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LONG CYCLES IN CERTAIN GRAPHS OF LARGE DEGREE

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ABSTRACT. Let G be a connected graph of order n and $X = \{x \in V : d(x) \geq n/2\}$. Suppose $|X| \geq 3$ and G satisfies the modified Fan's condition. We show that the vertices of the block B of G containing X form a cycle. This generalizes a result of Fan. We also give an efficient algorithm to obtain such a cycle. The complexity of this algorithm is $O(n^2)$. In case G is 2-connected, the condition $|X| \geq 3$ can be removed and G is hamiltonian.

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1. Introduction. We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard and can be found in [4, 8]. Let $G = (V,E)$ be a graph of order n $(= |V|)$. For each vertex $x \in V$, let $N(x) = \{v \in V : v$ is adjacent to $x\}$. Then $d(x) = |N(x)|$ is the degree (valency) of x in G. Denote by $\operatorname{dist}(x,y)$ the distance between x and y in G $(x,y \in V)$.

A graph G is said to satisfy Fan's condition, if $\min\{\max\{d(x),d(y)\} : \operatorname{dist}(x,y) = 2\ (x,y \in V)\} \geq n/2$. In [7], it was shown that a 2-connected graph which satisfies Fan's condition is hamiltonian. Fan's theorem is a direct generalization of Dirac's theorem [4, page 54, Theorem 4.3] and it opened an entirely new approach to study hamiltonian graphs. In [3], Fan's theorem was strengthened, where the same conditions were shown to imply the graph is pancyclic, with a few minor exceptions (also see [1]). Some generalizations of Fan's theorem can be found in [2, 5, 11, 12]. A similar result is obtained for bipartite graphs [13].

The purpose of this paper is to generalize Fan's theorem and give an algorithm to find a hamiltonian cycle. Let $X = \{x \in V : d(x) \geq n/2\}$. Suppose G is a connected graph and $|X| \geq 3$. We show that X is contained in a cycle C of G. Hence X is contained in a block B of G (Lemma 2.3). If, in addition, G satisfies the modified Fan's condition, then the vertices of B form a cycle (Theorem 2.10). From this proof, we obtain an algorithm to find such a cycle. The complexity of the algorithm is $O(n^2)$. If G is 2-connected, the condition $|X| \geq 3$ can be removed and G is hamiltonian (Corollary 2.12).

2. Existence of long cycles. Let $G = (V,E)$ be a connected graph of order n and $X = \{x \in V : d(x) \geq n/2\}$.

LEMMA 2.1. *If $d(u)+d(v) \geq n$ $(u,v \in V)$ and u is not adjacent to v, then $|N(u) \cap N(v)| \geq 2$. Hence u and v are contained in a 4-cycle and* $\operatorname{dist}(u,v) = 2$.

Proof. Suppose the lemma is not true. Then $|N(u)\cap N(v)| \le 1$ and so

$$n \ge d(u)+d(v)-1+|\{u,v\}| \ge n-1+2 = n-1. \tag{2.1}$$

This is impossible and so the lemma is true. □

The following lemma is probably known. For completeness, we give a proof.

Lemma 2.2. *Let $P = u_1u_2\cdots u_k$ $(k \ge 3)$ be a path in G. If $d(u_1)+d(u_k) \ge n$, then the vertices of P are contained in a cycle C of G.*

Proof. If u_1 is adjacent to u_k, then the lemma is true. Hence we can assume that u_1 is not adjacent to u_k. If $d(u_1) = 1$, then $d(u_k) \le n-2$, which is impossible. Hence $d(u_1) \ge 2$ and $d(u_k) \ge 2$. If there exists a vertex $x \notin P$ such that x is adjacent to both u_1 and u_k, then the lemma is true. Therefore, we can assume that $N(u_1)\cap N(u_k) \subset P$. By Lemma 2.1, $|N(u_1)\cap N(u_k)| \ge 2$ and so $k \ge 4$. Hence there exists some $u_t \in N(u_1)$ $(t \ne 2$ or $k)$ such that $u_{t-1} \in N(u_k)$; otherwise $d(u_k) \le (n-1)-d(u_1)$, which is impossible. Therefore, $C = u_1u_2\cdots u_{t-1}u_ku_{k-1}\cdots u_t$ is the required cycle. □

A block of a graph is a subgraph that has no cut vertices and is maximal with respect to this property (see [4, page 44]).

Lemma 2.3. *Suppose $|X| \ge 3$. Then the vertices of X are contained in a cycle C of G. Hence X is contained in a block of G.*

Proof. Let $u,v,w \in X$. If these three vertices are adjacent to each other, then we have a triangle uvw. If u is not adjacent to v, then by Lemma 2.1, we have a 4-cycle containing u and v. Hence we can assume that there is a cycle C in G containing at least two vertices $u,v \in X$. Suppose there exists a vertex $w \in X$ and $w \notin C$. We show that there exists a cycle C' containing the vertices $X\cap C$ and $\{w\}$. First we claim that there is a path P of length at most 4 passing through w and connecting two vertices of C that is internally disjoint from C.

Case 1. Assume w is adjacent to u. If w is also adjacent to v, then our claim is clearly true. Hence we can assume that w is not adjacent to v. Then by Lemma 2.1, there is a 4-cycle containing v and w. Therefore, there exists a vertex x such that x is adjacent to v and w. If $x \in C$, then $P = xwu$; otherwise $P = vxwu$.

Case 2. Assume that w is not adjacent to u or v. Then by Lemma 2.1, $|N(w)\cap N(u)| \ge 2$ and $|N(w)\cap N(v)| \ge 2$. Hence there exist two vertices x and y such that x is adjacent to w and u, and y is adjacent to w and v. If both x and $y \notin C$, then $P = uxwyv$; otherwise we have a shorter path. This proves our claim.

Let $\{w_i,w_j\} = P\cap C$. If the section of cycle C from w_i to w_j contains all vertices of $X\cap C$, then $C' = w_i\cdots w_j \cup P$ is the required cycle. Hence we can assume that the section of C from w_i to w_j contains a vertex $u \in X\cap C$ and the section of C from w_j to w_i contains a vertex $v \in X\cap C$. Furthermore, we can assume that the section of C from u to w_j on C contains no interior vertex which is in X and the section of C from v to w_i contains no interior vertex which is in X. Since $u,v \in X$, it follows from Lemma 2.2 that the path $P' = u\cdots w_i\cdots w\cdots w_j\cdots v$ is contained in cycle C' of G. Since X is finite, we can always obtain a cycle containing all vertices of X. This completes the proof. □

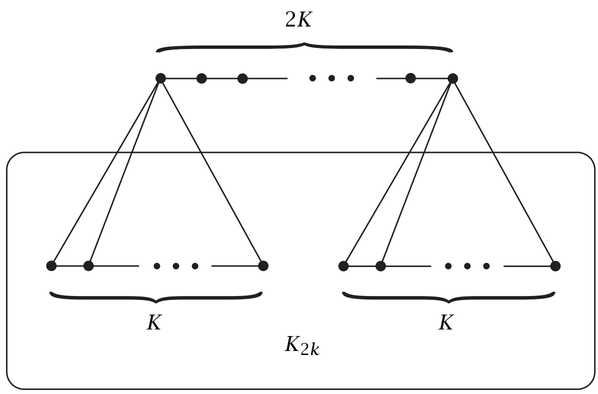

FIGURE 2.1.

REMARK 2.4. If $|X| \le 2$, Lemma 2.3 is not true. Let G be the graph with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that v_1 is adjacent to v_2, v_3, and v_4; v_2 is adjacent to v_3; v_4 is adjacent to v_5 and v_6; v_5 is adjacent to v_6. Then $|X| = 2$ and there is no cycle containing X in G.

REMARK 2.5. Suppose G is 2-connected. Then every two vertices of G is contained in a cycle. Hence we can always find a cycle containing all vertices of X by the proof of Lemma 2.3. Therefore, the condition $|X| \ge 3$ in Lemma 2.3 can be removed, if G is assumed to be 2-connected.

A graph G is said to satisfy the *modified Fan's condition*, if for any vertex $w \in V$ with $d(w) \ge 3$, we have $x, y \in N(w)$ implies either x is adjacent to y or $\max\{d(x), d(y)\} \ge n/2$.

REMARK 2.6. Let G be the graph with $V = \{v_1, v_2, v_3, v_3, v_4, v_5, v_6\}$ such that v_1 is adjacent to v_3, v_4, v_5, v_6; v_2 is adjacent to v_3, v_4, v_5, v_6; v_3 is adjacent to v_4. Then $|X| \ge 3$, but G is not hamiltonian and G does not satisfy the modified Fan's condition.

REMARK 2.7. If G satisfies Fan's condition, then G satisfies the modified Fan's condition, but not vise versa.

EXAMPLE 2.8. Let $n = 4k$ $(k \ge 2)$ and G the graph given in Figure 2.1. Then G satisfies the modified Fan's condition, but not the Fan's condition. It is easy to see that the diameter of G is equal to $k+1$. However, the diameter of any graph satisfying Fan's condition is less than or equal to 6.

LEMMA 2.9. *Suppose* $|X| \ge 3$ *and* G *satisfies the modified Fan's condition. Let* C *be a cycle containing* X*. If* $z \notin C$ *and* $|N(z) \cap C| \ge 2$*, then there exists a cycle* C' *containing* z *and* C*.*

PROOF. Write $C = w_1 w_2 \cdots w_k$ $(k \ge 3)$. Since $z \notin C$, $d(z) < n/2$. Let w_i and w_j $(i < j)$ be two vertices of C which are adjacent to z. If z is adjacent to w_{i+1}, then $C' = w_i \cdots w_1 \cdots w_{i+1} z$ is the required cycle. Therefore, we can assume that z is not adjacent to $w_{i+1}, w_{i-1}, w_{j+1}$, or w_{j-1}. Since $d(w_i) \geq 3$ and z is not adjacent to w_{i+1}, we have $d(w_{i+1}) \geq n/2$. Similarly, $d(w_{j+1}) \geq n/2$. Let $P = w_{i+1} \cdots w_j z w_i w_{i-1} \cdots w_{j+1}$. Then by Lemma 2.2, there exists a cycle C' containing P. This proves Lemma 2.9. □

Theorem 2.10. *Suppose $|X| \geq 3$ and G satisfies the modified Fan's condition. Let B be the block of G containing X. Then the vertices of B form a cycle.*

Proof. By Lemma 2.3, there exists a cycle C containing the vertices of X. Let $A = V(B) - V(C)$ and suppose $A \neq \emptyset$. We show that there is a larger cycle C' containing all the vertices of C. Let $a = \max\{d(v) : v \in A\}$ and write $C = w_1 w_2 \cdots w_k$ $(k \geq 3)$.

Case 1. Suppose $a = 2$. Hence $d(v) = 2$ for each $v \in A$. Let $z \in A$. Then by constructing a tree rooted at z, we can find a path $Q = u_1 u_2 \cdots u_t$ with $u_1 = w_i$, $u_t = w_j$ and $u_l \in A$ $(1 < l < t)$. If $j = i-1$ or $i+1$, then $C' = w_j \cdots w_i u_2 \cdots u_{t-1}$. Hence we can assume that w_i and w_j are nonconsecutive vertices of C. Since $d(u_2) = 2$, u_2 is not adjacent to w_{i+1}. Since $d(w_i) \geq 3$ and $d(u_2) < n/2$, it follows that $d(w_{i+1}) \geq n/2$. Similarly, $d(w_{j+1}) \geq n/2$. Let $P = w_{i+1} \cdots w_j u_{t-1} \cdots u_2 w_i \cdots w_{j+1}$. Then by Lemma 2.2, P is contained in a cycle C' of B.

Case 2. Suppose $a \geq 3$ and choose $z \in A$ such that $d(z) = a \geq 3$. By Lemma 2.9, we can assume $|N(z) \cap C| \leq 1$. We show that $N(z) \cap C \neq \emptyset$. Suppose this is not true. Then $d(u) < n/2$ for all $u \in N(z)$ and so $N(z) \cup \{z\}$ forms a complete subgraph of G. Hence $d(u) \geq d(z)$ and by the maximality of $d(z)$, we have $d(u) = d(z)$ for all $u \in N(z)$. But G is connected and this is impossible. Hence $|N(z) \cap C| = 1$.

Let $\{w_i\} = N(z) \cap C$. Then the set $H = (N(z) - \{w_i\}) \cup \{z\}$ forms a complete subgraph of G. Hence $d(u) \geq d(z) - 1$ for all $u \in H$ and $d(u) = d(z) - 1$ if and only if u is adjacent only to vertices of H. Since w_i is not a cut vertex of B, there exists some vertex $v \in H$ such that $v \notin N(w_i)$ and $d(v) > d(z) - 1$. By the maximality of $d(z)$, we have $d(v) = d(z)$. Then there exists a $y \notin H \cup \{w_i\}$ such that y is adjacent to v. Suppose $d(y) < n/2$. Since $d(z) < n/2$ and $d(v) = d(z) \geq 3$, it follows that y is adjacent to z. Hence $y \in H$, which is impossible. Therefore $d(y) \geq n/2$ and so $y \in C$. Write $y = w_j$. Since $d(v) = d(z)$, v is not adjacent to w_{j-1} or w_{j+1}. Let $Q = w_i z v w_j$. Then by the proof of Case 1, we can find a larger cycle C' containing all vertices of C.

Since B is finite, we can always obtain a cycle containing all vertices of B. This completes the proof of the theorem. □

Remark 2.11. If $|X| \leq 2$, then Theorem 2.10 is not true. See the example given in Remark 2.4.

By using Remark 2.5 and the proof of Theorem 2.10, we have the following result which generalizes Fan's theorem.

Corollary 2.12. *Suppose G is 2-connected and satisfies the modified Fan's condition, then G is hamiltonian.*

3. An algorithm.

In this section, let G be represented by an adjacent list (see [6, page 173] or [10, page 17]) and let $P = u_1 u_2 \cdots u_k$ $(k \geq 3)$ be a path in G.

Algorithm 3.1. (If $d(u_1) + d(u_k) \geq n$, we find a cycle C containing all the vertices of P.)

Step 1. Let $f(v) \neq 0$ for each $v \in N(u_1)$, $g(v) \neq 0$ for each $v \in N(u_k)$ and $h(u_i) \neq 0$ $(i = 1, 2, \ldots, k)$.

Step 2. If $g(u_1) \neq 0$, then $u_1 \in N(u_k)$ and stop.

Step 3. If there exists a vertex $x \in N(u_1)$ such that $g(x) \neq 0$ and $h(x) = 0$, then $x \in N(u_1) \cap N(u_k)$ and $x \notin P$. Let $C \leftarrow u_1 u_2 \cdots u_k x$ and stop.

Step 4. Find t $(t \neq 2$ or $k)$ such that $f(u_t) \neq 0$ and $g(u_{t-1}) \neq 0$. Let $C \leftarrow u_1 u_2 \cdots u_{t-1} u_k u_{k-1} \cdots u_t$ and stop.

The correctness of the algorithm follows from Lemma 2.2 and its complexity is clearly $O(n)$.

Algorithm 3.2. (Assume $|X| \geq 3$. Then the vertices of X is contained in a cycle C of G.)

Step 1. Find a cycle C containing at least two vertices in X.

Step 2. Let $X' = X - C$. If $X' = \emptyset$, stop. Otherwise let $w \in X'$ and $u, v \in X$.

Step 3. Find a path P of length at most 4 passing through w and connecting two vertices $\{w_i, w_j\}$ of C that is internally disjoint from C.

Step 4. Let $Q_1 \leftarrow w_i \cdots w_j$ and $Q_2 \leftarrow w_j \cdots w_i$. If $Q_2 \cap X = \emptyset$, let $C \leftarrow Q_1 \cup P$ and go to Step 2. If $Q_1 \cap X = \emptyset$, let $C \leftarrow Q_2 \cup P$ and go to Step 2.

Step 5. Choose $u \in Q_1$ and $v \in Q_2$ such that the sections $u \cdots w_j$ and $v \cdots w_i$ of C contains no interior vertex in X. Let $P \leftarrow u \cdots w_i \cdots w \cdots w_j \cdots v$.

Step 6. Apply Algorithm 3.1 to the path P and obtain a cycle C containing all vertices of P. Go to Step 2.

The correctness of the algorithm follows from Lemma 2.3.

We show that Algorithm 3.2 can be implemented in $O(n^2)$ time. Pick u, $v, w \in X$ and set $f(x) \neq 0$ for all $x \in N(u)$ and $g(x) \neq 0$ for all $x \in N(v)$. If $f(v) \neq 0$, $f(w) \neq 0$ and $g(w) \neq 0$, then let $C \leftarrow uvw$. Otherwise, we can assume v is not adjacent to u, that is, $f(v) = 0$. Then by Lemma 2.1, there exist $x, y \in N(u)$ such that $g(x) \neq 0$ and $g(y) \neq 0$. Let $C \leftarrow uxvy$. Hence Step 1 takes $O(n)$ time.

Let $f(z) \neq 0$ for all $z \in N(u)$, $g(z) \neq 0$ for all $z \in N(v)$, $h(z) \neq 0$ for all $z \in N(w)$, and $H(z) \neq 0$ for all $z \in C$. Suppose first that u is adjacent to w; that is, $h(u) \neq 0$. If $h(v) \neq 0$, then let $P \leftarrow uvw$. If $h(v) = 0$, then by Lemma 2.1, we can find a vertex $x \in N(w)$ such that $g(x) \neq 0$. If $H(x) \neq 0$, then let $P \leftarrow xwu$; otherwise let $P \leftarrow vxwu$. Now assume that u and v are not adjacent to w, that is, $h(u) = 0$ and $h(v) = 0$. Then there exists two vertices x and y such that $x \in N(u)$ and $h(x) \neq 0$, $y \in N(v)$ and $h(y) \neq 0$. If both $H(x) = 0$ and $H(y) = 0$, let $P \leftarrow uxwyv$. Otherwise we have a shorter path P. Hence Step 3 takes $O(n)$ time.

It is easy to see that Steps 4, 5, and 6 can be implemented in $O(n)$ time. Combine this with Step 2, we have an $O(n^2)$ algorithm.

Let B be the block of G containing X. Then B can be found using the depth-first search for blocks algorithm by Hopcroft and Tarjan (see [6] or [10]). The complexity of this algorithm is $O(\max(n, |E|))$. For our graphs, $|E| = O(n^2)$. Therefore it takes $O(n^2)$ time to find the block B of G.

Algorithm 3.3. (Assume $|X| \geq 3$ and G satisfies the modified Fan's condition. We find a cycle containing the vertices of B.)

Step 1. Find a cycle C containing the vertices of X (use Algorithm 3.2).

Step 2. Let $A \leftarrow V(B) - V(C)$. If $A = \emptyset$, stop.

Step 3. Let $a \leftarrow \max\{d(v) : v \in A\}$ and $z \in A$ with $d(z) = a$. Write $C = w_1 w_2 \cdots w_k$ $(k \geq 3)$. If $a \geq 3$, go to Step 7.

Step 4 $(a = 2)$**.** Find a path $Q = u_1 u_2 \cdots u_t$ such that $u_1 = w_i$, $u_t = w_j$, and $u_l \in A$ $(1 < l < t)$.

Step 5. If $j = i - 1$ or $i + 1$, let $C \leftarrow w_j \cdots w_i u_2 \cdots u_{t-1}$ and go to Step 2.

Step 6. Assume u_i and u_j are nonconsecutive vertices of C. Let $P \leftarrow w_{i+1} \cdots w_j$ $u_{t-1} \cdots u_2 w_i \cdots w_{j+1}$. Use Algorithm 3.1 to find a cycle C containing all vertices of P and go to Step 2.

Step 7 $(a \geq 3)$**.** If $|N(z) \cap C| = 1$, go to Step 10.

Step 8 $(|N(z) \cap C| \geq 2)$**.** Let w_i and w_j be two vertices of C which are adjacent to z. If $z \in N\ (w_{i+1})$, then let $C \leftarrow w_i \cdots w_1 \cdots w_{i+1} z$ and go to Step 2.

Step 9 (assume z is not adjacent to w_{i+1}, w_{i-1}, w_{j+1} or w_{j-1})**.** Let $P \leftarrow w_{i+1} \cdots w_j$ $z w_i w_{i-1} \cdots w_{j+1}$. Apply Algorithm 3.1 to find a cycle C containing the vertices of P and go to Step 2.

Step 10. Let $\{w_i\} = N(z) \cap C$ and $H \leftarrow (N(z) - \{w_i\}) \cup \{z\}$. Find a vertex $v \in H$ such that $v \notin N(w_i)$ and $d(v) = d(z)$.

Step 11. Find $w_j \in N(v)$ and $w_j \notin H \cup \{w_i\}$. Let $u_2 \leftarrow z$, $u_3 \leftarrow v$, $t \leftarrow 4$ and $Q \leftarrow w_i u_2 u_3 w_j$. Go to Step 5.

The correctness of the algorithm follows from Theorem 2.10.

We show that the algorithm can be implemented in $O(n^2)$ time. Step 1 uses Algorithm 3.2 and so it takes $O(n^2)$ time. Step 3 can be done in $O(n)$ time. (If we use F-heaps data structure [9], it takes $O(\log n)$ time.) In Step 4, we construct a tree rooted at z. As soon as we find two vertices w_i and $w_j \in C$, we stop. Since $a = 2$, each vertex of Q has degree 2. Hence Q can be constructed in $O(n)$ time. It is clear that Steps 5, 6, 7, 8, 9, 10, and 11 can be done in $O(n)$ time. Combine this with Step 2, we have an $O(n^2)$ algorithm.

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