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A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS

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Abstract. We give a new invariant characteristic property of Möbius transformations.

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1. Introduction.

Throughout this paper, we let $w = f(z)$ be a nonconstant meromorphic function in $\mathbb{C}$ unless otherwise stated.

We consider the following properties.

Property 1.1. $w = f(z)$ transforms circles in the z-plane onto circles in the w-plane, including straight lines among circles.

Property 1.2. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ and if $A'B'C'D'$ is a quadrilateral on the w-plane which is not self-intersecting, then the following hold

$$\angle A + \angle C = \angle A' + \angle C', \qquad \angle B + \angle D = \angle B' + \angle D'. \tag{1.1}$$

The following is a well-known principle of circle transformation of Möbius transformations.

Theorem 1.3. *$w = f(z)$ satisfies Property 1.1 if and only if $w = f(z)$ is a Möbius transformations.*

In [1], it is shown that Property 1.1 implies Property 1.2 and a new invariant characteristic property of Möbius transformations is given as follows.

Theorem 1.4. *Let α be an arbitrary fixed real number such that $0 < \alpha < 2\pi$. Suppose that $w = f(z)$ is analytic and univalent in a nonempty simply connected domain $\mathbb{R}$ on the z-plane. Let $ABCD$ be an arbitrary quadrilateral (not self-intersecting) contained in $\mathbb{R}$ satisfying*

$$\angle A + \angle C = \alpha. \tag{1.2}$$

If $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$ is a quadrilateral on the w-plane which is not self-intersecting, then the only function which satisfies

$$\angle A' + \angle C' = \alpha \tag{1.3}$$

is a Möbius transformation.

Theorem 1.4 gives an alternative proof of "the only if part" of Theorem 1.3. Motivated by the above results, we consider the following property.

Property 1.5. Let k be an arbitrary positive real number. For three arbitrary distinct points a, b, and c in $\mathbb{R}$ satisfying

$$\left|\frac{a-b}{c-b}\right| = k, \tag{1.4}$$

we have

$$\left|\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)}\right| = k. \tag{1.5}$$

In Section 3, we prove the following result concerning the mapping property of an analytic and univalent function on a connected domain.

Theorem 1.6. *Let k be an arbitrary positive real number. Let $w = f(z)$ be analytic and univalent in a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$. Then f satisfies Property 1.5 if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

2. Lemmas

Definition 2.1. Let f be a complex-valued function. The Schwarzian derivative of f is defined as follows:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{2.1}$$

Similar to Schwarzian derivative, we have the following.

Definition 2.2. Let f be a complex-valued function. We define the *Newton derivative* of f as follows:

$$N_f(z) = \left(z - \frac{f(z)}{f'(z)}\right)' = \frac{f(z)f''(z)}{(f'(z))^2}. \tag{2.2}$$

Remark 2.3. Note that $N_f(z)$ is the first derivative of Newton's method of f.

Remark 2.4. Let f be a complex-valued function. It is well known that $S_f(z) = 0$ if and only if f is a Möbius transformation.

From Remark 2.4, we have observed that a similar result holds true when we replace Schwarzian derivative by the Newton derivative.

Lemma 2.5. *Let f be a complex-valued function. Then $N_f(z) = 2$ if and only if f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$.*

Proof. Let f be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$, then it is easily checked that $N_f(z) = 2$. Let f be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left(z - \frac{f(z)}{f'(z)}\right)' = 2 \tag{2.3}$$

which implies that

$$z - \frac{f(z)}{f'(z)} = 2z - c_1, \tag{2.4}$$

where c_1 is a complex constant, thus

$$\frac{f(z)}{f'(z)} = -z + c_1 \tag{2.5}$$

or

$$\frac{1}{f(z)} \frac{df(z)}{dz} = \frac{1}{-z + c_1}. \tag{2.6}$$

From which it follows by a simple calculation that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

3. Main result. In this section, we assume that $w = f(z)$ is analytic and univalent on a nonempty connected domain $\mathbb{R}$ on the z-plane such that $f(z) \neq 0$ for all $z \in \mathbb{R}$.

Proof of Theorem 1.6. Let $f(z)$ be a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. Let a, b, and c be arbitrary three distinct points in $\mathbb{R}$ such that

$$\left| \frac{a-b}{c-b} \right| = k. \tag{3.1}$$

We observe that

$$\frac{a-b}{c-b} \tag{3.2}$$

is the cross-ratio of a, b, c, and d, where d is the point at infinity. Since $f(z) = u/(z+v)$, $u \neq 0$, we have $f(d) = 0$. Since Möbius transformations preserve the cross-ratio, we obtain

$$\frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a-b}{c-b} \tag{3.3}$$

which implies that

$$\left| \frac{f(a)-f(b)}{f(c)-f(b)} \cdot \frac{f(c)}{f(a)} \right| = \left| \frac{a-b}{c-b} \right| = k. \tag{3.4}$$

Therefore, any Möbius transformation of the form $u/(z+v)$, $u \neq 0$ satisfies Property 1.5.

Conversely, let x be an arbitrary fixed point in $\mathbb{R}$. Then there exists a positive real number r such that the r circular neighborhood $N_r(x)$ of x is contained in $\mathbb{R}$.

Throughout the proof let $A = x + ky$, $B = x$, $C = x - y$. Since $\mathbb{R}$ is a nonempty connected domain on the z-plane, there exists a positive real number s such that if

$$0 < |y| < s, \tag{3.5}$$

then A, B, and C are contained in $N_r(x)$.

Since $w = f(z)$ is univalent in $\mathbb{R}$, $f(A) = f(x+ky)$, $f(B) = f(x)$, and $f(C) = f(x-y)$ are distinct points. By assumption, we have

$$\left| \frac{f(x+ky)-f(x)}{f(x-y)-f(x)} \cdot \frac{f(x-y)}{f(x+ky)} \right| = k \tag{3.6}$$

for all y such that $0 < |y| < s$.

Let

$$h(y) = \frac{f(x+ky)-f(x)}{f(x-y)-f(x)} \cdot \frac{f(x-y)}{f(x+ky)}. \tag{3.7}$$

Then

$$|h(y)| = k \tag{3.8}$$

for all y such that $0 < |y| < s$. The function $h(y)$ extends analytically at zero by $h(0) = -k$. Hence, by the maximum modulus principle, we have $h(y) = -k$ for all y with $|y| < s$. In other words, we have

$$\frac{f(x+ky)-f(x)}{f(x-y)-f(x)} \cdot \frac{f(x-y)}{f(x+ky)} = -k \tag{3.9}$$

in $|y| < s$. This equality implies that

$$\big(f(x+ky)-f(x)\big)f(x-y) = -k\big(f(x-y)-f(x)\big)f(x+ky). \tag{3.10}$$

Differentiate this equality twice with respect to y and then set $y = 0$, we obtain

$$-k(k+1)\big(2\big(f'(x)\big)^2 - f(x)f''(x)\big) = 0 \tag{3.11}$$

which implies that

$$2\big(f'(x)\big)^2 - f(x)f''(x) = 0 \tag{3.12}$$

or

$$\frac{f(x)f''(x)}{\big(f'(x)\big)^2} = 2. \tag{3.13}$$

By the identity theorem and Lemma 2.5, we conclude that f is a Möbius transformation of the form $u/(z+v)$, $u \neq 0$. □

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