Internat. J. Math. & Math. Sci.
Vol. 23, No. 12 (2000) 849–854
S0161171200003604

RIGHT SIMPLE SUBSEMIGROUPS AND RIGHT SUBGROUPS OF COMPACT CONVERGENCE SEMIGROUPS

PHOEBE HO and SHING S. SO

(Received 1 July 1999)

ABSTRACT. Clifford and Preston (1961) showed several important characterizations of right groups. It was shown in Roy and So (1998) that, among topological semigroups, compact right simple or left cancellative semigroups are in fact right groups, and the closure of a right simple subsemigroup of a compact semigroup is always a right subgroup. In this paper, it is shown that such results can be generalized in convergence semigroups. In the discussion of maximal right simple subsemigroups and maximal right subgroups of semigroups, generalization of the results that no two maximal right simple subsemigroups and maximal right subgroups of a convergence semigroup intersect, is also established.

Keywords and phrases. Convergence semigroups, right simple semigroups, maximal right simple subsemigroups, left cancellative semigroups, right zero semigroups, right groups, maximal right subgroups.

2000 Mathematics Subject Classification. Primary 22A15.

1. Introduction. Discussion of convergence spaces, compactification of convergence spaces, and compact convergence semigroups can be found in [2, 3, 4, 5]; however, a brief summary of essential results will be repeated here.

DEFINITION 1.1. A *convergence semigroup* is a convergence space S together with a continuous function $m : S\times S \to S$ such that S is Hausdorff and m is associative.

The following notations are useful in the discussion of convergence semigroups:

(i) For $a, b \in S$, $ab = m(a,b)$.

(ii) For $A, B \subseteq S$, $AB = m(A\times B) = \{ab \mid a \in A \text{ and } b \in B\}$. In particular, $A\{b\}$ will be denoted Ab.

(iii) $\mathcal{F}\times\mathcal{G}$ is the filter on $S\times S$ with $\{F\times G \mid F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$ as its base.

(iv) $\mathcal{F}\cdot\mathcal{G}$ is the filter on S with $m(\mathcal{F}\times\mathcal{G})$ as its base.

LEMMA 1.2. *If $\mathcal{F}$ and $\mathcal{G}$ are filters on a convergence semigroup S such that $\mathcal{F} \to x$ and $\mathcal{G} \to y$, then $\mathcal{F}\cdot\mathcal{G} \to xy$.*

LEMMA 1.3. *If S is a compact convergence semigroup, then S contains an idempotent.*

2. Main results. Let S be a semigroup. Then S is *left cancellative* provided that $zx = zy$ implies $x = y$ for all $x, y, z \in S$; S is *right simple* if it contains no proper right ideal or $aS = S$ for all $a \in S$; S is a *right group* if S is both left cancellative and right simple; S is a *right zero semigroup* if $xy = y$ for all $x, y \in S$.

In [1], Clifford and Preston showed that a semigroup S is a right group if and only if S is right simple and contains an idempotent.

Using this result and Lemma 1.3, the next four results in compact convergence semigroups can be obtained in exactly the same way as in the topological setting.

Theorem 2.1. *Let S be a compact convergence semigroup. Then the following statements are equivalent.*

(i) *S is right simple.*

(ii) *S is a right group.*

(iii) *S is left cancellative.*

Corollary 2.2. *Every compact convergence simple semigroup is a group.*

Corollary 2.3. *Every compact convergence cancellative semigroup is a group.*

Corollary 2.4. *Every closed right simple or closed left cancellative subsemigroup of a compact convergence semigroup is a right group.*

The example in [6] indicates that right subgroups of compact topological semigroups are closely related to their right simple subsemigroups, but not left cancellative subsemigroups. Thus the following discussion focuses only on the relationship between right simple subsemigroups and right subgroups of compact convergence semigroups.

In [6], it is shown that the closure of a right simple subgroup of a compact topological semigroup is always a right group. The next two theorems show that similar results can be obtained in compact convergence semigroups.

Theorem 2.5. *If S is a compact convergence semigroup and R is a right simple subsemigroup. Then $\mathrm{Cl}_S R$, the closure of R, is also a right simple subsemigroup of S.*

Proof. Let $a,b \in \mathrm{Cl}_S R$. There exist filters $\mathcal{F}$ and $\mathcal{G}$ such that $R \in \mathcal{F}\cap\mathcal{G}$, $\mathcal{F} \to a$, and $\mathcal{G} \to b$.

Since R is right simple, for $F \in \mathcal{F}$ and $G \in \mathcal{G}$, let $X_{FG} = \{x \in R : g = xf,\ f \in F,\ g \in G\}$ and let χ be the filter with $\mathcal{B}$ as base where $\mathcal{B} = \{X_{FG} : F \in \mathcal{F},\ G \in \mathcal{G}\}$. Then $\chi \cdot \mathcal{F}$ is the filter on S with $m(\chi\times\mathcal{F})$ as its base.

Since S is compact, there exists an ultrafilter $\mathcal{Y} \geq \chi$ such that $\mathcal{Y} \to y$ where $y \in \mathrm{Cl}_S R$. Thus $\chi\cdot\mathcal{F} \leq \mathcal{Y}\cdot\mathcal{F}$ and $\mathcal{Y}\cdot\mathcal{F} \to ya$. On the other hand, for each $F \in \mathcal{F}$, $G \subset X_{FG}\cdot F$ for all $G \in \mathcal{G}$. It follows that $X_{FG}\cdot F \in \mathcal{G}$ and $\chi\cdot\mathcal{F} \leq \mathcal{G}$. Let $\mathcal{U}$ be an ultrafilter containing $\chi\cdot\mathcal{F}$. Then $\mathcal{U} \to ya$ and $\mathcal{U} \to b$. Thus $b = ya$ and it follows that $\mathrm{Cl}_S R$ is a right simple subsemigroup of S. □

Definition 2.6. Let R be a right simple subsemigroup of a semigroup S. Then R is called a *maximal right simple subsemigroup* of S if and only if $R \neq S$ and no proper right simple subsemigroup of S properly containing R.

Definition 2.7. Let R be a right subgroup of a semigroup S. Then R is called a *maximal right subgroup* of S if and only if $R \neq S$ and no proper right subgroup of S properly containing R.

Suppose S is a compact convergence semigroup and R is a right simple subsemigroup of S. Let $\mathcal{S} = \{T \mid R \subset T, T$ is a proper right simple subsemigroup of $S\}$. Partially order $\mathcal{S}$ by set inclusion. By the Hausdorff maximal principle, there is a maximal chain $\mathcal{C}$ of $\mathcal{S}$. Let $M = \cup\mathcal{C}$.

Let $x, y \in M$. Then $x \in T$ and $y \in T^*$ for some $T, T^* \in \mathcal{C}$. Let $T' = \max\{T, T^*\}$. Then $x, y \in T'$ and T' being right simple implies $y \in xT' \subset xM$. Therefore, M is a right simple subsemigroup of S.

Suppose M^* is a proper right simple subsemigroup of $\mathcal{S}$ such that $M \subset M^*$. Then $M^* \notin \mathcal{C}$ so $\mathcal{C} \subset \mathcal{C} \cup \mathcal{M}^*$, which contracts the fact that $\mathcal{C}$ the maximal chain of $\mathcal{S}$. Therefore, M is the maximal right simple subsemigroup of S containing R. Since S is compact, by Theorems 2.1 and 2.5, M is a compact maximal right subgroup of S containing R. Therefore, the following theorem is proved.

Theorem 2.8. *If R is a right simple subsemigroup of a compact convergence semigroup S, then either S is a right group or R is contained in a unique maximal right subgroup M of S such that M is compact.*

Similarly, the following corollaries concerning convergence semigroups can be obtained.

Corollary 2.9. *Every right simple subsemigroup R of a compact convergence semigroup S, with $R \neq S$, is contained in a unique maximal right subsemigroup M such that M is closed.*

Corollary 2.10. *Every right subgroup R of a compact convergence semigroup S, with $R \neq S$, is contained in a unique maximal right subgroup M such that M is closed.*

Clifford and Preston [1] showed that a semigroup S is a right group if and only if S is isomorphic to the direct product of $G \times E$ where G is a group and E is a right zero semigroup, denoted by $S \cong G \times E$. In fact, E is the set of all idempotent of S and $G = Se$ for some $e \in E$. This result suggests a different way of analyzing compact right simple convergence semigroups.

Let S be a compact right simple or left cancellative convergence semigroup. It follows from Theorem 2.1 that S is a right group. Since S is compact and $G = Sg$ for some $g \in E$, G is compact. Since S is a right group, $ef = f$ for $e, f \in E$. Thus E is a right zero semigroup. Since E is a closed subset of S, E is compact.

Let Z be a right zero subsemigroup of a compact semigroup S. By Theorem 2.5, $\mathrm{Cl}_S Z$ is a subsemigroup of S.

Let $x, y \in \mathrm{Cl}_S Z$. Then there exist filters $\mathcal{F}$ and $\mathcal{G}$ such that $Z \in \mathcal{F} \cap \mathcal{G}$, $\mathcal{F} \to x$, and $\mathcal{G} \to y$.

Consider $\mathcal{H} = \{Z \cap \mathcal{F} : \mathcal{F} \in \mathcal{F}\}$ and $\mathcal{K} = \{Z \cap \mathcal{G} : \mathcal{G} \in \mathcal{G}\}$. Then $\mathcal{H}$ and $\mathcal{K}$ are filter bases of some filter $\mathcal{H}^*$ and $\mathcal{K}^*$, respectively. Note that $\mathcal{H}^*$ and $\mathcal{K}^*$ contain $\mathcal{F}$ and $\mathcal{G}$, respectively. Thus $\mathcal{F} \cdot \mathcal{G} \le \mathcal{H}^* \cdot \mathcal{K}^* \to xy$.

On the other hand, for $H \in \mathcal{H}^*$ and $K \in \mathcal{K}^*$, there exists $F \in \mathcal{F}$ such that $(Z \cap F)(Z \cap G) = Z \cap F \subset HK$. Thus $\mathcal{H}^* \cdot \mathcal{K}^* \le \mathcal{F}$ and $\mathcal{F} \to xy$. It follows from $xy = y$ that $\mathrm{Cl}_S Z$ is a right zero subsemigroups.

The next two lemmas follow from the above discussion.

Lemma 2.11. *Let S be a compact right simple or left cancellative convergence semigroup. Then $S \cong G \times E$ where G is a compact group and E is a compact right zero semigroup.*

Lemma 2.12. *Let Z be a right zero subsemigroup of a compact convergence semigroup S. Then $\mathrm{Cl}_S Z$ is also a right zero subsemigroup of S.*

Using Hausdorff's maximal principle and Lemma 2.12, the following lemma can be easily obtained.

Lemma 2.13. *Let S be a compact convergence semigroup and Z be a right zero subsemigroup of S. Then Z is contained in a maximal right zero subsemigroup of S.*

The next theorem can be proved in the same way as in the topological setting.

Theorem 2.14. *Let S be a compact convergence semigroup and R be a right subgroup of S such that $R \cong G_R \times E_R$. Then there exist G_M, the maximal subgroup of S containing G_R, and E_M, the maximal right zero subgroup of S containing E_R, such that $G_M \times E_M$ is isomorphic to a maximal right subgroup M of S containing R.*

Proof. Since $R \cong G_R \times E_R$, there exists a unique maximal subgroup G_M of S containing G_R and a unique maximal right zero subsemigroup E_M of S containing E_R by Lemma 2.13. Let M be the isomorphic image of $G_M \times E_M$ in S. Then M is a right subgroup of S.

Suppose M^* is a maximal right subgroup containing R. Then $M^* \cong G_M^* \times E_M^*$. By the maximality of M^*, $M \subseteq M^*$. Since $R \subset M^*$, $G_R \subset G_{M^*}$ and $E_R \subset E_{M^*}$. By the maximality of G_M and E_M, $G_{M^*} \subseteq G_M$ and $E_{M^*} \subseteq E_M$. By Lemma 2.16, $M^* \subseteq M$ so $M^* = M$.

It is a well-known result that no two maximal subgroups of a semigroup intersect and the following are its generalizations. In Theorem 2.15, we generalize it for maximal right subgroups and the proof can be found in [6]. Theorem 2.17 is a partial generalization of Theorem 2.15 in commutative semigroups. □

Theorem 2.15. *No two maximal right subgroups of a semigroup intersect.*

Lemma 2.16. *If M_1 and M_2 are distinct maximal right simple subsemigroups of S such that $M_1M_2 = M_2M_1$, then either $M_1 \cap M_2 = \varnothing$ or $M_1 \cdot M_2 = S$.*

Proof. Suppose that $M_1 \cap M_2 \neq \varnothing$ and $M_1 \cdot M_2 \neq S$. Let $a, b \in M_1 \cdot M_2$. Then $ab \in M_1 \cdot M_2$ since $M_1M_2 = M_2M_1$. Thus M_1M_2 is a subsemigroup of S.

In fact, the following argument shows that $M_1 \cdot M_2$ is a right simple subsemigroup of S containing both M_1 and M_2. For $a, b \in M_1 \cdot M_2$, $a = a_1a_2$ and $b = b_1b_2$ for some $a_1, b_1 \in M_1$, $a_2, b_2 \in M_2$. Then

$$\begin{aligned}
a &= a_1a_2 \\
&= (b_1b_1^*)a_2 \text{ for some } b_1^* \in M_1 \text{ since } M_1 \text{ is simple} \\
&= b_1(a_2^*b_1^{**}) \text{ for some } b_1^{**} \in M_1 \text{ and } a_2^* \in M_2 \text{ since } M_1M_2 = M_2M_1 \\
&= b_1(b_2a_2^{**})b_1^{**} \text{ for some } a_2^{**} \in M_2 \text{ since } M_2 \text{ is simple} \\
&= (b_1b_2)m \text{ where } m = a_2^{**}b_1^{**} \in M_2M_1 = M_1M_2 \\
&= bm \text{ for some } m \in M_1 \cdot M_2.
\end{aligned} \tag{2.1}$$

Therefore, $M_1 \cdot M_2$ is right simple. Since $M_1 \cap M_2 \neq \varnothing$, and M_1, M_2 are distinct, $M_1 \subset M_1 \cdot M_2$. By the maximality of M_1 and the fact that $M_1 \neq M_2$, $M_1 \cdot M_2 = S$, which is a contradiction. □

The following theorem concerning maximal simple subsemigroups of commutative semigroups follows immediately from Lemma 2.16.

Theorem 2.17. *No two maximal simple subsemigroups of a commutative semigroup intersect.*

Theorem 2.18. *Let $\mathcal{D}$ be a partition of a maximal right simple subsemigroup M of a compact convergence semigroup S such that D is a monoid for each $D \in \mathcal{D}$. Then $D = Me$ for some $e \in E(M)$ for each $D \in \mathcal{D}$.*

Proof. Since M is a maximal right simple subsemigroup of S and S is compact, M is a maximal right subgroup of S and M can be written as union of the decomposition $\{Me : e \in E(M)\}$. For each $D \in \mathcal{D}$, let e_D be the identity of D. Then $e_D \in Me$ for some $e \in E(M)$. In fact, $e = e_D \cdot e = e \cdot e_D = e_D$ since e is the identity of Me. It follows that $D \subset M_{e_D}$.

Suppose $Me_D \not\subset D$. Then there exists $D^* \in \mathcal{D}$ such that $D^* \cap (Me_D - D) \neq \emptyset$. Let e_{D^*} be the identity of D^*. Then it follows from the discussion above $D^* \subset Me_{D^*}$ which implies $D^* \cap (Me_D - D) \subset D^* \cap Me_D \subset Me_{D^*} \cap Me_D$. This contradicts the fact that $\{Me : e \in E(M)\}$ is a decomposition of M. Therefore $D = Me_D = Me$ for each $D \in \mathcal{D}$. □

Definition 2.19. Let R be a right group. For each idempotent e of R, let $f_e : R \to R$ be defined by $f_e(x) = (ex)^{-1}$. Then R is called a *convergence right group* if and only if f_e is continuous for every idempotent e in R.

Theorem 2.20. *Let S be a compact pseudotopological semigroup.*

(i) *If S is right simple, then S is a right convergence group.*

(ii) *Either every maximal right simple subsemigroup of S is closed and hence a convergence right subgroup of S or S itself is a convergence right group.*

Proof. (i) By Theorem 2.1, S is a right group. For each idempotent e of S, let $f_e : S \to S$ be defined by $f_e(x) = (ex)^{-1}$ and let $\mathcal{F}$ be a filter such that $\mathcal{F} \to x$. Then $f_e(\mathcal{F})$ is a filter in Se. Let $\mathcal{U}$ be an ultrafilter such that $\mathcal{U} \geq f_e(\mathcal{F})$. Since Se is compact, $\mathcal{U} \to y$ for some y in Se. Since for every $U \in \mathcal{U}$, $U \cap f(\mathcal{F}) \neq \emptyset$ for all $F \in \mathcal{F}$. Then $e \in UF$ for all $U \in \mathcal{U}$ and $F \in \mathcal{F}$ where e is the identity of the group Se. Now $\mathcal{U}\mathcal{F} \to y$, $\mathcal{U}\mathcal{F} = \dot{e}$, and $\dot{e} \to e$ imply $yx = e$. Thus $y = (ex)^{-1}$. Since S is pseudotopological, $f(\mathcal{F}) \to (ex)^{-1}$. Thus the result follows.

(ii) Suppose M is a maximal right simple subsemigroup of S such that $M \neq \mathrm{Cl}_S M$. Then $\mathrm{Cl}_s M = S$ by the maximality of M and Theorem 2.5. The result follows from Theorem 2.1 and part (i) of this theorem. □

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, American Mathematical Society, Providence, R.I., 1961. MR 24#A2627. Zbl 111.03403.

[2] P. Ho, P. Plummer, and S. So, *Some results on compact convergence semigroups defined by filters*, Internat. J. Math. Math. Sci. **21** (1998), no. 1, 153–158. CMP 1 486 971. Zbl 888.54038.

[3] D. Kent, *Convergence functions and their related topologies*, Fund. Math. **54** (1964), 125–133. MR 28#4509. Zbl 122.41301.

[4] D. C. Kent and G. D. Richardson, *Compactifications of convergence spaces*, Internat. J. Math. Math. Sci. **2** (1979), no. 3, 345–368. MR 80i:54004. Zbl 418.54005.

[5] G. D. Richardson and D. C. Kent, *The star compactification*, Internat. J. Math. Math. Sci. **4** (1981), no. 3, 451–472. MR 82g:54038. Zbl 479.54002.

[6] K. Roy and S. S. So, *Right simple subsemigroups and right subgroups of a compact semigroup*, J. Inst. Math. Comput. Sci. Math. Ser. **11** (1998), no. 2, 121–125. MR 99i:22008. Zbl 990.21715.

Ho: Central Missouri State University, Warrensburg, Missouri, USA

So: Central Missouri State University, Warrensburg, Missouri, USA
E-mail address: so@cmsuvmb.cmsu.edu