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MULTIPLIERS ON SOME WEIGHTED L^p-SPACES

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ABSTRACT. Let G be a locally compact abelian group with Haar measure dx, and let ω be a symmetric Beurling weight function on G (Reiter, 1968). In this paper, using the relations between p_i and q_i, where $1 < p_i,\ q_i < \infty$, $p_i \neq q_i$ $(i = 1,2)$, we show that the space of multipliers from $L^p_\omega(G)$ to the space $S(q_1', q_2', \omega^{-1})$, the space of multipliers from $L^{p_1}_\omega(G) \cap L^{p_2}_\omega(G)$ to $L^q_\omega(G)$ and the space of multipliers $L^{p_1}_\omega(G) \cap L^{p_2}_\omega(G)$ to $S(q_1', q_2', \omega^{-1})$.

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1. Introduction. Let G be a locally compact abelian group with Haar measure dx. In this paper, $C_0(G)$ denotes the space of all continuous complex-valued functions on G with compact support. Let $y \in G$. Then the translation operator τ_y is defined by

$$\tau_y f = f(y). \tag{1.1}$$

For a Beurling weight function on G (see Reiter [6]), i.e., a continuous function ω satisfying $\omega(x) \geq 1$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in G$.

We set, for $1 \leq p \leq \infty$, $L^p_\omega(G) = \{f \mid f\omega \in L^p(G)\}$. This is a Banach space under the norm

$$\|f\|_{p,\omega} = \left(\int_G |f(x)\omega(x)|^p dx\right)^{1/p}. \tag{1.2}$$

$L^1_\omega(G)$ is a Banach algebra with respect to convolution under the norm $\|\cdot\|_{1,\omega}$. It is called Beurling algebra. If $(1/p) + (1/p') = 1$, then the conjugate space of $L^p_\omega(G)$ is $L^{p'}_{\omega^{-1}}(G)$.

Let $1 < p_i < \infty$ $(i = 1,2)$ and let $S(p_1, p_2, \omega)$ be the set of all complex-valued functions g which can be written as

$$g = g_1 + g_2 \quad \text{with } (g_1, g_2) \in L^{p_1}_\omega(G) x L^{p_2}_\omega(G). \tag{1.3}$$

We define a norm on $S(p_1, p_2, \omega)$ by

$$\|g\|_s = \inf\left(\|g_1\|_{p_1,\omega} + \|g_2\|_{p_2,\omega}\right), \tag{1.4}$$

where the infimum is taken over all such decompositions of g. $S(p_1, p_2, \omega)$ is a Banach space under this norm (see Liu and Wang [4]).

Similarly, if $D(p_1, p_2, \omega)$ denotes the set of all complex-valued functions defined on G which are in $L^{p_1}_\omega(G) \cap L^{p_2}_\omega(G)$, we introduce a norm by

$$\|f\|_{D_p} = \max\left(\|f\|_{p_1,\omega}, \|f\|_{p_2,\omega}\right). \tag{1.5}$$

Then, $D(p_1,p_2,\omega)$ is also a Banach space with the norm $\|\cdot\|_{D_p}$.

It is not hard to see that $D(p_1,p_2,\omega)$ is a Banach $L^1_\omega(G)$ module. It is known that $D(p_1,p_2,\omega)$ and $S(p_1,p_2,\omega)$ are reflexive Banach spaces and the following duality relations hold:

$$\begin{aligned} D(p_1,p_2,\omega)^* &\cong S(p_1',p_2',\omega^{-1}),\\ D(p_1',p_2',\omega^{-1})^* &\cong S(p_1,p_2,\omega), \end{aligned} \tag{1.6}$$

where $(1/p_i)+(1/p_i')=1$, $i=1,2$ (see Murthy-Unni [5] and Liu and Wang [4]).

Let $f\in L^{q_1}_\omega(G)\cap L^{q_2}_\omega(G)$ with $1\le q_1\le q_2\le\infty$. Then, $f\in L^q_\omega(G)$ for all $q_1\le q\le q_2$ and

$$\|f\|_{q,\omega}\le\|f\|^{\alpha}_{q_1,\omega}\|f\|^{1-\alpha}_{q_2,\omega}, \tag{1.7}$$

where $1/q=(\alpha/q_1)+(1-\alpha)/q_2$, $0\le\alpha\le 1$ (see Brezis [1]).

Now, we define the space $K(p,q,q_1,q_2,\omega)$ to be set of all functions h which can be written in the form

$$h=\sum_{i=1}^{\infty}f_i^*g_i, \tag{1.8}$$

where $f_i\in C_c(G)\subset L^p_\omega(G)$ and $g\in D(q_1,q_2,\omega)$ with $\Sigma_{i=1}^{\infty}\|f_i\|_{p,\omega}\|g_i\|_D<\infty$. We define a norm on $K(p,q,q_1,q_2,\omega)$ by

$$\|h\|=\inf\left(\sum_{i=1}^{\infty}\|f_i\|_{p,\omega}\|g_i\|_{D_q}\right), \tag{1.9}$$

where the infimum is taken over all such representations of h. Then $K(p,q,q_1,q_2,\omega)$ is a Banach space in this norm. Since

$$\|f^*g\|_{r,\omega}\le\|f\|_{p,\omega}\|g\|_{q,\omega}\le\|f\|_{p,\omega}\|g\|_{D_q} \tag{1.10}$$

for $f\in C_c(G)\subset L^p_\omega$, $g\in L^{q_1}_\omega(G)\cap L^{q_2}_\omega(G)$, where $(1/p)+(1/q)\ge 1$, $(1/r)=(1/p)+(1/q)-1$, and condition (1.7) holds, it follows that $K(p,q,q_1,q_2,\omega)\subset L^r_\omega(G)$ and that the topology on $K(p,q,q_1,q_2,\omega)$ is not weaker than the topology induced from $L^r_\omega(G)$.

We say that T is a multiplier from $L^p_\omega(G)$ to $S(q_1',q_2',\omega^{-1})$ if T is a bounded linear operator on $L^p_\omega(G)$ which commutes with translation. The space of all multipliers from $L^p_\omega(G)$ to $S(q_1',q_2',\omega^{-1})$ is denoted by $M[L^p_\omega(G),S(q_1',q_2',\omega^{-1})]$.

2. Multipliers from $L^p_\omega(G)$ to $S(q_1',q_2',\omega^{-1})$.

We have the following.

Theorem 2.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function. If condition (1.7) is satisfied and $(1/p)+(1/q)\ge 1$, $(1/p)+(1/q)-1=1/r$, then the space of multipliers $M[L^p_\omega(G),S(q_1',q_2',\omega^{-1})]$ is isometrically isomorphic to the dual $K(p,q,q_1,q_2,\omega)^*$ of $K(p,q,q_1,q_2,\omega)$.*

Proof. For any $T\in M[L^p_\omega(G),S(q_1',q_2',\omega^{-1})]$, define

$$t(h)=\sum_{i=1}^{\infty}Tf_i^*g_i(0) \tag{2.1}$$

for $h = \Sigma_{i=1}^{\infty} f_i^* g_i$ in $K(p,q,q_1,q_2,\omega)$. First, we show that t is well defined. To this end, it is sufficient to show that if $h = \Sigma_{i=1}^{\infty} f_i^* g_i = 0$ in $K(p,q,q_1,q_2,\omega)$ and $\Sigma_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{D_q} < \infty$, then $\Sigma_{i=1}^{\infty} T f_i^* g_i(0) = 0$.

It is known that $L_\omega^p(G)$ has approximate identities bounded in $L_\omega^1(G)$ with compactly supported (see Murthy-Unni [5]). Let $(\phi_\alpha)_{\alpha\in I}$ be an approximate identity for $L_\omega^p(G)$ with $\|\phi_\alpha\|_1 = 1$ and $\|\phi_\alpha\|_{1,\omega} \le K$ $(K > 0)$. Then, for each $f \in L_\omega^p(G)$, we have

$$\lim_\alpha \|\phi_\alpha^* f - f\|_{p,\omega} = 0. \tag{2.2}$$

Therefore, using (2.2) and the fact that T is a multiplier for all $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$, we obtain

$$\left| T(\phi_\alpha^* f_i)^* g_i(0) - T f_i^* g_i(0) \right| \le \|T\| \|\phi_\alpha^* f_i - f_i\|_{p,\omega} \|g_i\|_{D_q} \longrightarrow 0, \tag{2.3}$$

so that

$$\lim_\alpha T(\phi_\alpha^* f_i)^* g_i(0) = T f_i^* g_i(0). \tag{2.4}$$

Also, for each $\phi_\alpha \in C_c(G)$ and $f_i \in C_c(G)$, we have

$$T(\phi_\alpha^* f_i) = T\phi_\alpha^* f_i. \tag{2.5}$$

(see Larsen [2]).

Since $u = \Sigma_{i=1}^{\infty} f_i^* g_i = 0$ and the series $\Sigma_{i=1}^{\infty} f_i^* g_i$ converges uniformly and using equality (2.5), we get

$$\begin{aligned} \sum_{i=1}^{\infty} T(\phi_\alpha^* f_i)^* g_i(0) &= \sum_{i=1}^{\infty} \int_G (T\phi_\alpha)(-y)(f_i^* g_i)(y)\,dy \\ &= \int_G (T\phi_\alpha)(-y) \sum_{i=1}^{\infty} (f_i^* g_i)(y)\,dy = 0. \end{aligned} \tag{2.6}$$

We show that $\Sigma_{i=1}^{\infty} T(\phi_\alpha^* f_i)^* g_i(0)$ converges uniformly with respect to α.

$$\begin{aligned} \left| \sum_{i=1}^{\infty} T(\phi_\alpha^* f_i)^* g_i(0) \right| &\le \sum_{i=1}^{\infty} \|T(\phi_\alpha^* f_i)\|_s \|g_i\|_{D_q} \\ &\le \|T\| \sum_{i=1}^{\infty} \|\phi_\alpha^* f_i\|_{p,\omega} \|g_i\|_{D_q} \\ &= \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{D_q} < \infty. \end{aligned} \tag{2.7}$$

The convergence of $\Sigma_{i=1}^{\infty} T(\phi_\alpha^* f_i)^* g_i(0)$ is uniform with respect to α. Hence,

$$\lim_\alpha \sum_{i=1}^{\alpha} T(\phi_\alpha^* f_i)^* g_i(0) = \sum_{i=1}^{\infty} T f_i^* g_i(0) = 0 \tag{2.8}$$

using (2.4) and (2.6). Thus, t is well defined.

It is obvious that the mapping $T \to t$ is linear. Now we show that it is an isometry. In fact,

$$|t(u)| = \left|\sum_{i=1}^{\infty} Tf_i^* g_i(0)\right| \le \sum_{i=1}^{\infty} \|Tf_i\|_s \|g_i\|_{D_q} \le \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{D_q} \tag{2.9}$$

implies that

$$|t(u)| \le \|T\|\|u\|. \tag{2.10}$$

Hence, $\|t\| \le \|T\|$. On the other hand,

$$\begin{aligned} \|T\| &= \sup\left\{|Tf^*g(0)| : \|f\|_{p,\omega} \le 1, \|g\|_{D_q} \le 1\right\} \\ &= \sup\left\{|t(f^*g)| : \|f\|_{p,\omega} \le 1, \|g\|_{D_q} \le 1\right\} \le \|t\|. \end{aligned} \tag{2.11}$$

Therefore, $\|t\| = \|T\|$.

Finally, we show that the mapping $T \to t$ is onto. Suppose that $t \in K(p,q,q_1,q_2,\omega)^*$ and that $f \in C_c(G) \subset L^p_\omega(G)$. Define, for all $g \in L^{q_1}_\omega(G) \cap L^{q_2}_\omega(G)$,

$$g \longrightarrow t(f^*g). \tag{2.12}$$

Then,

$$|t(f^*g)| \le \|t\|\||f^*g|\| \le \|t\|\|f\|_{p,\omega}\|g\|_{D_q} \tag{2.13}$$

implies that this mapping gives a bounded linear functional on $L^{q_1}_\omega(G) \cap L^{q_2}_\omega(G)$. Hence, there exists a unique element, denoted by Tf, in $S(q_1', q_2', \omega^{-1})$ such that

$$Tf^*g(0) = t(f^*g) \tag{2.14}$$

for all $g \in L^{q_1}_\omega(G) \cap L^{q_2}_\omega(G)$, and

$$|Tf^*g(0)| \le \|t\|\|f\|_{p,\omega}\|g\|_{D_q}. \tag{2.15}$$

Therefore,

$$\|Tf\|_s \le \|t\|\|f\|_{p,\omega} < \infty. \tag{2.16}$$

Hence, T is a bounded operator from $C_c(G)$ into $S(q_1', q_2', \omega^{-1})$. Clearly, T is linear. Since $C_c(G)$ is dense $L^p_\omega(G)$. It can be extended uniquely as a bounded linear operator on $L^p_\omega(G)$. We have to prove that this extended T is a multiplier. Let $y \in G$ and $f \in C_c(G) \subset L^p_\omega(G)$. If $g \in L^{q_1}_\omega(G) \cap L^{q_2}_\omega(G)$, then

$$\tau_y Tf^*g(0) = Tf^*\tau_y g(0) = t(f^*\tau_y g) = t(\tau_y f^*g) = T\tau_y f^*g(0) \tag{2.17}$$

holds for all functions g in $L^{q_1}_\omega(G) \cap L^{q_2}_\omega(G)$. Hence,

$$\tau_y Tf = T\tau_y f. \tag{2.18}$$

Thus, T belongs to $M[L^p_\omega(G), S(q_1', q_2', \omega^{-1})]$ and our assertion is proved. □

3. Multipliers from $L_\omega^{p_1} \cap L_\omega^{p_2}(G)$ to $L_\omega^q(G)$. Let $f \in L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$, with $1 < p_1 \le p_2 < \infty$, then $f \in L_\omega^p(G)$ for all $p_1 \le p \le p_2$ and

$$\|f\|_{p,\omega} \le \|f\|_{p_1,\omega}^{\alpha} \|f\|_{p_2,\omega}^{1-\alpha}, \tag{3.1}$$

where $1/p = (\alpha/p_1) + (1-\alpha)/p_2$, $0 \le \alpha \le 1$.

Now, we define the space $K(p_1, p_2, p, q, \omega)$ to be set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i^* g_i, \tag{3.2}$$

where $f_i \in C_c(G) \subset L_\omega^{p_1} \cap L_\omega^{p_2}(G)$ and $g_i \in L_{\omega^{-1}}^{q'}(G)$ with $\Sigma_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{q',\omega^{-1}} < \infty$. We define a norm on $K(p_1, p_2, p, q, \omega)$ by

$$|||h||| = \inf \left(\sum_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{q',\omega^{-1}} \right), \tag{3.3}$$

where the infimum is taken over all such representations of h. Then $K(p_1, p_2, p, q, \omega)$ is a Banach space in this norm. Since

$$\|f^* g\|_{r,\omega^{-1}} \le \|f\|_{p,\omega} \|g\|_{q',\omega^{-1}} \le \|f\|_{D_p} \|g\|_{q',\omega^{-1}} < \infty \tag{3.4}$$

for $f \in C_c(G) \subset L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ and $g \in L_{\omega^{-1}}^{q'}(G)$, where condition (3.1) holds and $1 < p \le q < \infty$, $1/r = (1/p) - (1/q)$, it follows that $K(p_1, p_2, p, q, \omega) \subset L_{\omega^{-1}}^{r'}(G)$.

Theorem 3.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function under condition (3.1) and $1 < p \le q < \infty$, $1/r = (1/p) - (1/q)$, then the space of multipliers $M[L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G),\ L_\omega^q(G)]$ is isometrically isomorphic to the dual $K(p_1, p_2, p, q, \omega)^*$ of $K(p_1, p_2, p, q, \omega)$.*

Proof. Using the same method in the proof of Theorem 2.1, we can show our assertion. □

4. Multipliers from $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to the space $S(q_1', q_2', \omega^{-1})$. Suppose that $(1/p_i) + (1/q_i) \ge 1$, $1 < p_i$, $q_i < \infty$ and $(1/p_i) + (1/q_i) - 1 = 1/r_i$ $(i = 1, 2)$. Let $L_\omega^{r_1, r_2}(G)$ denote the set of all complex-valued functions defined on G which are in $L_\omega^{r_1}(G) \cap L_\omega^{r_2}(G)$. We introduce a norm by

$$\|f\|_\omega^{r_1, r_2} = \max(\|f\|_{r_1,\omega}, \|f\|_{r_2,\omega}). \tag{4.1}$$

Then $L_\omega^{r_1, r_2}(G)$ is also a Banach space with norm $\|\cdot\|_\omega^{r_1, r_2}$ (see Liu and Wang [3]).

To obtain the space of multipliers from $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to $S(q_1', q_2', \omega^{-1})$ as a dual space, we define the space $K(p_1, p_2, q_1, q_2, \omega)$ to be the set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i^* g_i, \tag{4.2}$$

where $f_i \in C_c(G) \subset L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ and $g_i \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$ with $\Sigma_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{D_q} <$

∞. It is not hard to see that $C_c(G)$ is dense in $L_\omega^{p_1}(G)\cap L_\omega^{p_2}(G)$. Define a norm $h \to |||h|||$ by

$$|||h||| = \inf\left(\sum_{i=1}^{\infty}\|f_i\|_{D_p}\|g_i\|_{D_q}\right), \tag{4.3}$$

where the infimum is taken over all such representations of h. It is easy to verify that $|||\cdot|||$ defines a norm on $K(p_1,p_2,q_1,q_2,\omega)$ and that the latter is a Banach space.

Now, let $f \in C_c(G) \subset L_\omega^{p_1}(G)\cap L_\omega^{p_2}(G)$ and $g \in L_\omega^{q_1}(G)\cap L_\omega^{q_2}(G)$. It follows that $f^*g \in L_\omega^{r_1}(G)$,

$$\|f^*g\|_{r_1,\omega} \le \|f\|_{p_1,\omega}\|g\|_{q_1,\omega} \le \|f\|_{D_p}\|g\|_{D_q} \tag{4.4}$$

and $f^*g \in L_\omega^{r_2}(G)$,

$$\|f^*g\|_{r_2,\omega} \le \|f\|_{p_2,\omega}\|g\|_{q_2,\omega} \le \|f\|_{D_p}\|g\|_{D_q} \tag{4.5}$$

so that

$$\|f^*g\|_\omega^{r_1,r_2} \le \|f\|_{D_p}\|g\|_{D_q}. \tag{4.6}$$

From this, it is clear that $K(p_1,p_2,q_1,q_2,\omega) \subset L_\omega^{r_1,r_2}(G)$ and that the topology on $K(p_1,p_2,q_1,q_2,\omega)$ is not weaker than the topology induced by $L_\omega^{r_1,r_2}(G)$.

Theorem 4.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function. If $(1/p_i)+(1/q_i) \ge 1$, $(1/p_i)+(1/q_i)-1 = (1/r_i)$, $i = 1,2$, then the space of multipliers $M[L_\omega^{p_1}(G)\cap L_\omega^{p_2}(G), S(q_1',q_2',\omega)]$ is isometrically isomorphic to $K(p_1,p_2,q_1,q_2,\omega)^*$, the dual space of $K(p_1,p_2,q_1,q_2,\omega)$.*

Proof. Use the same method employed in the proof of Theorem 2.1. □

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