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NEW CHARACTERIZATIONS OF SOME L^p-SPACES

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Abstract. For a complete measure space (X,Σ,μ), we give conditions which force $L^p(X,\mu)$, for $1 \le p < \infty$, to be isometrically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on (X,μ). Also, we give some new characterizations which yield the inclusion $L^p(X,\mu) \subset L^q(X,\mu)$ for $0 < p < q$.

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1. Introduction. Suppose X is a nonempty set, Σ is σ-algebra of subsets of X, μ a positive measure on Σ. For each positive number p, let $L^p(X,\mu)$ denote the space of all real valued Σ-measurable functions f on X such that $\int_x |f|^p \, d\mu < \infty$, and $L^\infty(X,\mu)$ denote the space of all essentially bounded, real valued Σ-measurable functions on X. In [2, 3, 5] some characterizations of the positive measure μ on (X,Σ) for which $L^p(X,\mu) \subseteq L^q(X,\mu)$, $0 < p < q$, are given. The purpose of this note is to give some new characterizations of such measure μ which yield the inclusion $L^p(X,\mu) \subseteq L^q(X,\mu)$ for $0 < p < q$. Our proofs are more transparent, direct, and work even if the measure μ is not σ-finite. Further we show that in a situation when $L^p(X,\mu) \subseteq L^q(X,\mu)$ for some pair p,q with $0 < p < q$, then $L^p(X,\mu)$, for $1 \le p < \infty$, is isometrically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on the measure space (X,Σ,μ).

2. Preliminaries. Throughout the following (X,Σ,μ) is a positive measure space. We assume that the measure μ is complete. For the sake of simplicity, we write $L^p(\mu)$ for $L^p(X,\mu)$ and $L^\infty(\mu)$ for $L^\infty(X,\mu)$. A set $A \in \Sigma$ is called an *atom* if $\mu(A) > 0$ and for every $E \subset A$ with $E \in \Sigma$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. A measurable subset E with $\mu(E) > 0$ is *nonatomic* if it does not contain any atom. We say that two atoms A_1 and A_2 are *distinct* if $\mu(A_1 \cap A_2) = 0$. We say that two atoms A_1 and A_2 are *indistinguishable* if $\mu(A_1 \cap A_2) = \mu(A_1) = \mu(A_2)$. A measurable space (X,Σ,μ) is said to be *atomic* if every measurable set of positive measure contains an atom. For more information on measurable spaces and related topics we refer to [1, 2, 4]. We collect some interesting and useful properties of atomic and nonatomic sets in the following proposition.

Proposition 2.1. *Let (X,Σ,μ) be a complete measure space.*

(a) *If $\{A_n\}$ is a sequence of distinct atoms, then there exists a sequence $\{B_n\}$ of disjoint atoms such that for each n, $B_n \subseteq A_n$ and $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$.*

(b) *If $\{A_n\}$ is a sequence of distinct atoms, and A is an atom contained in $\cup A_n$, then there exists a unique m such that A is indistinguishable from A_m.*

(c) *If A is a nonatomic set of positive measure, then there exists a sequence $\{E_n\}$ of disjoint measurable subsets of A of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$.*

(d) *If $f \in L^p(\mu)$ and A is an atom in Σ, then f is constant almost everywhere (a.e.) on A.*

Proof. (a) Let $B_1 = A_1$ and $B_n = A_n \backslash \cup_{k=1}^{n-1} A_k$. Obviously B_i's are disjoint and $\cup A_n = \cup B_n$. Also $\mu(B_n) = \mu(A_n \backslash \cup_{k=1}^{n-1} A_k)$ is either zero or is equal to $\mu(A_n)$. If $\mu(B_n) = 0$, then $\mu(A_n) = \mu(A_n \cap (\cup_{k=1}^{n-1} A_k)) \le \sum_{k=1}^{n-1} \mu(A_n \cap A_k)$. Since A_k's are distinct atoms, this implies $\mu(A_n) = 0$ which is absurd. Hence $\mu(B_n) = \mu(A_n)$.

(b) Suppose A is contained in $\cup A_n$. From part (a) of the proposition, there exists a sequence $\{B_n\}$ of disjoint atoms such that $B_n \subseteq A_n$ for each n and $\cup A_n = \cup B_n$. Obviously

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap B_n). \tag{2.1}$$

Clearly $\mu(A \cap B_n)$ is either zero or $\mu(A)$ for each n. Hence by (2.1), there exists a unique m such that $\mu(A \cap B_m) = \mu(A)$. Since A and B_m are indistinguishable, $B_m \subset A_m$, it follows that A and A_m are indistinguishable.

(c) Suppose A is a nonatomic set of positive measure and $\mu(A) = \delta$. There exists a measurable subset E_1 of A such that $0 < \mu(E_1) < \delta/2$. Since $A \backslash E_1$ is nonatomic, there exists a measurable subset E_2 of $A \backslash E_1$ such that $0 < \mu(E_2) < \delta/4$. Having chosen $E_1, E_2, \dots, E_{n-1}$, choose a measurable subset E_n of $A \backslash (E_1 \cup E_2 \cup \cdots \cup E_{n-1})$ such that $\mu(E_n) < \sigma/2^n$. Obviously E_n's are disjoint and $\mu(E_n) \to 0$ as $n \to \infty$.

(d) Since A is an atom, it is enough to show that if f is integrable then f is constant a.e. on A. Choose a real number c such that $c\mu(A) = \int_A f(x)\,d\mu$. Let $B = \{x \in A \mid f(x) \neq c\}$. We claim $\mu(B) = 0$. Obviously $B = \{x \in A \mid f(x) < c\} \cup \{x \in A \mid f(x) > c\}$. First, we show that $\mu(\{x \in A \mid f(x) > c\}) = 0$. We can use a similar argument to show that $\mu(\{x \in A \mid f(x) > c\}) = 0$. We note that $\{x \in A \mid f(x) > c\} = \cup_{i=1}^{\infty} B_i \cup B_0$, where $B_i = \{x \in A \mid c + 1/(1+i) \le f(x) < c + (1/i)\}$ and $B_0 = \{x \in A \mid f(x) \ge c + 1\}$. Obviously all B_i's are disjoint. Since A is an atom, at most one of the B_i's can have a positive measure. If B_k is of positive measure for some k, $0 \le k < \infty$, then $c\mu(A) = \int_A f(x)\,d\mu(x) = \int_{B_k} f(x)\,dx \ge (c + (1/(k+1)))\mu(A)$. This is absurd. Therefore, $\mu(B_i) = 0$ for all $i \ge 0$. Hence $\{x \in A \mid f(x) > c\}$ is of measure zero. This completes the proof. □

The following lemmas are quite useful in the proof of the main result.

Lemma 2.2. *Let (X, Σ, μ) be a complete measure space.*

(a) *If $\{B_n\}$ is a sequence of measurable sets of positive measure and $\mu(B_n) \to 0$ as $n \to \infty$, then there exists a sequence $\{C_n\}$ of disjoint measurable sets of positive measure such that $\mu(C_n) \to 0$ as $n \to \infty$.*

(b) *If $\{E_n\}$ is a sequence of disjoint measurable sets of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$, then for any positive number $m > 1$ there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ and an increasing sequence $\{k_i\}$ of positive integers such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$.*

Proof. (a) Without loss of generality, we may assume that $\mu(B_n) < 1$ for each n. If for some positive integer k, B_k is nonatomic, by using an argument similar to that of Proposition 2.1(c), we can construct a sequence C_n of disjoint measurable sets of positive measure such that $\mu(C_n) \to 0$ as $n \to \infty$. Suppose that B_k is atomic for each positive integer k, let A_1 be an atom contained in B_1. Since $\mu(B_n) \to 0$ as $n \to \infty$, $\mu(A_1 \cap B_k)$ can be positive only for finitely many $k > 1$. Let n_1 be the smallest positive integer such that $\mu(A_1 \cap B_{n_1}) = 0$. Now choose an atom A_2 contained in B_{n_1}. Obviously A_2 is indistinguishable from A_1. Also, $\mu(A_2 \cap B_k)$ can be positive for at most finitely many k greater than n_1. Let n_2 be the smallest positive integer greater than n_1 such that $\mu(A_2 \cap B_{n_2}) = 0$. Now choose an atom A_3 contained in B_{n_2}. Clearly A_3 is indistinguishable from A_1 and A_2. Continuing in this fashion, we get a sequence $\{A_k\}$ of atoms which are indistinguishable and $A_k \subseteq B_{n_{k-1}}$ for each $k \ge 2$. By Proposition 2.1(a), we may choose a sequence $\{E_k\}$ of disjoint atoms such that $E_k \subseteq A_k$. Clearly, $0 < \mu(E_k) = \mu(A_k) \le \mu(B_{n_{k-1}})$. This completes the proof of part (a).

(b) Let $\{E_n\}$ be a sequence of measurable sets of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $\{\mu(E_n)\}$ is a strictly decreasing sequence. Let $m > 1$. Let $k_0 > 2$ be a positive integer such that $1/2 < (k/(k+1))^{m-1}$ for all $k \ge k_0$. Clearly $(1/(\ell+1)^m, 1/(\ell+1)^{m-1}] \cap ((1/\ell)^m, (1/\ell)^{m-1}]$ is nonempty for each $\ell \ge k_0$. Since $\mu(E_n)$ is decreasing to zero, the set $\{\mu(E_n) \mid n \ge 1\}$ must have a nonempty intersection with an interval $((1/k)^m, (1/k)^{m-1}]$ for some $k \ge k_0$. Let k_1 be the smallest positive integer greater than k_0 such that $\{\mu(E_n) \mid n \ge 1\} \cap ((1/k_1)^m, (1/k_1)^{m-1}] \ne \emptyset$. Let n_1 be the smallest positive integer such that $\mu(E_{n_1}) \in ((1/k_1)^m, (1/k_1)^{m-1}]$. Next choose the smallest integer k_2 greater than k_1 such that $\{\mu(E_n) \mid n > n_1\} \cap ((1/k_2)^m, (1/k_2)^{m-1}] \ne \emptyset$. Let n_2 be the smallest integer greater than n_1 such that $\mu(E_{n_2}) \in ((1/k_2)^m, (1/k_2)^{m-1}]$. Continuing inductively in this way, we can choose strictly increasing sequences of positive integers $\{k_i\}$ and $\{n_i\}$ such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$. This completes the proof of part (b). □

Lemma 2.3. *If $L^p(\mu) \subseteq L^q(\mu)$ for $0 < p < q$, then there does not exist a disjoint sequence $\{E_n\}$ of measurable sets of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$.*

Proof. Suppose there exists a disjoint sequence $\{E_n\}$ of measurable sets of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$. Let

$$m = 3 - \frac{3p}{p-q} = -\frac{3q}{p-q}. \tag{2.2}$$

Clearly $m > 1$. By Lemma 2.2(b), there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ and a strictly increasing sequence of positive integers $\{k_i\}$ such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$. Define a function f from X into real numbers by $f(x) = (1/k_i)^{3/(p-q)}$ if $x \in E_{n_i}$ and $f(x) = 0$ for all $x \notin \cup_{i=1}^{\infty} E_{n_i}$. Then

$$\begin{aligned}\int_X |f(x)|^p d\mu &= \sum_{i=1}^{\infty} \int_{E_{n_i}} |f(x)|^p d\mu = \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3p/(p-q)} \mu(E_{n_i}) \\ &\le \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3p/(p-q)} \left(\frac{1}{k_i}\right)^{m-1} = \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^2 < \infty.\end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned}\int_X |f(x)|^q d\mu &= \sum_{i=1}^{\infty}\int_{E_{n_i}} |f(x)|^q d\mu = \sum_{i=1}^{\infty}\left(\frac{1}{k_i}\right)^{3q/(p-q)}\mu(E_{n_i})\\ &\geq \sum_{i=1}^{\infty}\left(\frac{1}{k_i}\right)^{3q/(p-q)}\left(\frac{1}{k_i}\right)^{m} = \infty.\end{aligned} \tag{2.4}$$

Thus $f \in L^p(\mu)$ but $f \notin L^q(\mu)$. This completes the proof of the lemma. □

3. Main results. For the sake of clarity, we first start with a definition. For any nonempty set Γ, and $p > 0$, we define $\ell^p(\Gamma)$ to be the set of all extended real valued functions f on Γ such that f is nonzero only on a countable subset of Γ and $\sum_\alpha |f(\alpha)|^p < \infty$.

When $p \geq 1$, $\ell^p(\Gamma)$ becomes a Banach space under the norm defined by $\| f \|_{\ell^p(\Gamma)} = (\sum_\alpha | f(\alpha) |^p)^{1/p}$. Now, we are ready to state the main result.

Theorem 3.1. *Let (X,Σ,μ) be a complete measure space. The following six conditions are equivalent:*

(1) $L^p(\mu) \subset L^q(\mu)$ *for some pair of real numbers p and q with $0 < p < q$.*

(2) $L^p(\mu) \subset L^\infty(\mu)$ *for some $p > 0$.*

(3) $L^p(\mu) \subset L^\infty(\mu)$ *for all positive numbers p.*

(4) $L^p(\mu) \subset L^q(\mu)$ *for all p and q with $0 < p < q$.*

(5) *There is no sequence $\{B_n\}$ in Σ such that $\mu(B_n) > 0$ for each n and $\mu(B_n) \to 0$ as $n \to \infty$.*

(6) (X,Σ,μ) *is atomic with $\inf_{A\in\Pi}\mu(A) > 0$, where Π is the set of all atoms in Σ.*

Moreover, these statements imply that: for each positive number $p \geq 1$, $L^p(\mu)$ is isomerically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on (X,Σ,μ).

Proof. Since the implication (4)⟹(1) is obvious, in order to prove the equivalence of the statements (1) through (6), it is enough to prove the following implications: (1)⟹(2), (2)⟹(3), (3)⟹(4), (4)⟹(5), (5)⟹(6), and (6)⟹(2).

(1)⟹(2): suppose that $L^p \subset L^q$ for some pair p,q with $0 < p < q$. We claim $L^p \subset L^\infty$. Suppose there is an f in L^p which is not essentially bounded. Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that for each $k \geq 1$, the set $E_k =: \{x \in X \mid n_k \leq |f(x)| < n_k + 1\}$ is of a positive measure. Obviously E_k's are disjoint. Since $\mu(E_k)n_k^p \leq \int_X |f|^p d\mu \leq \int_X |f|^p d\mu$, it follows $\mu(E_k) \to 0$. This is a contradiction in view of Lemma 2.2.

(2)⟹(3): suppose that $L^p(\mu) \subset L^\infty(\mu)$ for some $p > 0$. Let r be any positive real number. We show $L^r(\mu) \subset L^\infty(\mu)$. let $f \in L^r(\mu)$. If $A = \{x : |f(x)| > 1\}$ is of measure zero, then obviously $f \in L^\infty(\mu)$. Suppose that A is a positive measure. Let $g = X_A f$, where X_A is the characteristic function of the set A. Clearly, $g \in L^r(\mu)$ and $|g| \geq 1$ a.e. Since $|g|^{r/p} \in L^p, |g|^{r/p} \in L^\infty$. Let $M = \operatorname{ess\,sup}|g|^{r/p}$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $(M+\delta)^{p/r} - M^{p/r} < \epsilon$. Since $\{x : |g(x)| > M^{p/r} + \epsilon\} \subseteq \{x : |g(x)| > (M+\delta)^{p/r}\}$, and $\mu(\{x : |g(x)|^{r/p} > M+\delta\}) = 0$, it follows that $\operatorname{ess\,sup}|g| \leq M^{p/r}$.

(3)⟹(4): suppose that $L^p \subset L^\infty$ for all $p > 0$. Let $g \in L^p$. Write $A = \{x : |g(x)| > 1\}$. If A is a nonatomic set of positive measure, by Proposition 2.1(c), A contains a disjoint sequence $\{E_n\}$ of measurable subsets of A of positive measure such that $\mu(E_n) \to 0$ as $n \to \infty$. As is noted in the proof of Lemma 2.3, we can construct a function f in L^p which is not in L^∞. Hence A contains an atom. Since the measure of A is finite, in view of Proposition 2.1(a), A cannot contain infinitely many atoms. Therefore, A can be written as a finite disjoint union of atoms. Suppose that $A = \cup_{i=1}^n \theta_i$, where θ_i's are disjoint atoms. By Proposition 2.1(d), g is constant on each θ_i, Let g_{θ_i} be the value of g on θ_i. Then for any $q > p$,

$$\begin{aligned} \int_X |g|^q\,du &= \int_{X-A} |g|^q\,du + \int_A |g|^q\,du \\ &\le \int_{X-A} |g|^p\,du + \sum_{i-1}^{n} |g_{\theta_i}|^q \mu(\theta_i) \\ &\le \int_X |g|^p\,du + \sum_{i-1}^{n} |g_{\theta_i}|^q \mu(\theta_i) < \infty. \end{aligned} \tag{3.1}$$

Hence $L^p \subset L^q$ for $q > p$.

(4)⟹(5): this follows from Lemmas 2.2(a) and 2.3.

(5)⟹(6): Proposition 2.1(c) implies that the space (X,Σ,μ) is atomic. Since atoms are of positive measure, obviously statement (5) implies that $\inf_{A\in\pi}\mu(A) > 0$.

(6)⟹(2): Suppose (X,Σ,μ) is atomic with $\inf_{A\in\pi}\mu(A) > 0$. Let $p > 0$ and $g \in L^p(\mu)$. Suppose $B = \{x \mid g(x)| > 1\}$. If $\mu(B) = 0$, then clearly $g \in L^\infty$. Suppose $\mu(B) > 0$. Obviously $\mu(B)$ is finite. Since $\inf_{A\in\pi}\mu(A) > 0$, B cannot contain infinitely many atoms. Therefore, B can be written as finite disjoint union of atoms. Since g is constant on each atom, it follows that $g \in L^\infty$.

Finally, we show that for $p \ge 1$, one of the statements (1) through (6) (and hence all of them) imply statement (7). Let (X,Σ,μ) be a measure space such that $L^p(\mu) \subseteq L^q(\mu)$ for some $1 \le p < q$. Let $\{\theta_i\}_{i\in\Gamma}$ be the collection of all atoms in X where Γ is some index set. Let $f \in L^p(\mu)$ be an arbitrary nonzero element of f. By Proposition 2.1(d) f is constant almost everywhere on any atom. We denote the value of f on an atom θ lies in the support of f by f_θ. Since the support of f is σ-finite, and by statement (5) of the theorem any measurable set of finite measure is disjoint union of finitely many atoms, the support of f can be written as countable union of atoms. Let $\{\theta_n(f)\}$ be the set of all atoms that forms the support of f. We define $F : L^p(\mu) \to \ell^p(\Gamma)$ by

$$F(f)(\gamma) = \begin{cases} f_{\theta_n}(\mu(\theta_n))^{1/p}, & \text{if } \theta_\gamma = \theta_n(f) \text{ for some } n, \\ 0, & \text{if } \theta_\gamma \notin \{\theta_n(f)\} \end{cases} \tag{3.2}$$

for any nonzero f in $L^p(\mu)$. The function F is well defined since any two functions that are equal in $L^p(\mu)$ are equal almost everywhere and thus share the same support. It is straightforward to verify that F is a one-to-one linear operator from $L^p(\mu)$ into $\ell^p(\Gamma)$. Let $h \in \ell^p(\Gamma)$. Since h is nonzero only on a countable subset Γ_h of Γ, define f on X as follows:

$$f(x) = \begin{cases} \dfrac{h(\gamma)}{(\mu(\theta_\gamma))^{1/p}}, & \text{if } x \in \theta_\gamma,\ \gamma \in \Gamma_h, \\ 0, & \text{if } x \notin \displaystyle\bigcup_{\gamma\in\Gamma_h} \theta_\gamma. \end{cases} \tag{3.3}$$

Obviously, $f \in L^p(\mu)$ and $F(f) = h$. Thus F is an isomorphism from $L^p(\mu)$ onto $\ell^p(\Gamma)$. Further for any $f \in L^p(\mu)$,

$$\begin{aligned} \|F(f)\|^p_{\ell^p(\Gamma)} &= \sum_i \left| f_{\theta i}\big(\mu(\theta_i)\big)^{(1/p)} \right|^p = \sum_i |f_{\theta_i}|^p \mu(\theta_i) \\ &= \sum_i \int_{\theta_i} |f(x)|^p \, d\mu = \int_X |f(x)|^p \, d\mu = \|f\|^p, \end{aligned} \tag{3.4}$$

where the sum runs over $i \in \Gamma$ such that θ_i is in the support of f.

Therefore F is an isometry. This completes the proof of the theorem. □

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