Internat. J. Math. & Math. Sci.
Vol. 23, No. 2 (2000) 109–117
S0161171200001666

DIFFERENTIAL SUBORDINATIONS FOR FRACTIONAL-LINEAR TRANSFORMATIONS

YONG CHAN KIM, ADAM LECKO, JAE HO CHOI, and MEGUMI SAIGO

(Received 29 May 1998 and in revised form 29 June 1998)

ABSTRACT. We establish that the differential subordinations of the forms $p(z)+\gamma zp'(z) \prec h(A_1,B_1;z)$ or $p(z)+\gamma zp'(z)/p(z) \prec h(A_2,B_2;z)$ implies $p(z) \prec h(A,B;z)$, where $\gamma \geq 0$ and $h(A,B;z) = (1+Az)/(1+Bz)$ with $-1 \leq B < A$.

Keywords and phrases. Subordination, univalent functions, Carlson-Shaffer operator, Ruscheweyh derivative.

2000 Mathematics Subject Classification. Primary 30C45.

1. Introduction. For each $n \in \mathbb{N}$, let $\mathcal{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}$ instead of $\mathcal{A}(1)$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$ (see Srivastava and Owa [9]).

For analytic functions g and h on $\mathcal{U}$ with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function ω on $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = h(\omega(z))$ for $z \in \mathcal{U}$. We denote this subordination relation by

$$g \prec h \qquad \text{or} \qquad g(z) \prec h(z) \quad (z \in \mathcal{U}). \tag{1.2}$$

For each A and B such that $-1 \leq B < A$, let us define the function

$$h(A,B;z) = \frac{1+Az}{1+Bz}, \quad (z \in \mathcal{U}). \tag{1.3}$$

It is well known that $h(A,B;z)$, for $-1 \leq B \leq 1$, is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1-AB)/(1-B^2)$ and the radius $(A-B)/(1-B^2)$. The boundary circle cuts the real axis at the points $(1-A)/(1-B)$ and $(1+A)/(1+B)$. A function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{S}^*[A,B]$ if

$$\frac{zf'}{f} \prec h(A,B;z), \quad (z \in \mathcal{U}) \tag{1.4}$$

and in $\mathcal{K}[A,B]$ if

$$1 + \frac{zf''}{f'} \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{1.5}$$

Note that $f \in \mathcal{K}[A,B]$ if and only if $zf' \in \mathcal{S}^*[A,B]$.

In [3] Janowski introduced the class $\mathcal{P}(A,B)$ for $-1 \le B < A \le 1$

$$\mathcal{P}(A,B) = \left\{p : p(z) \prec h(A,B;z), z \in \mathcal{U}\right\}. \tag{1.6}$$

For fixed $n \in \mathbb{N} = \{1,2,3,\dots\}$ the subclass $\mathcal{P}_n(A,B)$ of $\mathcal{P}(A,B)$ containing functions p of the form $p(z) = 1 + p_n z^n + \cdots$, $z \in \mathcal{U}$, was defined by Stankiewicz and Waniurski [10].

Further subclasses of $\mathcal{P}(A,B)$ were considered by various authors. Janowski [3, 4], and Silverman and Silvia [8] studied the above-mentioned class $\mathcal{S}^*[A,B]$. The class $R_n(A,B)$ for $n \in \mathbb{N}$ of functions $f \in \mathcal{A}(n)$ such that $f' \in \mathcal{P}_n(A,B)$ was examined by Stankiewicz and Waniurski [10]. For $\gamma \ge 0$ the class

$$H(\gamma,A,B) = \{f \in \mathcal{A} : f' + \gamma z f'' \in \mathcal{P}(A,B)\} \tag{1.7}$$

was studied by Dinggong [11]. Notice that $H(0,A,B) = R_1(A,B)$.

Let the functions $f_j(z)$ be defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1}, \quad (j = 1,2). \tag{1.8}$$

We denote by $(f_1 * f_2)(z)$ the Hadamard product or convolution of two functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1}\, z^{n+1}. \tag{1.9}$$

Also, let the function $\phi(a,c;z)$ be defined by

$$\phi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \quad (z \in \mathcal{U}), \tag{1.10}$$

where $c \ne 0,-1,-2,\dots$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n \in \mathbb{N}). \end{cases} \tag{1.11}$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [2] defined a linear operator on $\mathcal{A}$ by

$$\mathcal{L}(a,c)f(z) = \phi(a,c;z) * f(z) \quad \text{for } f(z) \in \mathcal{A}. \tag{1.12}$$

Then $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself. Furthermore, if $a \ne 0,-1,-2,\dots$, $\mathcal{L}(c,a)$ is an inverse of $\mathcal{L}(a,c)$. (See also Owa and Srivastava [6].)

Ruscheweyh [7] introduced an operator $\mathcal{D}^\lambda : \mathcal{A} \to \mathcal{A}$ defined by the convolution

$$\mathcal{D}^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad (\lambda \ge -1; z \in \mathcal{U}) \tag{1.13}$$

which implies that

$$\mathcal{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.14}$$

We also note that

$$\mathcal{D}^\lambda f(z) = \mathcal{L}(\lambda+1,1)f(z), \tag{1.15}$$

$$z(\mathcal{D}^\lambda f)'(z) = (\lambda+1)\mathcal{D}^{\lambda+1} f(z) - \lambda \mathcal{D}^\lambda f(z). \tag{1.16}$$

For a function $f(z)$ belonging to the class $\mathcal{A}$, Bernardi [1] defined the integral operator $\mathcal{J}_c$,

$$(\mathcal{J}_c f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \quad (c > -1; z \in \mathcal{U}). \tag{1.17}$$

By the series expansion of the function $(\mathcal{J}_c f)(z)$, it is easily seen that

$$(\mathcal{J}_c f)(z) = \mathcal{L}(c+1, c+2) f(z) \quad \text{for } f \in \mathcal{A}. \tag{1.18}$$

In this paper, we consider some geometric properties of certain differential subordinations associated with the function $h(A,B;z)$. We also apply the Carlson-Shaffer operator and the Ruscheweyh derivative to such subordinations.

2. Main results.

The following lemma proved by Miller and Mocanu [5] is required in our investigation.

Lemma 1. *Let q be an analytic function on $\bar{\mathcal{U}}$ except for at most one pole on $\partial\mathcal{U}$, and univalent on $\bar{\mathcal{U}}$, and let p be an analytic function in $\mathcal{U}$ with $p(0) = q(0)$ and $p(z) \not\equiv p(0)$, $z \in \mathcal{U}$. If p is not subordinate to q, then there exist points $z_0 \in \mathcal{U}$ and $\xi_0 \in \partial\mathcal{U}$ and a number $m \geq 1$ for which*

(a) $p(\{z \in \mathbb{C} : |z| < |z_0|\}) \subset q(\mathcal{U})$,

(b) $p(z_0) = q(\xi_0)$,

(c) $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$.

After simple calculations, we have the following lemma.

Lemma 2. *If $-1 < B < A$, then*

$$\begin{aligned} \left|h'\left(A,B;e^{i\theta}\right)\right| &= \frac{A-B}{1+2B\cos\theta + B^2}, \\ \frac{A-B}{(1+|B|)^2} \leq \left|h'\left(A,B;e^{i\theta}\right)\right| &\leq \frac{A-B}{(1-|B|)^2}, \quad (\theta \in \mathbb{R}). \end{aligned} \tag{2.1}$$

Now, we prove the following theorem.

Theorem 1. *Let $\gamma \geq 0$, A and B be such that $-1 < B < A \leq 1$. Let $A_1(\gamma)$ and $B_1(\gamma)$ be defined by the system of equations*

$$\begin{aligned} \frac{1-A_1(\gamma)}{1-B_1(\gamma)} &= \frac{1-A}{1-B} - \gamma\frac{A-B}{(1+|B|)^2}, \\ \frac{1+A_1(\gamma)}{1+B_1(\gamma)} &= \frac{1+A}{1+B} + \gamma\frac{A-B}{(1+|B|)^2}. \end{aligned} \tag{2.2}$$

If p is an analytic function in $\mathcal{U}$ with $p(0) = 1$ and

$$p(z) + \gamma z p'(z) \prec h\Big(A_1(\gamma), B_1(\gamma); z\Big), \quad (z \in \mathcal{U}), \tag{2.3}$$

then

$$p(z) \prec h(A,B;z) \quad (z \in \mathcal{U}). \tag{2.4}$$

Proof. First, notice that $B_1(\gamma) = (2 - a_1 - b_1)/(b_1 - a_1)$ for $\gamma \geq 0$, where

$$a_1 = \frac{1-A}{1-B} - \gamma \frac{A-B}{(1+|B|)^2} \quad \text{and} \quad b_1 = \frac{1+A}{1+B} + \gamma \frac{A-B}{(1+|B|)^2}. \tag{2.5}$$

Then $b_1 > a_1$, $a_1 < 1$, $b_1 > 0$, and $-1 < B_1(\gamma) < 1$ for each $\gamma \geq 0$. Hence, the function $h(A_1(\gamma), B_1(\gamma); z)$ is analytic and univalent in $\mathcal{U}$, so that (2.3) is well defined.

To prove (2.4), we suppose that p is not subordinate to $h(A,B;z)(z \in \mathcal{U})$. Then, by Lemma 1, there exist points $z_0 \in \mathcal{U}$ and $\xi_0 = e^{i\theta}(\theta \in \mathbb{R})$, and $m \geq 1$ such that

$$p(z_0) = h(A,B;\xi_0), \qquad z_0 p'(z_0) = m e^{i\theta} h'(A,B;e^{i\theta}). \tag{2.6}$$

By Lemma 2 and by the fact that $m \geq 1$, we have

$$|z_0 p'(z_0)| \geq |h'(A,B;e^{i\theta})| = \frac{A-B}{1+2B\cos\theta + B^2} \tag{2.7}$$

and

$$\min_{\theta \in [0,2\pi]} |h'(A,B;e^{i\theta})| = \frac{A-B}{(1+|B|)^2}, \tag{2.8}$$

the minimum is achieved for $\theta = 0$ if $B \geq 0$ and for $\theta = \pi$ if $B < 0$.

From (2.2) it follows at once that the disk $h(A,B;\mathcal{U})$ is contained in the disk $h(A_1(\gamma), B_1(\gamma);\mathcal{U})$ and they have the same center. Also, the distance between the circle $\partial h(A_1(\gamma), B_1(\gamma);\mathcal{U})$ and the circle $\partial h(A,B;\mathcal{U})$ is a constant and equal to $\gamma(A-B)/(1+|B|)^2$.

On the other hand, $\xi_0 h'(A,B;\xi_0)$ is an outward normal to the circle $\partial h(A,B;\mathcal{U})$ at the point $h(A,B;\xi_0)$ of the length not less than $(A-B)/(1+|B|)^2$ as a consequence of (2.8). But $m \geq 1$ and the point $h(A,B;\xi_0) + \gamma m \xi_0 h'(A,B;\xi_0)$ is outside of the disk $h(A_1(\gamma), B_1(\gamma);\mathcal{U})$. Using Lemma 1, we finally obtain

$$p(z_0) + \gamma z_0 p'(z_0) = h(A,B;\xi_0) + \gamma m \xi_0 h'(A,B;\xi_0) \notin h(A_1(\gamma), B_1(\gamma);\mathcal{U}). \tag{2.9}$$

This is a contradiction to the assumption. □

In the following corollaries, we assume the conditions of Theorem 1 on constants γ, A, B, $A_1(\gamma)$, and $B_1(\gamma)$.

By setting $p(z) = f(z)/z$ for $f \in \mathcal{A}$ in Theorem 1, we obtain the following.

Corollary 1.1. *If $f \in \mathcal{A}$ and*

$$(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) \prec h\Big(A_1(\gamma), B_1(\gamma); z\Big), \quad (z \in \mathcal{U}), \tag{2.10}$$

then

$$\frac{f(z)}{z} \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{2.11}$$

Especially for $\gamma = 1$, we have the following.

Corollary 1.2. *If $f \in \mathcal{A}$ and*

$$f'(z) \prec h(A_1(1), B_1(1); z), \quad (z \in \mathcal{U}), \tag{2.12}$$

then

$$\frac{f(z)}{z} \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{2.13}$$

Setting $p(z) = f'(z)$ for $f \in \mathcal{A}$ in Theorem 1, we have the next corollary.

Corollary 1.3. *If $f \in \mathcal{A}$ and*

$$f'(z) + \gamma z f''(z) \prec h(A_1(\gamma), B_1(\gamma); z), \quad (z \in \mathcal{U}), \tag{2.14}$$

then

$$f'(z) \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{2.15}$$

Taking $p(z) = zf'(z)/f(z)$ for $f \in \mathcal{A}$ in Theorem 1, we have the following corollary.

Corollary 1.4. *If $f \in \mathcal{A}$ and*

$$\frac{zf'(z)}{f(z)}\left[1 + \gamma + \frac{zf''(z)}{f'(z)} - \gamma\frac{zf'(z)}{f(z)}\right] \prec h(A_1(\gamma), B_1(\gamma); z), \quad (z \in \mathcal{U}), \tag{2.16}$$

then

$$\frac{zf'(z)}{f(z)} \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{2.17}$$

By putting $p(z) = \mathcal{D}^{\lambda} f(z)/z$ and $\gamma = 1/(\lambda+1)$ for $f \in \mathcal{A}$ in Theorem 1, the relation (1.16) yields the following.

Corollary 1.5. *Let $\lambda > -1$. If $f \in \mathcal{A}$ and*

$$\frac{\mathcal{D}^{\lambda+1} f(z)}{z} \prec h\left(A_1\left(\frac{1}{\lambda+1}\right), B_1\left(\frac{1}{\lambda+1}\right); z\right), \quad (z \in \mathcal{U}), \tag{2.18}$$

then

$$\frac{\mathcal{D}^{\lambda} f(z)}{z} \prec h(A,B;z), \quad (z \in \mathcal{U}). \tag{2.19}$$

Remark 1. As was observed in the proof of Theorem 1, there holds the inclusion property

$$h(A,B;\mathcal{U}) \subset h(A_1(\gamma), B_1(\gamma); \mathcal{U}) \quad \text{for every } \gamma \ge 0. \tag{2.20}$$

Consequently, Theorem 1 and its corollaries can be improved results concerning inclusion relations between classes of analytic functions. For example, from Corollary 1.3 it follows that $H(\gamma, A, B) \subset H(0, A, B)$ for every $\gamma > 0$ in terms of the class $H(\gamma, A, B)$ in (1.7), which was proved in [11].

For $\gamma \ge 0$ such that $A_1(\gamma) \le 1$ and $B_1(\gamma) \le 1$, the statement of Corollary 1.3 can be written as $H(\gamma, A_1(\gamma), B_1(\gamma)) \subset H(0, A, B)$.

Theorem 2. *Let $\gamma \ge 0$. For $-1 < B < A \le 1$, let*

$$\Phi(A,B) = \frac{(A-B)(1+B)}{(1+A)(1+|B|)^2} \tag{2.21}$$

and let

$$\Psi(A,B) = \frac{\sqrt{(1-A^2)(1-B^2)}}{1-AB}. \tag{2.22}$$

Let $A_2(\gamma)$ and $B_2(\gamma)$ be defined by the system of equations

$$\begin{aligned}\frac{1-A_2(\gamma)}{1-B_2(\gamma)}&=\frac{1-A}{1-B}-\gamma\Phi(A,B)\Phi(A,B),\\ \frac{1+A_2(\gamma)}{1+B_2(\gamma)}&=\frac{1+A}{1+B}+\gamma\Phi(A,B)\Psi(A,B).\end{aligned} \tag{2.23}$$

If p is an analytic function in $\mathcal{U}$ with $p(0)=1$ and

$$p(z)+\gamma\frac{zp'(z)}{p(z)}\prec h(A_2(\gamma),B_2(\gamma);z),\quad (z\in\mathcal{U}), \tag{2.24}$$

then

$$p(z)\prec h(A,B;z),\quad (z\in\mathcal{U}). \tag{2.25}$$

Proof. By the same way as in the proof of Theorem 1, it is easily seen that the function $h(A_2(\gamma),B_2(\gamma);z)$ for $\gamma\geq 0$ is analytic and univalent in $\mathcal{U}$. Since for $\gamma=0$ the statement of the theorem is trivial, we can assume, for further considerations, that $\gamma>0$.

Let us assume that p is not subordinate to $h(A,B;z)(z\in\mathcal{U})$. Then, by Lemma 1, there exist points $z_0\in\mathcal{U}$ and $\xi_0\in\partial\mathcal{U}$, and $m\geq 1$ such that $p(z_0)=h(A,B;\xi_0)$, $z_0p'(z_0)=m\xi_0h'(A,B;\xi_0)$. From Lemma 2, we also have

$$|m\xi_0h'(A,B;\xi_0)|\geq\frac{A-B}{(1+|B|)^2}. \tag{2.26}$$

Since $|z|=1$ is mapped by $h(A,B;z)$ onto a circle centered at $c=(1-AB)/(1-B^2)$ with radius $r=(A-B)/(1-B^2)$, we see that

$$|h(A,B;z)|<\frac{1+A}{1+B},\quad (z\in\mathcal{U}). \tag{2.27}$$

If we put $\psi=\tan^{-1}\{(A-B)/\sqrt{(1-A^2)(1-B^2)}\}$, then we also have

$$|\arg h(A,B;z)|\leq\tan^{-1}\frac{r}{\sqrt{c^2-r^2}}=\psi,\quad (z\in\mathcal{U}). \tag{2.28}$$

By using (2.26) and (2.27), it is obvious that

$$\left|\frac{z_0p'(z_0)}{p(z_0)}\right|=\left|\frac{m\xi_0h'(A,B;\xi_0)}{h(A,B;\xi_0)}\right|\geq\Phi(A,B), \tag{2.29}$$

where $\Phi(A,B)$ is given by (2.21).

From (2.23) it follows that the disk $h(A,B;\mathcal{U})$ and $h(A_2(\gamma),B_2(\gamma);\mathcal{U})$ are concentric and $h(A,B;\mathcal{U})\subset h(A_2(\gamma),B_2(\gamma);\mathcal{U})$. Thus the distance between an arbitrary point of the circle $\partial h(A_2(\gamma),B_2(\gamma);\mathcal{U})$ and the circle $\partial h(A,B;\mathcal{U})$ is a constant and equal to $\gamma\Phi(A,B)\Psi(A,B)$.

Notice that $\xi_0h'(A,B;\xi_0)$ is an outward normal to the circle $\partial h(A,B;\mathcal{U})$ at the point $h(A,B;\xi_0)$. Therefore, $\xi_0h'(A,B;\xi_0)/h(A,B;\xi_0)$ is the vector of the length not less than $\Phi(A,B)$ by (2.29), rotated with respect to the normal vector $\xi_0h'(A,B;\xi_0)$ not more than the angle ψ in view of (2.28). Since $\Psi(A,B)=\cos\psi$, so an elementary geometric observation, and let us allow to assert that the point

$$h(A,B;\xi_0)+m\gamma\frac{\xi_0h'(A,B;\xi_0)}{h(A,B;\xi_0)} \tag{2.30}$$

lies in the outside of the disk $h(A_2(\gamma),B_2(\gamma);\mathcal{U})$. Hence, we finally obtain

$$p(z_0)+\gamma\frac{z_0p'(z_0)}{p(z_0)}=h(A,B;\xi_0)+m\gamma\frac{\xi_0h'(A,B;\xi_0)}{h(A,B;\xi_0)}\notin h(A_2(\gamma),B_2(\gamma);\mathcal{U}). \tag{2.31}$$

This is a contradiction to the assumption. □

By taking $p(z)=zf'(z)/f(z)$ for $f\in\mathcal{A}$ in Theorem 2, we have the following.

Corollary 2.1. *Let $\gamma\geq 0$, $-1<B<A\leq 1$, $A_2(\gamma)$ and $B_2(\gamma)$ are given by (2.23). If $f\in\mathcal{A}$ satisfies*

$$(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\prec h\Big(A_2(\gamma),B_2(\gamma);z\Big),\quad (z\in\mathcal{U}), \tag{2.32}$$

then $f(z)\in\mathcal{S}^*[A,B]$.

Next, we consider the case $\gamma=1$ in Corollary 2.1.

Corollary 2.2. *Let $-1<B<A\leq 1$ and $A_2(1)$, $B_2(1)$ are defined by (2.23). If $f(z)\in\mathcal{K}[A_2(1),B_2(1)]$, then $f(z)\in\mathcal{S}^*[A,B]$.*

By using the definition (1.12) and Theorem 2 we prove the following theorem.

Theorem 3. *Let*

$$a>0,\quad -1<B<A\leq 1,\qquad \textit{and}\qquad A_2\left(\frac{1}{a}\right),\quad B_2\left(\frac{1}{a}\right) \tag{2.33}$$

be defined by (2.23). If $f\in\mathcal{A}$, then

$$\frac{\mathcal{L}(a,c)f(z)}{z}+\frac{\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}-1\prec h\left(A_2\left(\frac{1}{a}\right),B_2\left(\frac{1}{a}\right);z\right),\quad (z\in\mathcal{U}) \tag{2.34}$$

implies

$$\frac{\mathcal{L}(a,c)f(z)}{z}\prec h(A,B;z),\quad (z\in\mathcal{U}). \tag{2.35}$$

Proof. The function

$$p(z)\frac{\mathcal{L}(a,c)f(z)}{z},\quad (z\in\mathcal{U}) \tag{2.36}$$

is analytic in $\mathcal{U}$ with $p(0)=1$. Since

$$\begin{gathered} z(\mathcal{L}(a,c)f(z))'=a\mathcal{L}(a+1,c)f(z)-(a-1)\mathcal{L}(a,c)f(z),\\ \frac{zp'(z)}{p(z)}=\frac{a\mathcal{L}(a+1,c)f(z)}{\mathcal{L}(a,c)f(z)}-a. \end{gathered} \tag{2.37}$$

Therefore, the hypothesis (2.34) is equivalent to

$$p(z)+\frac{zp'(z)}{ap(z)}\prec h\left(A_2\left(\frac{1}{a}\right),B_2\left(\frac{1}{a}\right);z\right). \tag{2.38}$$

Hence, by Theorem 2 with $\gamma=1/a$, the proof of Theorem 3 is completed. □

Setting $a = \lambda + 1$ and $c = 1$ in Theorem 3 and owing to the relation (1.15), we have the following.

Corollary 3.1. *Let*

$$\lambda > -1, \quad -1 < B < A \le 1, \qquad \text{and} \qquad A_2\left(\frac{1}{(\lambda+1)}\right), \quad B_2\left(\frac{1}{(\lambda+1)}\right) \tag{2.39}$$

be determined by (2.23). If $f \in \mathcal{A}$ *and*

$$\frac{\mathcal{D}^\lambda f(z)}{z} + \frac{\mathcal{D}^{\lambda+1} f(z)}{\mathcal{D}^\lambda f(z)} - 1 \prec h\left(A_2\left(\frac{1}{\lambda+1}\right), B_2\left(\frac{1}{\lambda+1}\right); z\right), \quad (z \in \mathcal{U}), \tag{2.40}$$

then

$$\frac{\mathcal{D}^\lambda f(z)}{z} \prec h(A, B; z), \quad (z \in \mathcal{U}). \tag{2.41}$$

From Theorem 3 and the relation (1.18), we obtain the next corollary

Corollary 3.2. *Let* $c > -1$, $-1 < B < A \le 1$, $A_2(1/(c+1))$, *and* $B_2(1/(c+1))$ *be determined by (2.23). If* $f \in \mathcal{A}$ *and*

$$\frac{(\mathcal{J}_c f)(z)}{z} + \frac{f(z)}{(\mathcal{J}_c f)(z)} - 1 \prec h\left(A_2\left(\frac{1}{c+1}\right), B_2\left(\frac{1}{c+1}\right); z\right), \quad (z \in \mathcal{U}), \tag{2.42}$$

then

$$\frac{(\mathcal{J}_c f)(z)}{z} \prec h(A, B; z), \quad (z \in \mathcal{U}), \tag{2.43}$$

where the integral operator $\mathcal{J}_c$ *is defined by (1.17).*

Acknowledgement. This work was partially supported by KOSEF, BSRI-98-1401 and TGRC-KOSEF.

References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446. MR 38#1243. Zbl 172.09703.

[2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), no. 4, 737–745. MR 85j:30014. Zbl 567.30009.

[3] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math. **28** (1973), 297–326. MR 48 6401. Zbl 275.30009.

[4] ———, *Some extremal problems for certain families of analytic functions. I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **21** (1973), 17–25. MR 47 3659. Zbl 252.30021.

[5] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), no. 2, 157–171. MR 83c:30017. Zbl 456.30022.

[6] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39** (1987), no. 5, 1057–1077. MR 89f:30021. Zbl 611.33007.

[7] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115. MR 51 3418. Zbl 303.30006.

[8] H. Silverman and E. M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math. **37** (1985), no. 1, 48–61. MR 86j:30015. Zbl 574.30015.

[9] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Co., Inc., River Edge, NJ, 1992. MR 94b:30001. Zbl 970.22308.

[10] J. Stankiewicz and J. Waniurski, *Some classes of functions subordinate to linear transformation and their applications*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **28** (1974), 85–94. MR 56 5858. Zbl 441.30031.

[11] D. Yang, *Properties of a class of analytic functions*, Math. Japon. **41** (1995), no. 2, 371–381. MR 96h:30020. Zbl 833.30003.

Kim: Department of Mathematics, Yeungnam University, Gyongsan 712-749, Korea
E-mail address: kimyc@ynucc.yeungnam.ac.kr

Lecko: Department of Mathematics, Technical University of Rzeszów, Rzeszów 35-959, Poland

Choi and Saigo: Department of Mathematics, Fukuoka University, Fukuoka 814-0180, Japan