspace of $X^{**}_{A^{\circ\bullet}}$. By the open map theorem it suffices to prove that X_A, endowed with the restriction of $p_{A^{\circ\bullet}}$ to X_A, is complete because we have the relation

$$p_{A^{\circ\bullet}}(x) \le p_A(x) \quad \text{for all } x \in X_A. \tag{3.10}$$

To see this, let (x_n) be a Cauchy sequence in $X^{**}_{A^{\circ\bullet}}$ for which $x_n \in X_A$ for all $n \in \mathbb{N}$. In particular, (x_n) is bounded for $p_{A^{\circ\bullet}}$. Then there is a constant $c > 0$ such that $x_n \in cA^{\circ\bullet}$ for all $n \in \mathbb{N}$. Clearly, (x_n) converges in $X^{**}_{A^{\circ\bullet}}$ to a limit $x^{**} \in cA^{\circ\bullet}$, and therefore, $x^{**} = \lim x_n$ in X^{**} too. Since X is closed in X^{**} it follows that $x^{**} \in X$. This shows that $x^{**} \in cA^{\circ\bullet}$. As $A^{\circ\bullet} = A$, the theorem is established. $\square$

Remark 3.4. (a) It is known that the convergence for the norm $\|\cdot\|$ does not imply the convergence for p_A. We are going to show an easy example. Let $X = L^1[0,1]$ and $A = \overline{\text{aco}}\{\mathscr{X}_{[k-1/n,k/n]} : 1 \le k \le n, n \in \mathbb{N}\}$. Obviously, A is contained in the range of some X-valued measure of bounded variation, then $j_A \in \mathscr{R}_{bv}(X_A, X)$. By Piñeiro's theorem [6], j_A maps every null sequence from X_A into a sequence lying in a countable sum of segments. But again Piñeiro [6] proved that the null sequence $(\mathscr{X}_{[k-1/n,k/n)]})_{n\in\mathbb{N},1\le k\le n}$ is not contained in any countable sum of segments. Then this sequence does not converge in X_A.

We do not know whether every Banach disc $A \subset X$ lying in the range of some X^{**}-valued measure of bounded variation has property (P).

(b) If A is the unit ball of a closed subspace Y of X, then X_A is isometric to Y and $X^*_{A^\circ}$ is isometric to the quotient space $X^*/Y^\perp$. So $X^*_{A^\circ}$ is a Banach space. In this case, we have the following complete result.

Theorem 3.5. *Let X be a Banach space and Y a closed subspace of X. B_Y has property (P) if and only if B_Y lies inside the range of some X^{**}-valued measure having bounded variation. If this is the case, Y is isomorphic to a Hilbert space.*

(c) Unfortunately, Theorem 3.2 is not true when $X^*_{A^\circ}$ is not complete. For example, consider the set $A = B_{L^\infty[0,1]}$ in $L^1[0,1]$. A is a Banach disc in $L^1[0,1]$ that lies in the range of some $L^1[0,1]$-valued measure of bounded variation since the identity operator from $L^\infty[0,1]$ into $L^1[0,1]$ is Pietsch integral (see Diestel and Uhl [3]). But X_A is isometric to $L^\infty[0,1]$, so it cannot be isomorphic to a Hilbert space.

From now on in this section we suppose that the space under consideration belongs to a particular class. To start, we consider the Hilbert case.

Theorem 3.6. *Let X be a Hilbert space and $A \subset X$ a Banach disc such that X_A is isomorphic to a Hilbert space. Then the set A has property (P) if and only if*

$$\sum_{i\in I} \sup_{a\in A} \left| \left(\frac{e_i}{a} \right) \right|^2 < +\infty \tag{3.11}$$

for some orthonormal basis $(e_i)_{i\in I}$ in X.

Proof. Let $A \subset X$ be a Banach disc for which X_A is isomorphic to a Hilbert space H. If $J : H \to X_A$ is an isomorphism and $(e_i)_{i\in I}$ an orthonormal basis in X, we have

$$\sum_{i\in I} \sup_{a\in A} \left| \left(\frac{e_i}{a} \right) \right|^2 = \sum_{i\in I} \|(j_A)^* e_i\|^2 \le \|(J^*)^{-1}\|^2 \sum_{i\in I} \|J^* \circ (j_A)^* e_i\|^2, \tag{3.12}$$

on the other hand,

$$\sum_{i\in I} \|J^* \circ (j_A)^* e_i\|^2 \le \|J^*\|^2 \sum_{i\in I} \|(j_A)^* e_i\|^2 = \|J^*\|^2 \sum_{i\in I} \sup_{a\in A} \left| \left(\frac{e_i}{a} \right) \right|^2. \tag{3.13}$$

This proves that the map $(j_A \circ J)^* : X \to H$ is Hilbert-Schmidt if and only if (3.11) holds. As J is an isomorphism, it follows that $(j_A)^*$ is 1-summing only in this case. $\square$

Corollary 3.7. *Let X be a Hilbert space and $A = \{\Sigma\alpha_n x_n : (\alpha_n) \in B_{\ell_2}\}$, where (x_n) is a sequence belonging to $\ell^2_\omega(X)$. The set A has property (P) if and only if $\Sigma\|x_n\|^2 < +\infty$.*

Proof. In fact, if $(e_i)_{i\in I}$ is an orthonormal basis in X, by Parseval's identity we have

$$\begin{aligned}\sum_{i\in I}\sup_{a\in A}\left|\left(\frac{e_i}{a}\right)\right|^2 &= \sum_{i\in I}\sup_{\alpha\in B_{\ell_2}}\left|\left(\frac{e_i}{\sum_{n=1}^{\infty}\alpha_n x_n}\right)\right|^2 = \sum_{i\in I}\sup_{\alpha\in B_{\ell_2}}\left|\sum_{n=1}^{\infty}\alpha_n\left(\frac{e_i}{x_n}\right)\right|^2 \\ &= \sum_{i\in I}\sum_{n=1}^{\infty}\left|\left(\frac{e_i}{x_n}\right)\right|^2 = \sum_{n=1}^{\infty}\sum_{i\in I}\left|\left(\frac{e_i}{x_n}\right)\right|^2 = \sum_{n=1}^{\infty}\|x_n\|^2.\end{aligned} \tag{3.14}$$

□

Now we are going to obtain an interesting result in case X^* is a G. T.-space. Recall that a Banach space is called a G. T.-space if $\Pi_1(X,H) = \mathcal{L}(X,H)$ for all Hilbert space H (see Pisier [9]).

Theorem 3.8. *Let X be a Banach space. The following statements are equivalent:*
(i) *X^* is a G. T.-space.*
(ii) *A Banach disc $A \subset X$ is contained in the range of some X^{**}-valued measure with bounded variation whenever X_A is isomorphic to a Hilbert space.*

Proof. The implication (i)⇒(ii) follows directly from Theorem 2.1 and the definition of G. T.-spaces. So, let X be a Banach space satisfying (ii). By Piñeiro [7], we only need to prove that $\Pi_1(\ell_1,X) = \mathcal{I}(\ell_1,X)$. To this end, let (x_n) be a sequence in X such that the operator $T : (\alpha_n) \in \ell_1 \to \Sigma\alpha_n x_n \in X$ is 1-summing. By [7], there exists a sequence $(y_n) \in \ell^2_\omega(X)$ such that

$$x_n \in \left\{\sum \alpha_n y_n : (\alpha_n) \in B_{\ell_2}\right\}. \tag{3.15}$$

Put $A = \{\Sigma\alpha_n y_n : (\alpha_n) \in B_{\ell_2}\}$. As X_A is isomorphic to a Hilbert space, by hypothesis A is contained in the range of some X^{**}-valued measure with bounded variation. In particular, so is (x_n). Again by Piñeiro [7], T is integral. □

Next theorem proves that in $\mathcal{L}_1$-spaces a Hilbert disc has the property (P) if and only if it is contained in some countable sum of segments in X.

Theorem 3.9. *Let X be an $\mathcal{L}_1$ space and $A \subset X$ a Hilbert disc. The following statements are equivalent:*
(i) *A has property (P).*
(ii) *$j_A : X_A \to X$ is nuclear.*
(iii) *A is contained in some countable sum of segments in X.*

Proof. (i)⇒(ii) If $A \subset X$ has property (P), Theorem 3.1 tells us that A lies inside the range of some X^{**}-valued measure with bounded variation. By Theorem 2.1, $(j_A)^* : X^* \to (X_A)^*$ is 1-summing. Since X^* is an $\mathcal{L}_\infty$-space it follows that $(j_A)^*$ is integral. Finally, $(j_A)^*$ is nuclear because $(X_A)^*$ is a dual space with Radon-Nikodym property (X_A is isomorphic to a Hilbert space). As Hilbert spaces have the approximation property, j_A itself is nuclear (see Diestel and Uhl [3]).

The other implications are obvious. □

4. (μ,c)-uniformly dominated sets. We finish our work by obtaining characterization, in terms of ranges of vector measures, of (μ,c)-uniformly dominated subsets of $\Pi_1(X,Y)$.

Theorem 4.1. *Let X and Y be Banach spaces, and M a subset of $\Pi_1(X,Y)$. M is uniformly dominated if and only if there exists some X^*-valued measure m of bounded variation such that*

$$T^*(B_{Y^*}) \subset \mathrm{rg}(m) \quad \textit{for all } T \in M. \tag{4.1}$$

Proof. $(\Rightarrow)$ Let $M \subset \Pi_1(X,Y)$ be a (μ,c)-uniformly dominated set, i.e., we have

$$\|Tx\| \le c\int_{B_{X^*}} |\langle x,x^*\rangle|\,d\mu(x^*) \quad \text{for all } x \in X,\ T \in M, \tag{4.2}$$

where μ is a regular Borel probability measure on B_{X^*}. This yields

$$\sup_{a\in A}|\langle x,a\rangle| \le c\int_{B_{X^*}} |\langle x,x^*\rangle|\,d\mu(x^*) \quad \text{for all } x \in X, \tag{4.3}$$

where $A = \cup_{T\in M}T^*(B_{Y^*})$. By (4.3), the operator

$$U : x \in X \to \big(\langle x,a\rangle\big)_{a\in A} \in \ell_\infty(A) \tag{4.4}$$

is 1-summing. So is $U^{**} : X^{**} \to \ell_\infty(A)$, in particular, U^{**} is integral. Now it suffices to notice that $U^{**}(x^{**}) = (\langle a,x^{**}\rangle)_{a\in A}$. Since there is a projection from X^{***} in X^*, it follows from Theorem 3.2 that A is contained in the range of some X^*-valued measure of bounded variation.

Conversely, suppose $M \subset \Pi_1(X,Y)$ is a set satisfying $T^*(B_{Y^*}) \subset \mathrm{rg}(m)$ for all $T \in M$, m being some X^*-valued measure with bounded variation. Put $A = \overline{\mathrm{aco}}(\cup_{T\in M}\ T^*(B_{Y^*}))$. A is a Banach disc in X^* that lies inside the range of some X^*-valued measure with bounded variation. According to Theorem 3.2, the operator $x^{**} \in X^{**} \to (\langle a,x^{**}\rangle)_{a\in A} \in \ell_\infty(A)$ is integral. So is $x \in X \to (\langle x,a\rangle)_{a\in A} \in \ell_\infty(A)$. By Grothendieck-Pietsch's theorem there exists a regular Borel positive measure μ on B_{X^*} such that

$$\sup_{a\in A}|\langle x,x^*\rangle| \le \int_{B_{X^*}} |\langle x,x^*\rangle|\,d\mu(x^*) \quad \text{for all } x \in X. \tag{4.5}$$

This yields

$$\|Tx\| \le \int_{B_{X^*}} |\langle x,x^*\rangle|\,d\mu(x^*) \quad \text{for all } x \in X \text{ and all } T \in M. \tag{4.6}$$

□

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