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SPATIAL NUMERICAL RANGES OF ELEMENTS OF SUBALGEBRAS OF $C_0(X)$

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To Professor Junzo Wada on his retirement from Waseda University

Abstract. When A is a subalgebra of the commutative Banach algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact Hausdorff space X, the spatial numerical range of element of A can be described in terms of positive measures.

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1. Introduction and results. Let X be a locally compact Hausdorff space and $C_0(X)$ the commutative Banach algebra (with supremum norm $\|\|_\infty$) of all continuous complex-valued functions on X which vanish at infinity. Let A be a subalgebra (not necessarily closed) of $C_0(X)$, A^* the dual space of A and $f \in A$. If A is unital, then

$$V_1(A,f) \equiv \{m(f) : m \in A^* \text{ and } \|m\| = m(1) = 1\} \tag{1.1}$$

is called the (algebra) numerical range of f and it is a nonempty compact convex subset of the complex plane C (cf. [1, page 52]). However if A is nonunital, then the above definition is not meaningful. In this case, Gaur and Husain [2] introduced the following set:

$$V(A,f) = \{m(fg) : \exists m \in A^* \text{ and } g \in A \text{ such that } \|m\| = \|g\|_\infty = m(g) = 1\} \tag{1.2}$$

and studied the spatial numerical range in a nonunital algebra. The set $V(A,f)$ is equal to $V_1(A,f)$ whenever A is unital.

In [2], Gaur and Husain proved the following result.

Theorem 1.1. *Let f be an element of $C_0(X)$. Then*

$$\operatorname{co} R(f) \subseteq V(C_0(X),f) \subseteq \overline{\operatorname{co}}\, R(f), \tag{1.3}$$

where co *and* $\overline{\text{co}}$ *denote the convex hull and the closed hull, respectively, and $R(f)$ is the range of the function f.*

In this paper, we describe spatial numerical ranges of elements of subalgebras of $C_0(X)$ in terms of positive measures and show that Theorem 1.1 also holds for the subalgebra of $C_0(X)$. Let $M(X)$ denote the measure space of all bounded regular Borel measures on X. Our main result is the following theorem.

THEOREM 1.2. *Let A be a subalgebra of $C_0(X)$ and $f \in A$. Then*

(i) $V(A,f) = \{\int f d|\mu| : \exists \mu \in M(X)$ *and* $\exists g \in A$ *such that* $\|\mu\| = \|g\|_\infty = \int g d\mu = 1\} \subseteq \overline{\text{co}}\, R(f)$, *where* $|\mu|$ *denotes the total variation of* μ.

(ii) *If A has the following property (#), then* $\text{co}\, R(f) \subseteq V(A,f)$.

(#) *For any finite set* $\{x_1,\dots,x_n\}$ *in X, there exists $g \in A$ such that* $\|g\|_\infty = 1$ *and* $g(x_1) = \cdots = g(x_n) = 1$.

COROLLARY 1.3. *Let A be a $*$-subalgebra of $C_0(X)$ and $f \in A$. Then*

(i)

$$V(A,f) = \left\{ \int f\, d\mu : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = 1,\ \mu \geq 0,\ 0 \leq g \leq 1 \text{ and } \int g\, d\mu = 1 \right\}. \tag{1.4}$$

(ii) *If A has the following property (##), then*

$$V(A,f) = \left\{ \int f\, d\mu : 0 \leq \mu \in M(X),\ \|\mu\| = 1 \text{ and } \operatorname{supp}(\mu) \text{ is compact} \right\}. \tag{1.5}$$

(##) *For any compact set $E \subseteq X$, there exists $g \in A$ such that $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in E$. Here* $\operatorname{supp}(\mu)$ *denotes the support of* μ.

REMARK 1.4. If $A = C_0(X)$, then A satisfies the desired properties appeared in Theorem 1.2 and Corollary 1.3. Hence, we have

$$V(C_0(X),f) = \left\{ \int f\, d\mu : 0 \leq \mu \in M(X),\ \|\mu\| = 1 \text{ and } \operatorname{supp}(\mu) \text{ is compact} \right\} \tag{1.6}$$

and

$$\text{co}\, R(f) \subseteq V(C_0(X),f) \subseteq \overline{\text{co}}\, R(f). \tag{1.7}$$

2. Proofs of results

PROOF OF THEOREM 1.2. (i) By the Hahn-Banach extension theorem, we have

$$V(A,f) = \left\{ \int fg\, d\mu : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g\, d\mu = 1 \right\}. \tag{2.1}$$

Now suppose $\mu \in M(X)$, $g \in A$ and $\|\mu\| = \|g\|_\infty = \int g d\mu = 1$. Let $\mu = h \cdot |\mu|$ be the polar decomposition of μ (cf. [4, Corollary 19.38]), then $|h| = 1$. Since

$$1 = \int g\, d\mu = \int gh\, d|\mu| \leq \sqrt{\int |g|^2\, d|\mu|}\sqrt{\int |h|^2\, d|\mu|} \leq \|\mu\| = 1, \tag{2.2}$$

it follows that

$$\left| \int gh\, d|\mu| \right| = \sqrt{\int |g|^2\, d|\mu|}\sqrt{\int |h|^2\, d|\mu|} \tag{2.3}$$

and hence there exists a scalar $\lambda \in C$ such that $\overline{g(x)} = \lambda h(x)|\mu|$ a.e. on X. Therefore we have

$$1 = \int gh\,d|\mu| = \bar{\lambda}\int |h|^2\,d|\mu| = \bar{\lambda}\|\mu\| = \bar{\lambda}, \tag{2.4}$$

and so

$$\int fg\,d\mu = \int fgh\,d|\mu| = \int f|h|^2\,d|\mu| = \int f\,d|\mu|. \tag{2.5}$$

Consequently, we obtain that

$$V(A,f) = \left\{\int f\,d|\mu| : \exists\mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g\,d\mu = 1\right\}. \tag{2.6}$$

Next consider the following set:

$$S = \left\{v \in M(X) : v \geq 0 \text{ and } \|v\| = \int |g|^2 dv = 1\right\}. \tag{2.7}$$

Then S is a weak $*$-closed set. Also note that $\sqrt{\int |g|^2 d|\mu|} = 1$ by the above arguments and hence $|\mu| \in S$. Moreover, we can easily see that any extreme point of S is also an extreme point of $\{v \in M(X) : v \geq 0,\ \|v\| \leq 1\}$. But since the extreme points of $\{v \in M(X) : v \geq 0,\ \|v\| \leq 1\}$ consist of 0 and $\{\delta_x : x \in X\}$, it follows that the extreme points of S are contained in $\{\delta_x : x \in X\}$, where δ_x denotes the Dirac measure at $x \in X$. Then by the Krein-Milman theorem, we have $S \subseteq \overline{\text{co}}\{\delta_x : x \in X\}$ and so $|\mu| \in \overline{\text{co}}\ \{\delta_x : x \in X\}$. Hence $\int f d|\mu| = \lim_\lambda \int f dv_\lambda$ for some net $\{v_\lambda\}$ in $\text{co}\{\delta_x : x \in X\}$, and so $\int f d|\mu| \in \overline{\text{co}}\ R(f)$. Therefore, we have

$$\left\{\int f\,d|\mu| : \exists\mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g\,d\mu = 1\right\} \subseteq \overline{\text{co}}\ R(f). \tag{2.8}$$

(ii) Let $x_1,\dots,x_n \in X$ and $\lambda_1 \geq 0,\dots,\lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$, and set $\mu = \lambda_1\delta_{x_1} + \cdots + \lambda_n\delta_{x_n}$. Then μ is a positive measure on X with norm of one such that $\int f d\mu = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$. Assume that A has the property (#). Then we can take an element $g \in A$ such that $\|g\|_\infty = 1$ and $g(x_1) = \cdots = g(x_n) = 1$. Therefore $\int g d\mu = 1$ and hence we conclude that $\text{co}R(f) \subseteq V(A,f)$. □

Proof of Corollary 1.3. Assume A is a^*-subalgebra of $C_0(X)$.

(i) Set

$$W = \left\{\int f\,d\mu : \exists\mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = 1,\ \mu \geq 0,\ 0 \leq g \leq 1 \text{ and } \int g\,d\mu = 1\right\}. \tag{2.9}$$

Then $W \subseteq V(A,f)$ by Theorem 1.2. Now, suppose $\mu \in M(X)$, $g \in A$ and $\|\mu\| = \|g\|_\infty = \int g d\mu = 1$. Then $\||\mu|\| = 1$ and $0 \leq |g|^2 \leq 1$. Also since A is a $*$-subalgebra of $C_0(X)$, we have $|g|^2 = g\bar{g} \in A$. Moreover, we have $\sqrt{\int |g|^2 d|\mu|} = 1$ and hence $\int |g|^2 d|\mu| = 1$ as observed in the proof of Theorem 1.2. Hence we conclude that $V(A,f) \subseteq W$ by Theorem 1.2 again and hence $V(A,f) = W$.

(ii) Let $\mu \in M(X)$, $g \in A$ and $\|\mu\| = \|g\|_\infty = \int g\,d\mu = 1$. Then we have $\int (1-|g|^2)d|\mu| = 0$ as observed in the proof of (i). It follows that $|g(x)| = 1$ $|\mu|$ a.e. on X and hence $\operatorname{supp}(|\mu|)$ is compact. Therefore, we have

$$V(A,f) \subseteq \left\{ \int f\,d\mu : 0 \le \mu \in M(X),\ \|\mu\| = 1 \text{ and } \operatorname{supp}(\mu) \text{ is compact} \right\}. \tag{2.10}$$

Now, suppose that $0 \le v \in M(X)$, $\|v\| = 1$ and $\operatorname{supp}(v)$ is compact and that A has the property (##). Then we can take an element $g \in A$ such that $0 \le g \le 1$ and $g(x) = 1$ for all $x \in \operatorname{supp}(v)$. Therefore $\|v\| = \|g\| = \int g\,dv = 1$ and hence, by Theorem 1.2, we have

$$\left\{ \int f\,d\mu : 0 \le \mu \in M(X),\ \|\mu\| = 1 \text{ and } \operatorname{supp}(\mu) \text{ is compact} \right\} \subseteq V(A,f) \tag{2.11}$$

□

3. Examples. Let $X = (0,1]$, the half open interval and let $h \in C_0(X)$ be such that $h(x) \ne 0$ for all $x \in X$. Define

$$A = \{hg : g \in C_0(X)\}. \tag{3.1}$$

Then A is an ideal (and hence subalgebra) of $C_0(X)$. In this case, A is neither closed nor unital. Also A has the desired property: for any compact set $E \subseteq X$, there exists $g \in A$ such that $\|g\|_\infty = 1$ and $g(x) = 1$ for all $x \in E$. In fact, let $t_E = \min\{x : x \in E\}$ and so $0 < t_E \le 1$. Put

$$g_0(x) = \begin{cases} \dfrac{x\varphi(x)}{t_E\varphi(t_E)h(x)}, & \text{if } 0 < x \le t_E, \\ \dfrac{1}{h(x)}, & \text{if } t_E < x \le 1, \end{cases} \tag{3.2}$$

where $\varphi(x) = \min(|h(t_E)|x/t_E, |h(x)|)$ $(x \in X)$. Since $|g_0(x)| \le x/t_E\varphi(t_E)$ for $0 < x \le t_E$, a function g_0 must be in $C_0(X)$. Set $g = hg_0$ and hence $g \in A$ and $g(x) = 1$ for all $x \in E$. Also since $|g(x)| = (x/t_E)\cdot(\varphi(x)/\varphi(t_E)) \le 1\cdot 1 = 1$ for $0 < x \le t_E$, it follows that $\|g\|_\infty = 1$. Therefore A has the desired property and so by Theorem 1.2, we have

$$V(A,f) = \left\{ \int f\,d|\mu| : \exists \mu \in M(X) \text{ and } g \in A \text{ such that } \|\mu\| = \|g\|_\infty = \int g\,d\mu = 1 \right\} \tag{3.3}$$

and

$$\operatorname{co} R(f) \subseteq V(A,f) \subseteq \overline{\operatorname{co}}\, R(f) \tag{3.4}$$

for every $f \in A$. In particular, if $f \in A$ is real-valued, then we have

$$V(A,f) = \begin{cases} [\alpha,\beta], & \text{if } \{x \in X : f(x) = 0\} \ne \emptyset, \\ (0,\beta] \text{ or } [\alpha,0), & \text{if } \{x \in X : f(x) = 0\} = \emptyset, \end{cases} \tag{3.5}$$

where $\alpha = \inf\{f(x) : x \in X\}$ and $\beta = \sup\{f(x) : x \in X\}$.

Of course, this holds even if $A = C_0(X)$, so we have the spatial numerical range of the function $f(x) = x(x \in X)$ with respect to $C_0(X)$ is equal to $X = (0, 1]$. This fact has been observed in [2, Example 4.2].

Also, A is not generally a $*$-subalgebra of $C_0(X)$. But if h is real-valued, then A becomes a $*$-subalgebra of $C_0(X)$ and so A has the property (##).

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