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UNIQUENESS OF WEAK SOLUTION FOR NONLINEAR ELLIPTIC EQUATIONS IN DIVERGENCE FORM

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ABSTRACT. We study the uniqueness of weak solutions for quasilinear elliptic equations in divergence form. Some counterexamples are given to show that our uniqueness result cannot be improved in the general case.

Keywords and phrases. Uniqueness, weak solution, quasilinear elliptic equation, divergence form, comparison theorem.

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1. Introduction. In this paper, we demonstrate the uniqueness of weak solution of the Dirichlet problem for divergence structure elliptic equations of the form

$$L[u] \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(a_i(x,u,\nabla u)\big) + b(x,u) = 0 \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$. In [1, 2, 3, 7, 8, 5], the uniqueness of classical solutions of problem (1.1) is treated under various hypotheses. Here, we consider the same problem for weak solutions. Especially, we give some counterexamples to show that our result cannot be improved in the general case.

To conclude this section, we would like to point out that after this paper had been submitted for publication, it came to our attention that a similar (uniqueness) result had been given in [4]. However, there is no further discussion in [4] as we do in Section 4.

2. Statement of the main results. Suppose that, for any $(x,z_1,z_2,\eta) \in \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$,

$$\sum_{i,j} \frac{\partial(a_i(x,z,\eta))}{\partial \eta_j}\xi_i\xi_j \geq |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{2.1}$$

$$\big(b(x,z_1)-b(x,z_2)\big)(z_1-z_2) \geq 0, \tag{2.2}$$

$$\big|a_i(x,z_1,\eta)-a_i(x,z_2,\eta)\big| \leq \big|a(z_1,z_2)\big|\,|z_1-z_2|\,(1+|\eta|), \tag{2.3}$$

where $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R})$.

Remark 2.1. Unlike the previous works (cf. [1, 5, 7, 8] and so on), we do not assume that $b(\cdot,\cdot)$ is Lipschitz continuous in its second argument.

We need the following two definitions, which can be found in many references (cf. [3] and so on).

Definition 2.2. $u \in H^1(\Omega)\cap L^\infty(\Omega)$ is said to satisfy $L[u] \geq (\leq) 0$ in Ω in the weak sense if

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(x,u,\nabla u)\frac{\partial w}{\partial x_i} + b(x,u)w \right\} dx \geq (\leq) 0, \tag{2.4}$$

for any $w \in H_0^1(\Omega)$ such that $w \geq 0$ a.e. in Ω.

Definition 2.3. $u, v \in H^1(\Omega)$ are said to satisfy $u \leq v$ on $\partial\Omega$ in the weak sense if $(u-v)^+ \equiv \max\{u-v, 0\} \in H_0^1(\Omega)$.

Now, we can state our main results.

Theorem 2.4 (Comparison theorem). *Let the hypotheses (2.1), (2.2), and (2.3) hold, and let $u_1, u_2 \in H^1(\Omega)\cap L^\infty(\Omega)$ satisfy*

$$L[u_1] \geq 0, \quad L[u_2] \leq 0 \quad \text{in } \Omega, \qquad u_2 \leq u_1 \quad \text{on } \partial\Omega \tag{2.5}$$

in the weak sense. Then we have $u_1 \geq u_2$ a.e. in Ω.

If, furthermore, $a \in L^\infty(\mathbb{R}\times\mathbb{R})$ and $u_1, u_2 \in H^1(\Omega)$ satisfy condition (2.5), then the same conclusion holds.

Theorem 2.5 (Uniqueness Theorem). *Let the hypotheses (2.1), (2.2), (2.3), and (2.4) hold. Then the problem (1.1) admits at most one weak solution $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$.*

If, furthermore, $a \in L^\infty(\mathbb{R}\times\mathbb{R})$, then (1.1) admits at most one weak solution $u \in H_0^1(\Omega)$.

3. Proof of the main results

Proof of Theorem 2.4. Assume that $u_1, u_2 \in H^1(\Omega)\cap L^\infty(\Omega)$ satisfy condition (2.5) in the weak sense, that is

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(x,u_1,\nabla u_1)\frac{\partial w}{\partial x_i} + b(x,u_1)w \right\} dx \geq 0 \tag{3.1}$$

and

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(x,u_2,\nabla u_2)\frac{\partial w}{\partial x_i} + b(x,u_2)w \right\} dx \leq 0, \tag{3.2}$$

for any $w \in H_0^1(\Omega)$ such that $w \geq 0$, a.e. in Ω.

Put

$$\Omega_1 = \{x \in \Omega; u_1(x) < u_2(x)\}. \tag{3.3}$$

We assert that $|\Omega_1| = 0$ ($|\Omega_1|$ denotes the Lebesgue measure of Ω_1). In fact, for any $\varepsilon > 0$, we write

$$E_\varepsilon = \{x \in \Omega_1 ; u_2 - u_1 > \varepsilon\}, \qquad v_\varepsilon = \min\left(\varepsilon, (u_2 - u_1)^+\right). \tag{3.4}$$

Note that $v_\varepsilon = 0$ and $\nabla v_\varepsilon = 0$, a.e. in $\Omega \setminus \Omega_1$, and that $v_\varepsilon = \varepsilon$ and $\nabla v_\varepsilon = 0$, a.e. in E_ε (see, e.g., [9]).

Lemma 3.1. *Let $u \in H_0^1(\Omega)$. Then $|u|$, u^+, $\min(\varepsilon, u) \in H_0^1(\Omega)$ for any nonnegative constant ε.*

Proof. Let $u_n \in C_0^\infty(\Omega)$ be such that $u_n \to u$ in $H_0^1(\Omega)$. Set

$$v_n = \left(u_n^2 + \frac{1}{n^2}\right)^{1/2} - \frac{1}{n}. \tag{3.5}$$

It is easy to see that $v_n \in C_0^\infty(\Omega)$. By a direct calculation, we can check that $\{v_n\}$ is a Cauchy sequence in $H_0^1(\Omega)$. Thus, there is a function $v \in H_0^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$. On the other hand, it is obvious that for an appropriate subsequence (still denoted by itself) $u_n \to u$, a.e. in Ω and, hence,

$$v_n \longrightarrow |u|, \quad \text{a.e. in } \Omega. \tag{3.6}$$

So, we obtain that $|u| = v \in H_0^1(\Omega)$. Similarly, we can prove that $u^+, \min(\varepsilon, u) \in H_0^1(\Omega)$ for any nonnegative constant ε. This completes the proof of the lemma. □

Now, we can return to the proof of Theorem 2.4. Since $u_1, u_2 \in H^1(\Omega)$ and $u_2 \le u_1$ on $\partial\Omega$ in the weak sense, we have $(u_2 - u_1)^+ \in H_0^1(\Omega)$ by Definition 2.3 and $v_\varepsilon \in H_0^1(\Omega)$ by the above lemma. By (3.1) and (3.2), we then obtain

$$\begin{aligned}
0 &\ge \int_\Omega \sum_i \left(a_i(x, u_2, \nabla u_2) - a_i(x, u_1, \nabla u_1)\right) \frac{\partial w}{\partial x_i}\, dx \\
&\quad + \int_\Omega \left(b(x, u_2) - b(x, u_1)\right) w\, dx \\
&= \int_\Omega \sum_{i,j} \int_0^1 \frac{\partial a_i(x, u_2, \nabla u_1 + s(\nabla u_2 - \nabla u_1))}{\partial \eta_j}\, ds\, \frac{\partial (u_2 - u_1)}{\partial x_j} \frac{\partial w}{\partial x_i}\, dx \\
&\quad + \int_\Omega \sum_i \left(a_i(x, u_2, \nabla u_1) - a_i(x, u_1, \nabla u_1)\right) \frac{\partial w}{\partial x_i}\, dx \\
&\quad + \int_\Omega \left(b(x, u_2) - b(x, u_1)\right) w\, dx,
\end{aligned} \tag{3.7}$$

for any $w \in H_0^1(\Omega)$ satisfying $w \ge 0$ a.e. in Ω. Take $w = v_\varepsilon$ in the above inequality. Note that $v_\varepsilon \ge 0$ and that $u_2 > u_1$ whenever $v_\varepsilon > 0$. By condition (2.2), we see that $b(x, u_2) - b(x, u_1) \ge 0$ whenever $v_\varepsilon > 0$. Thus,

$$\int_\Omega \left(b(x, u_2) - b(x, u_1)\right) v_\varepsilon\, dx \ge 0. \tag{3.8}$$

So, we have

$$
\begin{aligned}
0 \geq & \int_{\Omega} \sum_{i,j} \int_0^1 \frac{\partial a_i(x, u_2, \nabla u_1 + s(\nabla u_2 - \nabla u_1))}{\partial \eta_j}\, ds\, \frac{\partial (u_2 - u_1)}{\partial x_j} \frac{\partial v_\varepsilon}{\partial x_i}\, dx \\
& + \int_{\Omega} \sum_i (a_i(x, u_2, \nabla u_1) - a_i(x, u_1, \nabla u_1)) \frac{\partial v_\varepsilon}{\partial x_i}\, dx.
\end{aligned}
\tag{3.9}
$$

Since $v_\varepsilon = u_2 - u_1$ a.e. in $\Omega_1 \setminus E_\varepsilon$ and $\nabla v_\varepsilon = 0$ a.e. in $\Omega \setminus \{\Omega_1 \setminus E_\varepsilon\}$, by the above inequality, we get

$$
\begin{aligned}
0 \geq & \int_{\Omega_1 \setminus E_\varepsilon} \sum_{i,j} \int_0^1 \frac{\partial a_i(x, u_2, \nabla u_1 + s(\nabla u_2 - \nabla u_1))}{\partial \eta_j}\, ds\, \frac{\partial v_\varepsilon}{\partial x_j} \frac{\partial v_\varepsilon}{\partial x_i}\, dx \\
& + \int_{\Omega_1 \setminus E_\varepsilon} \sum_i (a_i(x, u_2, \nabla u_1) - a_i(x, u_1, \nabla u_1)) \frac{\partial v_\varepsilon}{\partial x_i}\, dx \\
\geq & \int_{\Omega_1 \setminus E_\varepsilon} |\nabla v_\varepsilon|^2\, dx + \int_{\Omega_1 \setminus E_\varepsilon} \sum_i (a_i(x, u_2, \nabla u_1) - a_i(x, u_1, \nabla u_1)) \frac{\partial v_\varepsilon}{\partial x_i}\, dx \\
= & \int_{\Omega} |\nabla v_\varepsilon|^2 dx + \int_{\Omega_1 \setminus E_\varepsilon} \sum_i (a_i(x, u_2, \nabla u_1) - a_i(x, u_1, \nabla u_1)) \frac{\partial v_\varepsilon}{\partial x_i}\, dx,
\end{aligned}
\tag{3.10}
$$

where we have used condition (2.1) at the second step. By (2.3) and the above inequality, we get

$$
\begin{aligned}
\int_{\Omega} |\nabla v_\varepsilon|^2 dx & \leq \left| \int_{\Omega_1 \setminus E_\varepsilon} \sum_i (a_i(x, u_1, \nabla u_1) - a_i(x, u_2, \nabla u_1)) \frac{\partial v_\varepsilon}{\partial x_i}\, dx \right| \\
& \leq \varepsilon C_1 \left(\int_{\Omega_1 \setminus E_\varepsilon} (1 + |\nabla u_1|)^2 dx \right)^{1/2} ||\nabla v_\varepsilon||_{L^2(\Omega)}
\end{aligned}
\tag{3.11}
$$

for some constant C_1 (since u_1, u_2 are bounded). Thus,

$$
||\nabla v_\varepsilon||_{L^2(\Omega)} \leq \varepsilon C_1 \left(\int_{\Omega_1 \setminus E_\varepsilon} (1 + |\nabla u_1|)^2 dx \right)^{1/2}. \tag{3.12}
$$

But also, using the Sobolev-Poincaré inequality on Ω, we obtain

$$
\begin{aligned}
|E_\varepsilon| & = \varepsilon^{-2} \int_{E_\varepsilon} \varepsilon^2 dx \leq \varepsilon^{-2} \int_{\Omega} |v_\varepsilon|^2 dx \\
& \leq C \varepsilon^{-2} \int_{\Omega} |\nabla v_\varepsilon|^2 dx \leq C C_1^2 \int_{\Omega_1 \setminus E_\varepsilon} (1 + |\nabla u_1|)^2 dx.
\end{aligned}
\tag{3.13}
$$

But $0 < u_2 - u_1 \leq \varepsilon$ in $\Omega_1 \setminus E_\varepsilon$. So, obviously,

$$
\lim_{\varepsilon \to 0} (\Omega_1 \setminus E_\varepsilon) = \emptyset. \tag{3.14}
$$

Thus, $|E_\varepsilon| \to 0$ as $\varepsilon \to 0$. On the other hand, clearly, $|E_\varepsilon|$ is nondecreasing. Thus, $|E_\varepsilon| = 0$ for any $\varepsilon > 0$ and in turn $u_2 \leq u_1$, a.e. in Ω.

We omit the details of the proof of the second statement since it is similar to that of the first part. $\square$

Proof of Theorem 2.5. Assume that $u_1, u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ are two weak solutions of problem (1.1). By Theorem 2.4, we must have

$$u_1 \geq u_2, \qquad u_2 \geq u_1, \quad \text{a.e. in } \Omega. \tag{3.15}$$

So, $u_1 = u_2$ a.e. in Ω, which proves the first conclusion.

We omit the details of the proof of the second statement since it is similar to that of the first part. □

4. Some remarks and counterexamples

Remark 4.1. Uniqueness does not hold for equations of nondivergence form,

$$-\sum_{i,j=1}^{n} a_{ij}(x, u, \nabla u) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x, u) = 0 \quad \text{in } \Omega, \tag{4.1}$$

even when $b \equiv 0$ and a_{ij} is independent of ∇u, as shown by Meyers [6]. In particular, he gave an example of a nondivergence equation with analytic coefficients, which is uniformly elliptic and which has nonunique analytic solutions in a bounded domain with analytic boundary.

Remark 4.2. Condition (2.3) essentially says that $a_i(x, z, \eta)$ is locally Lipschitz continuous with respect to its second argument and, furthermore, that the Lipschitz constant is independent of x. This condition also cannot be removed. In fact, we have the following counterexamples.

(1) Consider the equation

$$-\sum \frac{\partial}{\partial x_i}\left(a(x, u)\delta_{ij} \frac{\partial u}{\partial x_j}\right) = 1 \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0, \tag{4.2}$$

where $a(x, u) = 2 - \sin\{\pi \operatorname{sgn}(u - u_1)\}$ and where $u_1 \in H_0^1(\Omega)$ solves the problem

$$-2\Delta u = 1 \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0. \tag{4.3}$$

It is easy to check that $u = u_1$ and $u = 2u_1$ are both solutions for problem (4.2), and that the minimum principle holds for (4.2) (for all of the following examples, maximum/minimum principle also holds). (Note that $a(x, u)$ is not continuous in its second argument.)

(2) In the previous example, we take

$$a(x, u) = \begin{cases} 2 - \sin(u - 2u_1)\sin\left((u - u_1)^{1/2}\right) & \text{if } u \geq u_1, \\ 2 \quad \text{if } u < u_1. \end{cases} \tag{4.4}$$

It is easy to check that both $u = u_1$ and $u = 2u_1$ are solutions of problem (4.2). (Note that $a(x, u)$ is not locally Lipschitz continuous in its second argument.)

(3) Take

$$a(x, u) = 2 - \sin\left\{\frac{\pi}{2} \frac{u(u - u_1)}{u_1^2}\right\} \tag{4.5}$$

in the previous example. It is easy to check that $u = u_1$ and $u = 2u_1$ are again solutions of problem (4.2).

Remark 4.3. If $b(x,u)$ is replaced by $b(x,u,\nabla u)$ in (1.1), we cannot obtain any satisfactory uniqueness result except for special cases such as $\partial b(\cdot,\cdot,\cdot)/\partial z \gg 0$ and where other restrictive conditions are assumed. In particular, consider the equation

$$\frac{d^2u}{dx^2} + \left| x\frac{du}{dx} \right| = 0 \quad \text{in } \Omega = (-1,1), \quad u(-1) = u(1) = 0. \tag{4.6}$$

We can check directly that $u = 0$ and

$$u(x) = \int_x^1 e^{s^2/2}\,ds\,\chi_{[0,1]}(x) + \int_{-1}^x e^{s^2/2}\,ds\,\chi_{[-1,0)}(x) \tag{4.7}$$

are both weak solutions of this problem.

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