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DUALITY IN THE OPTIMAL CONTROL OF HYPERBOLIC EQUATIONS WITH POSITIVE CONTROLS

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Abstract. We study the duality theory for hyperbolic equations. Also, we consider distributed control systems with positive control and convex cost functionals.

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1. Introduction. Recently Lions, motivated by various practical problems, started a program of studying the optimal control distributed systems. The developments of distributed systems are the establishing of the optimality systems which characterize the optimal control [2, 3, 4, 5]. Duality theory for the corresponding parabolic equations with positive control has been given by Chan [1]. But, in this paper, we study the duality theory for hyperbolic distributed control systems. In fact, we consider distributed control systems with positive control and convex cost functionals. The approach presented exploits the fundamental results of Lions [2] on the optimality system which characterizes the optimal control. The method can be used to construct dual optimal systems when the controls are positive.

2. Duality in the optimal control hyperbolic equations with second-order operator. Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ and $Q = \Omega \times (0,T)$. The norm on $L^2(Q)$ is denoted by $|\cdot|$ and the corresponding inner product by $(\cdot,\cdot)$. In Q, information on the state is given by

$$\begin{gathered}\frac{\partial^2 y}{\partial t^2} + \Delta y = u, \quad \text{in } Q,\ u \in U_{ad},\ y \in L^2(Q), \\ y(0) \in K_0, \qquad y = 0, \quad \text{on } \sum,\end{gathered} \tag{2.1}$$

where

$$\begin{aligned} U_{ad} &= \{u \mid u \in L^2(Q), u \geq 0 \text{ in } Q\}, \\ K_0 &= \{\phi \mid \phi \in H^{-1}(\Omega), \phi \geq 0 \text{ in } \Omega\}, \\ \sum &= \Gamma \times (0,T), \end{aligned} \tag{2.2}$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{2.3}$$

Given $u \in U_{ad}$ and y satisfying (2.1) we set

$$J(y,u) = \frac{1}{2}|y - z_d|^2 + \frac{1}{2}(Nu,u), \tag{2.4}$$

where z_d is given in $L^2(Q)$ and $N > 0$. The problem of optimal control is find $\inf J(y,u)$, $u \in U_{ad}, u$ and y being connected by (2.1). The above problem admits a unique solution $\{u,y\}$ which is characterized by the solution $\{u,y,p\}$ of the optimality system

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \Delta y = u \quad \text{for } u \geq 0,\\
&y = 0 \quad \text{on } \textstyle\sum,\\
&y(x,0;u) = y_0(x), \qquad \frac{\partial y}{\partial t}(x,0;u) = y_1(x) \quad \text{in } \Omega,\\
&u \geq 0 \quad \text{a.e., in } Q,
\end{aligned} \tag{α_1}$$

$$\begin{aligned}
&\frac{\partial^2 p}{\partial t^2} + \Delta p = y - \mathcal{Z}_d \quad \text{in } Q,\\
&p = 0 \quad \text{on } \textstyle\sum,\\
&p(T) = 0, \qquad \frac{\partial p}{\partial t}(T) = 0 \quad \text{in } \Omega,\\
&p + Nu \geq 0 \quad \text{in } Q,\\
&p(0) \geq 0,
\end{aligned} \tag{β_1}$$

$$\begin{aligned}
&u(p + Nu) = 0,\\
&\frac{\partial p(0)}{\partial t} y(0) = 0, \qquad \frac{\partial y(0)}{\partial t} p(0) = 0 \quad \text{in } \Omega.
\end{aligned} \tag{γ_1}$$

Next we prove the duality theorem.

Theorem 1. *Let* $J = 1/2|y - \mathcal{Z}_d|^2 + 1/2(Nu,u), K = -1/2|y|^2 + 1/2|\mathcal{Z}_d|^2 - 1/2(Nu, u)$. *Assume* y_0, u_0, p_0 *satisfy* (α_1), (β_1), (γ_1); y,u *in* J *satisfy* (α_1); *and* y, u *in* K *satisfy* (β_1). *Then*

$$\inf_{(\alpha_1)} J = J(y_0,u_0) = K(y_0,u_0) = \sup_{(\beta_1)} K. \tag{2.5}$$

Proof. (i) We begin by showing that $J = K$ at (y_0,u_0,p_0).

$$\begin{aligned}
J(y_0,u_0) &= J(y_0,u_0) - (u_0,p_0) - (u_0,Nu_0)\\
&= J(y_0,u_0) - \left(y_0, \frac{\partial^2 p_0}{\partial t^2} + \Delta p_0\right) - (u_0,Nu_0) = K(y_0,u_0).
\end{aligned} \tag{2.6}$$

(ii) To show $\inf(\alpha_1)J = J(y_0,u_0)$, we must check that $J(y,u) \geq J(y_0,u_0)$, where (y,u) satisfy (α_1) and (y_0,u_0,p_0) satisfy (α_1), (β_1), (γ_1). Now, we have

$$\begin{aligned}
J(y,u) - J(y_0,u_0) &\geq (y_0 - z_d, y - y_0) + (Nu_0, u - u_0)\\
&= \left(\frac{\partial^2 p_0}{\partial t^2} + \Delta p_0, y - y_0\right) + (Nu_0, u - u_0)\\
&= (p_0 + Nu_0, u - u_0) \geq 0.
\end{aligned} \tag{2.7}$$

Thus $\inf J = J(y_0,u_0)$.

(iii) To show $\sup_{(\beta_1)} K = K(y_0,u_0)$, we have to check that $K(y,u) \le K(y_0,u_0)$, where (y,u) satisfy (β_1). But we have

$$\begin{aligned} J(y_0,u_0)-J(y,u) &\ge (y-z_d,y_0-y)+(Nu,u_0-u) = \left(\frac{\partial^2 p}{\partial t^2}+\Delta p, y_0-y\right) \\ &\quad + (Nu,u_0-u)+\left(\frac{\partial^2 y_0}{\partial t^2}+\Delta y_0-u_0,p_0-p\right) \\ &= -(y-z_d,y)-(Nu,u)+(y_0-z_d,y_0)+(Nu_0,u_0) \\ &\quad + (u_0,Nu+p)-(u_0,p_0+Nu_0) \\ &\ge -(y-z_d,y)-(Nu,u)+(y_0-z_d,y_0)+(Nu_0,u_0). \end{aligned} \tag{2.8}$$

Therefore,

$$J(y_0,u_0)-(y_0-z_d,y_0)-(Nu_0,u_0) \ge J(y,u)-(y-z_d,y)-(Nu,u), \tag{2.9}$$

or

$$K(y_0,u_0) \ge K(y,u). \tag{2.10}$$

This completes the proof. □

Now, we define the cost functional as

$$J = \frac{1}{2}\left| y(T;u)-z_d \right|^2_{L^2(\Omega)} + \frac{1}{2}(Nu,u). \tag{2.11}$$

In $Q = \Omega\times(0,T)$, we consider the following system:

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2}+\Delta y = u, \\ &y = 0, \quad \text{on } \sum, \\ &y(x,0;u) = y_0, \qquad \frac{\partial y(x,0;u)}{\partial t} = y_1, \\ &u \ge 0 \quad \text{a.e., in } Q, \end{aligned} \tag{α_2}$$

$$\begin{aligned} &\frac{\partial^2 p}{\partial t^2}+\Delta p = 0 \quad \text{in } Q, \\ &p = 0 \quad \text{on } \sum, \\ &p(x,T;u) = 0 \quad \text{for } x\in\Omega, \\ &\frac{\partial p(x,T;u)}{\partial t} = y(x,T;u)-z_d \quad \text{for } x\in\Omega, \\ &-p+Nu \ge 0 \quad \text{in } Q, \end{aligned} \tag{β_2}$$

$$\begin{aligned} &u(-p+Nu) = 0 \quad \text{a.e., in } Q, \\ &\frac{\partial p(0)}{\partial t}\, y(0) = 0, \qquad \frac{\partial y(0)}{\partial t}\, p(0) = 0 \quad \text{on } \Omega. \end{aligned} \tag{γ_2}$$

Theorem 2. *Let* $J = 1/2|y(T;u) - z_d|^2_{L^2(\Omega)} + 1/2(Nu,u)$, $K = -1/2|y(T;u)|^2_{L^2(\Omega)} + 1/2|z_d|^2_{L^2(\Omega)} - 1/2(Nu,u)$. *Assume* y_0, u_0, p_0 *satisfy* (α_2), (β_2), (γ_2); y, u *in* J *satisfy* (α_2); *and* y, u *in* K *satisfy* (β_2). *Then*

$$\inf_{(\alpha_2)} J = J(y_0,u_0) = K(y_0,u_0) = \sup_{(\beta_2)} K. \tag{2.12}$$

Proof. (i) Now we prove that $J = K$ at (y_0,u_0,p_0). Then

$$\begin{aligned} J(y_0,u_0) &= J(y_0,u_0) + (p_0,u_0) - (Nu_0,u_0) \\ &= J(y_0,u_0) + \left(\frac{\partial^2 y_0}{\partial t^2} + \Delta y_0, p_0\right) - (Nu_0,u_0) \\ &= -\frac{1}{2}\,|\,y_0(x,T;u)\,|^2_{L^2(\Omega)} + \frac{1}{2}|z_d|^2_{L^2(\Omega)} - \frac{1}{2}(Nu_0,u_0) = K(y_0,u_0). \end{aligned} \tag{2.13}$$

(ii) We show that $J(y,u) \geq J(y_0,u_0)$, where (y,u) satisfy (α_2) and (y_0,u_0,p_0) satisfy (α_2), (β_2), (γ_2).

$$\begin{aligned} J(y,u) - J(y_0,u_0) &\geq (y_0(T;u) - z_d, y(T;u) - y_0(T;u))_{L^2(\Omega)} + (Nu_0, u - u_0) \\ &= (y_0(T;u) - z_d, y(T;u) - y_0(T;u))_{L^2(\Omega)} + (Nu_0, u - u_0) \\ &\quad - \left(\frac{\partial^2 p_0}{\partial t^2} + \Delta p_0, y(t;u) - y_0(t;u)\right) \\ &= (-p_0 + Nu_0, u - u_0) \geq 0. \end{aligned} \tag{2.14}$$

(iii) Now we claim that $\sup_{(\beta_2)} K = K(y_0,u_0)$.

$$\begin{aligned} J(y_0,u_0) - J(y,u) &\geq (y(T;u) - z_d, y_0(T;u) - y(T,u))_{L^2(\Omega)} + (Nu, u_0 - u) \\ &= (y(T;u) - z_d, y_0(T;u))_{L^2(\Omega)} - (y(T;u) - z_d, y(T;u))_{L^2(\Omega)} \\ &\quad + (Nu,u_0) - (Nu,u) - \left(\frac{\partial^2 y_0}{\partial t^2} + \Delta y_0 - u_0, p_0 - p\right) \\ &= -(y(T;u) - z_d, y(T;u))_{L^2(\Omega)} - (Nu,u) \\ &\quad + (y_0(T;u) - z_d, y_0(T;u))_{l^2(\Omega)} + (Nu_0,u_0) \\ &\quad + (-p + Nu, u_0) + (p_0 - Nu_0, u_0). \end{aligned} \tag{2.15}$$

Therefore,

$$\begin{aligned} &J(y_0,u_0) - (y_0(T;u) - z_d, y_0(T;u))_{L^2(\Omega)} - (Nu_0,u_0) \\ &\qquad \geq J(y,u) - (y(T;u) - z_d, y(T;u))_{L^2(\Omega)} - (Nu,u), \end{aligned} \tag{2.16}$$

and this implies

$$\sup_{(\beta_2)} K(y,u) = K(y_0,u_0). \tag{2.17}$$

Thus, the proof is complete. □

3. Duality in the optimal control of hyperbolic equation with fourth-order operator. Let us consider the fourth-order differential operator.

$$U_{ad} = \{u \mid u \in L^2(Q), u \geq 0 \text{ in } Q\}, \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{3.1}$$

We consider a function $a(x,t)$ such that

$$a \in C^1(]0,T]; L^\infty(\Omega)). \tag{3.2}$$

We introduce

$$V = \{\phi \mid \phi, \Delta\phi \in L^2(\Omega)\}, \qquad H = L^2(\Omega) \tag{3.3}$$

and

$$a(t;\phi,\psi) = \int_\Omega a(x,t)\Delta\phi\,\Delta\psi\,dx, \quad \forall \phi,\psi \in V, \tag{3.4}$$

given $u \in U_{ad}$ and we set

$$J(y,u) = \frac{1}{2}|y - z_d|^2 + \frac{1}{2}(Nu,u), \tag{3.5}$$

where $z_d \in L^2(Q), u \in U_{ad}$ and $N > 0$.

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \Delta(a\Delta y) = u,\\
&\Delta y = 0, \qquad \frac{\partial \Delta}{\partial n} y = 0, \quad \text{on } \sum,\\
&\Delta y(x,0;u) = y_0(x), \qquad \frac{\partial y}{\partial t}(x,0;u) = y_1(x), \quad x \in \Omega,\\
&u \geq 0, \qquad \partial y(0) \geq 0, \qquad \frac{\partial \Delta}{\partial n} y(0) \geq 0.
\end{aligned} \tag{α_3}$$

$$\begin{aligned}
&\frac{\partial p}{\partial t^2} + \Delta(a\Delta p) = y - z_d \quad \text{in } Q,\\
&\Delta p = 0, \qquad \frac{\partial \Delta p}{\partial n} = 0 \quad \text{on } \sum,\\
&p(x,T;u) = 0, \qquad \frac{\partial p(x,T;u)}{\partial t} = 0 \quad \text{on } \Omega,\\
&p + Nu \geq 0 \quad \text{in } Q.
\end{aligned} \tag{β_3}$$

$$\begin{aligned}
&u(p + Nu) = 0,\\
&\frac{\partial p(0)}{\partial t} y(0) = 0, \qquad \frac{\partial y(0)}{\partial t} p(0) = 0 \quad \text{on } \Omega,
\end{aligned} \tag{γ_3}$$

Now we claim the following.

Theorem 3. *Let $J = (1/2)|y - z_d|^2 + (1/2)(Nu,u)$, $K = -(1/2)|y|^2 + (1/2)|z_d|^2 - (1/2)(Nu,u)$. Assume y_0, u_0, p_0 satisfy (α_3), (β_3), (γ_3); y, u in J satisfy (α_3) and y, u in K satisfy (β_3). Then*

$$\inf_{(\alpha_3)} J = J(y_0,u_0) = K(y_0,u_0) = \sup_{(\beta_3)} K. \tag{3.6}$$

Proof. (i) We begin by showing that $J = K$ at (y_0, u_0, p_0).

$$
\begin{aligned}
J(y_0,u_0) &= J(y_0,u_0) - (u_0,p_0) - (Nu_0,u_0) \\
&= J(y_0,u_0) - \left(\frac{\partial^2 y_0}{\partial t^2} + \Delta(a\Delta y_0), p_0\right) - (Nu_0,u_0) \\
&= J(y_0,u_0) - (y_0, y_0 - z_d) - (Nu_0,u_0) = K(y_0,u_0).
\end{aligned} \tag{3.7}
$$

(ii) We show that $J(y,u) \geq J(y_0,u_0)$.

$$
\begin{aligned}
J(y,u) - J(y_0,u_0) &\geq (y_0 - z_d, y - y_0) + (Nu_0, u - u_0) \\
&= \left(\frac{\partial^2 p_0}{\partial t^2} + \Delta(a\Delta p_0), y - y_0\right) + (Nu_0, u-u_0) \\
&\quad + (Nu_0, u - u_0) = (p_0 + Nu_0, u - u_0) \geq 0.
\end{aligned} \tag{3.8}
$$

(iii) We prove that $K(y,u) \leq K(y_0,u_0)$.

$$
\begin{aligned}
J(y_0,u_0) - J(y,u) &\geq \left(y - z_d, y_0 - y\right) + (Nu, u_0 - u) \\
&= \left(\frac{\partial^2 p}{\partial t^2} + \Delta(a\Delta P), y_0 - y\right) + (Nu, u_0 - u) \\
&\quad + \left(\frac{\partial^2 y_0}{\partial t^2} + \Delta(a\Delta y_0) - u_0, p_0 - p\right) \\
&= -(y - z_d, y) + (y_0 - z_d, y_0) + (p + Nu, u_0) - (Nu_0 + p_0, u_0) \\
&\quad - (Nu,u) + (Nu_0,u_0) + (u_0,p) - (y - \mathcal{Z}_d, y_0) \\
&\geq -(y - z_d, y) + (y_0 - z_0, y_0) - (Nu,u) + (Nu_0,u_0).
\end{aligned} \tag{3.9}
$$

Therefore,

$$
K(y_0,u_0) \geq K(y,u). \tag{3.10}
$$

Now, we set the following cost function:

$$
J = \frac{1}{2}\left| y(T;u) - z_d \right|^2_{L^2(\Omega)} + \frac{1}{2}(Nu,u), \tag{3.11}
$$

where $z_d \in L^2(Q)$, $u \in U_{ad}$ and $N > 0$ and associated following systems:

$$
\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} + \Delta(a\Delta y) = u, \\
&\Delta y = 0, \qquad \frac{\partial \Delta}{\partial n} y = 0 \quad \text{on } \sum, \\
&\Delta y(x,0;u) = y_0(x), \qquad \frac{\partial y}{\partial t}(x,0;u) = y_1(x), \quad x \in \Omega, \\
&u \geq 0, \qquad \partial y(0) \geq 0, \qquad \frac{\partial \Delta}{\partial n} y(0) \geq 0.
\end{aligned} \tag{α_4}
$$

$$\begin{aligned}
&\frac{\partial^2 p}{\partial t^2}+\Delta(a\Delta p)=0 \quad \text{in } Q,\\
&\Delta p=0, \qquad \frac{\partial \Delta}{\partial n}p=0 \quad \text{on } \sum\\
&p(x,T;u)=0, \qquad \frac{\partial p}{\partial t}(x,T;u)=y(x,T;u)-z_d \quad \text{on } \Omega,\\
&-p+Nu\geq 0 \quad \text{in } Q,
\end{aligned} \tag{β_4}$$

$$\begin{aligned}
&u(p+Nu)=0,\\
&\frac{\partial p(0)}{\partial t}y(0)=0, \qquad \frac{\partial y(0)}{\partial t}p(0)=0 \quad \text{on } \Omega.
\end{aligned} \tag{γ_4}$$

we have the duality result. □

Theorem 4. *Let* $J=(1/2)|y(T;u)-z_d|^2_{L^2(\Omega)}+(1/2)(Nu,u)$, $K=-(1/2)|y(T;u)|^2_{L^2(\Omega)}+(1/2)|z_d|^2_{L^2(\Omega)}-(1/2)(Nu,u)$. *Assume* y_0, u_0, p_0 *satisfy* (α_4), (β_4), (γ_4); y, u *in* J *satisfy* (α_4) *and* y,u *in* K *satisfy* (β_4). *Then*

$$\inf_{(\alpha_4)} J=J(y_0,u_0)=K(y_0,u_0)=\sup_{(\beta_4)} K. \tag{3.12}$$

Proof. (i) We begin to prove that

$$\begin{aligned}
J(y_0,u_0)&=J(y_0,u_0)-(u_0,p_0)-(Nu_0,u_0)\\
&=J(y_0,u_0)-\left(\frac{\partial^2 y_0}{\partial t^2}+\Delta(a\Delta y_0),p_0\right)-(Nu_0,u_0)\\
&=J(y_0,u_0)-(y_0(T;u)-z_d,y_0(T;u))_{L^2(\Omega)}\\
&\quad+\left(y_0,\frac{\partial^2 p_0}{\partial t^2}+\Delta(a\Delta p_0)\right)-(Nu_0,u_0)\\
&=-\frac{1}{2}|y_0(T;u)|^2_{L^2(\Omega)}+\frac{1}{2}|z_d|^2_{L^2(\Omega)}-\frac{1}{2}(Nu_0,u_0)=K(y_0,u_0).
\end{aligned} \tag{3.13}$$

(ii) We claim that $J(y,u)\geq J(y_0,u_0)$.

$$\begin{aligned}
J(y,u)-J(y_0,u_0)&\geq (y_0(T;u)-z_d,y(T;u)-y_0(T;u))_{L^2(\Omega)}+(Nu_0,u-u_0)\\
&=(y_0(T;u)-z_d,y(T;u)-y_0(T;u))_{L^2(\Omega)}+(Nu_0,u-u_0)\\
&\quad-\left(\frac{\partial^2 p_0}{\partial t^2}+\Delta(a\Delta p_0),y(t;u)-y_0(t;u)\right)\\
&=(-p+Nu_0,u-u_0)\geq 0.
\end{aligned} \tag{3.14}$$

(iii) Now, we verify that $K(y,u)\leq K(y_0,u_0)$.

$$\begin{aligned}
J(y_0,u_0)-J(y,u)&\geq (y(T;u)-z_d,y_0(T;u)-y(T;u))_{L^2(\Omega)}+(Nu,u_0-u)\\
&=(y(T;u)-z_d,y_0(T;u))_{L^2(\Omega)}-(y(T;u)-z_d,y(T;u))_{L^2(\Omega)}
\end{aligned}$$

$$\begin{aligned}
&+(Nu,u_0)-(Nu,u)-\left(\frac{\partial^2 y_0}{\partial t^2}+\Delta(a\Delta y_0)-u_0,p_0-p\right) \qquad (3.15)\\
&=-(y(T;u)-z_d,y(T;u))_{L^2(\Omega)}-(Nu,u)\\
&+(y_0(T;u)-z_d,y_0(T;u))_{L^2(\Omega)}+(Nu_0,u_0)\\
&+(-p+Nu,u_0)+(p_0-Nu_0,u_0).
\end{aligned}$$

This implies that

$$\sup_{(\beta_4)} K = K(y_0,u_0). \tag{3.16}$$

□

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