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TAUBERIAN OPERATORS IN p-ADIC ANALYSIS

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Abstract. In archimedean analysis Tauberian operators and operators having property N were defined by Kalton and Wilansky (1976). We give several characterizations of p-adic Tauberian operators and operators having property N in terms of basic sequences. And, as its applications, we give some equivalent relations between these operators and p-adic semi-Fredholm operators.

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1. Introduction. Throughout this paper, K is a non-archimedean non-trivially valued complete field with a valuation $|\ |$. Let E and F be infinite-dimensional Banach spaces over K. Let $L(E,F)$ stand for the set of all continuous linear operators from E into F.

In this paper, we say that $T \in L(E,F)$ is semi-Fredholm if its range space, $R(T)$, is closed in F and its kernel, $N(T)$, is finite-dimensional. Further we recall that $T \in L(E,F)$ is Tauberian if $x'' \in E''$, $T''x'' \in E$ imply $x'' \in E$. And $T \in L(E,F)$ has a property N if $x'' \in E''$, $T''x'' = 0$ imply $x'' \in E$. If K is spherically complete, then E is a strongly polar (of course a polar) space, so the natural map $J_E : E \to E''$ is a linear homeomorphism into E'' (see [14]). Let A be a subset of E. If E is a polar space and if A is bounded and closed in E, then A is also bounded and closed in E'', respectively. We denote the closure of A in E or E'' by $\bar{A}$, the weak closure of A in E by $\bar{A}^w$ and the weak* closure A in E'' by $\bar{A}^{w^*}$. If B is a subset of E', then we denote the weak* closure of B by $\bar{B}^{w^*}$.

A subset $X = \{x_1, x_2, \dots, x_n, \dots\}$ of E is said to be a basis for E if every $x \in E$ has a unique representation in the form $x = \sum_{n=1}^{\infty} \alpha_n x_n$ $(\alpha_n \in K)$. And the subset X of E is said to be t-orthogonal if there exists a real number t, $0 < t \leqq 1$, such that for any integer n and for any $\alpha_i \in K$ $(i = 1, 2, \dots, n)$,

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geqq t \max\{\|\alpha_1 x_1\|, \dots, \|\alpha_n x_n\|\}. \tag{1.1}$$

It is known that if X is a basis for E, then there exists a real number t, $0 < t \leqq 1$, such that X is t-orthogonal (see [16, page 62]).

A sequence $\{x_n\}_{n \geqq 1}$ in E is said to be a basic sequence if $\{x_n\}_{n \geqq 1}$ is a basis for its closed linear span $[\{x_i : i = 1, 2, \dots, n, \dots\}]$. And a basic sequence $\{x_n\}_{n \geqq 1}$ is said to be of type l^+ if it is bounded and there exist a real number $\varepsilon > 0$ and $x' \in E'$ such that $|x'(x_n)| \geqq \varepsilon$ for all n.

A point x in E is said to be a weak (weak*) limit point of a sequence $\{x_n\}_{n\geqq 1}$ if every weak (weak*) neighborhood of x contains an element of $\{x_n\}_{n\geqq 1}$ different from x. Of course a weak (weak*) limit point of the sequence is in the weak (weak*) closure of $\{x_i : i = 1, 2, \dots, n, \dots\}$.

Let π denote an arbitrary fixed element of K with $0 < |\pi| < 1$. Other terms and symbols will be used in [16].

In archimedean analysis, many characterizations of Tauberian, semi-Fredholm operators and operators having the property N are given (e.g., [3, 4, 9]). Some of them are presented in terms of sequences. In this paper, we give the analogous results to them. Further, as applications of them, we give that equivalent relations among those operators.

2. Basic sequences.

In this section, we give some results on basic sequences. Before proceeding our discussions, we first recall the following two theorems.

Theorem 2.1 (see [8]). *If K is spherically complete, then every weakly convergent sequence in E is norm-convergent.*

Theorem 2.2 (see [19]). *Let K be locally compact. If A is a bounded subset of E'', then $\bar{A} = \bar{A}^{w^*}$ and $\bar{A}^{w^*}$ is weak* compact in E''.*

Now we need the following proposition.

Proposition 2.3. *Let $\{x_n\}_{n\geqq 1}$ be a sequence in E such that for each n $|\pi| \leqq \|x_n\| \leqq 1$. Then $\{x_n\}_{n\geqq 1}$ is a basic sequence if and only if there exists a constant $c \geqq 1$ so that for any $\alpha_i \in K$ and for any integers $m, n,\ m < n$,*

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\| \leqq c \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|. \tag{2.1}$$

Proof. Suppose that $\{x_n\}_{n\geqq 1}$ is a basic sequence for its closed linear span $[\{x_i : i = 1, 2, \dots, n, \dots\}]$. Then there exists a real number t, $0 < t \leqq 1$, such that the sequence $\{x_n\}_{n\geqq 1}$ is t-orthogonal. Hence we have

$$\begin{aligned} \left\| \sum_{i=1}^{m} \alpha_i x_i \right\| &\leqq \max\left(\|\alpha_1 x_1\|, \dots, \|\alpha_m x_m\|\right) \\ &\leqq \max\left(\|\alpha_1 x_1\|, \dots, \|\alpha_n x_n\|\right) \\ &\leqq \frac{1}{t} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|. \end{aligned} \tag{2.2}$$

Conversely, suppose that there exists a constant $c \geqq 1$ such that for any $\alpha_i \in K$ and for any integers $m, n,\ m < n$,

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\| \leqq c \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|. \tag{2.3}$$

If we have $\sum_{i=1}^{\infty}\alpha_i x_i = 0$, then for any $\varepsilon > 0$ there exists an integer n_0 such that for every integer $n \geqq n_0 \|\sum_{i=1}^{n}\alpha_i x_i\| < \varepsilon$. Hence we have

$$|\alpha_1|\|x_1\| \leqq c\left\|\sum_{i=1}^{n}\alpha_i x_i\right\| < c\varepsilon. \tag{2.4}$$

This implies that $\alpha_1 = 0$ and we have $\sum_{i=2}^{n}\alpha_i x_i = 0$. Proceeding with this way, we can conclude that for every integer n, $\alpha_n = 0$. It follows that if a vector $x \in E$ has a representation in the form $x = \sum_{i=1}^{\infty}\alpha_i x_i$, then this representation is unique. Next, we show that every $x \in [\{x_i : i = 1,2,\dots,n,\dots\}]$ has a representation in the form $\sum_{i=1}^{\infty}\alpha_i x_i$ $(\alpha_i \in K)$. Let $L(\{x_n\})$ denote the linear span of $\{x_n : n = 1,2,\dots\}$ and for each m let P_m denote a linear operator from $L(\{x_n\})$ into itself defined by

$$P_m\left(\sum \alpha_j x_j\right) = \sum_{i=1}^{m}\alpha_i x_i, \tag{2.5}$$

where $\sum \alpha_j x_j$ denotes a finite linear combination of $\{x_n : n = 1,2,\dots\}$. Then it holds that $\|P_m\| \leqq c$, so P_m is continuous. It follows that P_m has a continuous linear extension to $[\{x_n : n = 1,2,\dots\}]$, still called P_m. Further, let x_k' $(k = 1,2,\dots)$ be a coordinate functional defined on $L(\{x_n\})$ by $x_k'(\sum \alpha_j x_j) = \alpha_k$. Let $x = \sum \alpha_j x_j \in L(\{x_n\})$. Then for integer $k \geqq 2$, we have

$$||x_k'|| = \sup\frac{|\alpha_k|}{\|x\|} \leqq \frac{1}{|\pi|}\sup\frac{\|\alpha_k x_k\|}{\|x\|} = \frac{1}{\pi}\sup\frac{||(P_k - P_{k-1})(x)||}{\|x\|} \leqq \frac{c}{|\pi|}. \tag{2.6}$$

Obviously,

$$||x_1'|| \leqq \frac{1}{|\pi|}\|P_1\| \leqq \frac{c}{|\pi|}. \tag{2.7}$$

Hence for every integer k, $x_k' \in (L(\{x_n\}))'$. It follows that x_k' has a unique continuous linear extension to all of $[\{x_n : n = 1,2,\dots\}]$, still also called x_k'. By the continuity of x_k', P_k and their definitions on $L(\{x_n\})$, it is easy to see that for every $x \in [\{x_n : n = 1,2,\dots\}] x_1'(x)x_1 = P_1(x)$ and $x_k'(x)x_k = P_k(x) - P_{k-1}(x)$ $(k \geqq 2)$. Let $x \in [\{x_n : n = 1,2,\dots\}]$ and $\varepsilon' > 0$ be given. Thus

$$\begin{aligned} P_n(x) &= P_1(x) + (P_2(x) - P_1(x)) + \cdots + (P_n(x) - P_{n-1}(x)) \\ &= \sum_{k=1}^{n} x_k'(x)x_k. \end{aligned} \tag{2.8}$$

Then there exist an integer n_1 and an element y in the linear span of $\{x_1, x_2, \dots, x_{n_1}\}$ such that $\|x - y\| < \varepsilon'$. If $n \geqq n_1$, then we have

$$\begin{aligned} \|x - P_n(x)\| &\leqq \max(\|x - y\|, \|y - P_n(y)\|, \|P_n(y - x)\|) \\ &\leqq \max(\varepsilon', c\varepsilon') = c\varepsilon'. \end{aligned} \tag{2.9}$$

This implies that

$$x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty}\sum_{k=1}^{n} x_k'(x)x_k, \tag{2.10}$$

which completes the proof. □

Corollary 2.4. *Every t-orthogonal sequence $(0 < t \leqq 1)$ in a Banach space is a basic sequence.*

Proof. t-orthogonal sequences satisfy the condition of Proposition 2.3 as shown in the proof. □

Proposition 2.5. *Let E be a strongly polar space and $\{x_n\}_{n\geqq 1}$ be a basic sequence in E. If $x \in E$ is a weak limit point of $\{x_n\}_{n\geqq 1}$, then $x = 0$.*

Proof. Since $\{x_n\}_{n\geqq 1}$ is a basic for $[\{x_n : n = 1,2,\dots\}]$, there exists a real number t, $0 < t \leqq 1$, such that $\{x_n\}_{n\geqq 1}$ is t-orthogonal. For every k define a linear operator

$$g_k : [\{x_n : n = 1,2,\dots\}] \longrightarrow K \tag{2.11}$$

by $g_k(\sum_{i=1}^{\infty} \alpha_i x_i) = \alpha_k$ $(\alpha_i \in K,\ i = 1,2,\dots)$. Then it holds that $\|g_k\| \leqq 1/t$. Therefore $g_k \in [\{x_n : n = 1,2,\dots\}]'$. Since E is a strongly polar space, by [12, Theorem 4.2], g_k has an extension linear operator to all of E. Hence we may assume that $g_k \in E'$ and $g_k(x_j) = \delta_{kj}$. Since x is a weak limit point of the sequence $\{x_n\}_{n\geqq 1}$, we have

$$g_k(x) = \lim_{i\to\infty} g_k(x_{k_{ni}}) \quad (k = 1,2,\dots), \tag{2.12}$$

where $\{x_{k_{ni}}\}_{i\geqq 1}$ is some subsequence of $\{x_n\}_{n\geqq 1}$. But as $g_k(x_{k_{ni}}) = 0$ for $k_{ni} > k$, $g_k(x) = 0$ for all k. By [12, Corollary 4.9], $[\{x_n : n = 1,2,\dots\}]$ is weakly closed. Hence $x \in [\{x_n : n = 1,2,\dots\}]$. It follows that x has a unique expansion of the form $x = \sum_{i=1}^{\infty} \beta_i x_i$ $(\beta_i \in K,\ i = 1,2,\dots)$. Hence we have $x = \sum_{i=1}^{\infty} g_i(x) x_i = 0$, which completes the proof. □

Theorem 2.6. *Let E be a strongly polar space and let A be a subset of E. Suppose that $\bar{A} \neq \bar{A}^w$ and let $x_0 \in \bar{A}^w \setminus \bar{A}$. Then there exist a sequence $\{x_n\}_{n\geqq 1}$ in A and $x_0' \in E'$ which satisfy the following conditions.*

(1) $\lim_{n\to\infty} x_0'(x_n) = x_0'(x_0)$ *and* $|x_0'(x_0)| \geqq \|x_0\|/2$.

(2) $\{x_n - x_0\}_{n\geqq 1}$ *is a basic sequence.*

(3) *If* $x_0 \neq 0$, *then* $x_0 \notin [\{x_n - x_0 : n = 1,2,\dots\}]$.

Proof. Choose three sequences $\{r_n\}_{n\geqq 0}$, $\{s_n\}_{n\geqq 0}$, and $\{t_n\}_{n\geqq 0}$ in $\mathbb{R}$ (the set of real numbers) with the following properties:

(i) $0 < r_n < 1$ for all $n \geqq 0$,

(ii) whenever $1 \leqq p < q < \infty$, $\prod_{i=p}^{q-1}(1 - r_i) > 1 - r_0$,

(iii) $0 < s_n < r_n$ for all $n \geqq 0$,

(iv) $t_n = (1 - r_n)/(1 - s_n)$ for all $n \geqq 0$.

Take any $x_0' \in E'$ such that $|x_0'(x_0)| \geqq \|x_0\|/2$. Since $x_0 \notin \bar{A}$, there exists a real number $\delta > 0$ such that for every $z \in A, \|z - x_0\| \geqq \delta$. By hypothesis, there is a $y_1 \in A$ such that

$$|x_0'(y_1) - x_0'(x_0)| < 1. \tag{2.13}$$

Set $E_1 = [\{y_1 - x_0\}] = [\{a\}]$, where a is an element of $[\{y_1 - x_0\}]$ with $|\pi| \leqq \|a\| \leqq 1$ for π referred to in the introduction. For every $x \in E_1,\ x = \lambda a$ $(\lambda \in K)$, let $x_1'(x) = \lambda$.

Then we have

$$||x_1'|| = \frac{1}{\|a\|}. \tag{2.14}$$

Hence $x_1' \in E_1'$. Since E is a strongly polar space, for every $\varepsilon > 0$, x_1' has an extension $\overline{x_1'} \in E'$ such that

$$||\overline{x_1'}|| \leqq \frac{1+\varepsilon}{\|a\|}. \tag{2.15}$$

By hypothesis, there is a $y_2 \in A$ such that

$$|x_0'(y_2) - x_0'(x_0)| < \frac{1}{2}, \qquad |\overline{x_1'}(y_2) - \overline{x_1'}(x_0)| < \delta s_1 |\pi|. \tag{2.16}$$

Now, we show that for every $x \in E_1$ and for every $\alpha \in K$,

$$||x + \alpha(y_2 - x_0)|| > t_1(1 - s_1)\|x\|. \tag{2.17}$$

Towards this end it is sufficient to show that

$$||a + \alpha(y_2 - x_0)|| > t_1(1 - s_1)\|a\|. \tag{2.18}$$

If $|\alpha| \leqq 1/\delta$, then we have

$$\begin{aligned} ||a + \alpha(y_2 - x_0)|| &\geqq \frac{|\overline{x_1'}(a + \alpha(y_2 - x_0))|}{||\overline{x_1'}||} \\ &\geqq \frac{1}{||\overline{x_1'}||}\{|\overline{x_1'}(a)| - |\alpha|\,|\overline{x_1'}(y_2 - x_0)|\} \\ &> \frac{\|a\|}{1+\varepsilon}(1 - s_1|\pi|) \geqq \frac{1}{1+\varepsilon}(1 - s_1)\|a\|. \end{aligned} \tag{2.19}$$

Since ε is arbitrary, we have

$$||a + \alpha(y_2 - x_0)|| > (1 - s_1)\|a\| \geqq t_1(1 - s_1)\|a\|. \tag{2.20}$$

If $|\alpha| > 1/\delta$, then $|\alpha|\|y_2 - x_0\| > 1$. Since $\|a\| \leqq 1$, we have

$$||a + \alpha(y_2 - x_0)|| = |\alpha|||y_2 - x_0|| > \|a\| \geqq t_1(1 - s_1)\|a\|. \tag{2.21}$$

Thus, for every $\alpha \in K$, (2.18) is proved.

Next, set $E_2 = [\{y_1 - x_0,\ y_2 - x_0\}]$. Then E_2 is two dimensional. Hence there exist $b_1, b_2 \in E_2$ such that $E_2 = [\{b_1, b_2\}]$, $|\pi| \leqq \|b_i\| \leqq 1$ $(i = 1,2)$ and for every $\lambda, \mu \in K$,

$$||\lambda b_1 + \mu b_2|| \geqq t_2 \max(\|\lambda b_1\|, \|\mu b_2\|) \tag{2.22}$$

(see [16, page 66]).

For every $x \in E_2$, $x = \lambda b_1 + \mu b_2$, let $y_1'(x) = \lambda$ and $y_2'(x) = \mu$. Then we have

$$||y_1'|| \leqq \frac{1}{t_2||b_1||}, \qquad ||y_2'|| \leqq \frac{1}{t_2||b_2||}. \tag{2.23}$$

This means that $y_1', y_2' \in E_2'$. Hence, for every $\varepsilon > 0$, y_1' and y_2' have extensions $\overline{y_1'}$ and $\overline{y_2'}$, respectively, such that

$$\|\overline{y_1'}\| \leqq \frac{1+\varepsilon}{t_2\|b_1\|}, \qquad \|\overline{y_2'}\| \leqq \frac{1+\varepsilon}{t_2\|b_2\|}. \tag{2.24}$$

By hypothesis, there is a $y_3 \in A$ such that

$$|x_0'(y_3) - x_0'(x_0)| < \frac{1}{3}, \tag{2.25}$$

and

$$|\overline{y_1'}(y_3) - \overline{y_1'}(x_0)| < \delta s_2|\pi|, \qquad |\overline{y_2'}(y_3) - \overline{y_2'}(x_0)| < \delta s_2|\pi|. \tag{2.26}$$

We wish to show that for every $x \in E_2$ and for every $\alpha \in K$,

$$\|x + \alpha(y_3 - x_0)\| > t_2(1-s_2)\|x\|. \tag{2.27}$$

We may assume that $|\pi| \leqq \|x\| \leqq 1$. Let $x = \lambda b_1 + \mu b_2$ $(\lambda, \mu \in K)$. Suppose that $\max(\|\lambda b_1\|, \|\mu b_2\|) = \|\lambda b_1\|$. Then $t_2\|\lambda b_1\| \leqq \|x\| \leqq \|\lambda b_1\|$. If $|\alpha| \leqq 1/\delta$, then we have

$$\begin{aligned}
\|x + \alpha(y_3 - x_0)\| &\geqq \frac{|\overline{y_1'}(x) + \alpha\overline{y_1'}(y_3 - x_0)|}{\|\overline{y_1'}\|} \\
&\geq \frac{|\overline{y_1'}(x)| - |\alpha|\,|\overline{y_1'}(y_3 - x_0)|}{\|\overline{y_1'}\|} \\
&> \frac{t_2\|b_1\|}{1+\varepsilon}(|\lambda| - s_2|\pi|) \\
&= \frac{t_2}{1+\varepsilon}(\|\lambda b_1\| - s_2|\pi|\|b_1\|) \\
&\geqq \frac{t_2}{1+\varepsilon}(1-s_2)\|x\|.
\end{aligned} \tag{2.28}$$

Since ε is arbitrary, we have

$$\|x + \alpha(y_3 - x_0)\| \geqq t_2(1-s_2)\|x\|. \tag{2.29}$$

If $|\alpha| > 1/\delta$, then $|\alpha|\|y_3 - x_0\| > 1$. Since $\|x\| \leqq 1$, we have

$$\|x + \alpha(y_3 - x_0)\| = |\alpha|\|y_3 - x_0\| > 1 > t_2(1-s_2)\|x\|. \tag{2.30}$$

Hence, in the case $\max(\|\lambda b_1\|, \|\mu b_2\|) = \|\lambda b_1\|$, we showed the inequality (2.27). If $\max(\|\lambda b_1\|, \|\mu b_2\|) = \|\mu b_2\|$, then in a similar fashion, we can also show the inequality (2.27). Thus, for every $\lambda_1, \lambda_2, \lambda_3 \in K$ we have

$$\begin{aligned}
\|\lambda_1(y_1 - x_0) + \lambda_2(y_2 - x_0) + \lambda_3(y_3 - x_0)\| &\geqq t_2(1-s_2)\|\lambda_1(y_1 - x_0) + \lambda_2(y_2 - x_0)\| \\
&\geqq t_2(1-s_2)t_1(1-s_1)\|\lambda_1(y_1 - x_0)\|.
\end{aligned} \tag{2.31}$$

Proceeding thusly, we find a sequence $\{y_n\}_{n\geqq 1}$ in A such that all $n \geqq 1$

$$|x_0'(y_n) - x_0'(x_0)| < \frac{1}{n} \tag{2.32}$$

and for which given $1 \leqq p < q < \infty$ and $\alpha_1, \alpha_2, \dots, \alpha_q \in K$,

$$\begin{aligned}\left\|\sum_{i=1}^{q} \alpha_i (y_i - x_0)\right\| &\geqq t_{q-1}(1-s_{q-1})\left\|\sum_{i=1}^{q-1} \alpha_i (y_i - x_0)\right\| \geqq \cdots \\ &\geqq t_{q-1}t_{q-2}\cdots t_p(1-s_{q-1})(1-s_{q-2})\cdots(1-s_p)\left\|\sum_{i=1}^{p} \alpha_i (y_i - x_0)\right\| \\ &= (1-r_{q-1})(1-r_{q-2})\cdots(1-r_p)\left\|\sum_{i=1}^{p} \alpha_i (y_i - x_0)\right\| \\ &> (1-r_0)\left\|\sum_{i=1}^{p} \alpha_i (y_i - x_0)\right\|. \end{aligned} \tag{2.33}$$

By Proposition 2.3, the sequence $\{y_n - x_0\}_{n \geqq 1}$ is basic. Now, we wish to show that

$$\bigcap_{k=1}^{\infty} [\{y_k - x_0, y_{k+1} - x_0, \dots\}] = \{0\}. \tag{2.34}$$

To this end, suppose that there exists a non-zero element x in $\bigcap_{k=1}^{\infty}[\{y_k - x_0, y_{k+1} - x_0, \dots\}]$. Take a real number ε_1 such that $0 < \varepsilon_1 < \|x\|$. Then there exist $\alpha_1, \alpha_2, \dots, \alpha_n$, $\alpha_{n+1}, \dots, \alpha_j \in K$ such that

$$\begin{gathered} \|x - \alpha_1(y_1 - x_0) - \alpha_2(y_2 - x_0) - \cdots - \alpha_n(y_n - x_0)\| < \frac{1-r_0}{2}\varepsilon_1, \\ \|x - \alpha_{n+1}(y_{n+1} - x_0) - \alpha_{n+2}(y_{n+2} - x_0) - \cdots - \alpha_j(y_j - x_0)\| < \frac{1-r_0}{2}\varepsilon_1. \end{gathered} \tag{2.35}$$

Hence, we have

$$\begin{aligned}\frac{1-r_0}{2}\varepsilon_1 &> \|\alpha_1(y_1 - x_0) + \cdots + \alpha_n(y_n - x_0) + \alpha_{n+1}(y_{n+1} - x_0) + \cdots + \alpha_j(y_j - x_0)\| \\ &> (1-r_0)\|\alpha_1(y_1 - x_0) + \alpha_2(y_2 - x_0) + \cdots + \alpha_n(y_n - x_0)\| \\ &= (1-r_0)\|x\| > (1-r_0)\varepsilon_1. \end{aligned} \tag{2.36}$$

This is a contradiction and we showed that

$$\bigcap_{k=1}^{\infty} [\{y_k - x_0, y_{k+1} - x_0, \dots\}] = \{0\}. \tag{2.37}$$

Hence there exists m such that

$$x_0 \notin [\{y_m - x_0, y_{m+1} - x_0, \dots\}]. \tag{2.38}$$

Thus, for every positive integer n we put $x_n = y_{m+n}$, then the sequence $\{x_n\}_{n \geqq 1}$ is the required sequence, which completes the proof. □

From now on we assume that K is locally compact. Then we recall that every Banach space is strongly polar.

Theorem 2.7. *Let B be a subset of the dual E' of E. Suppose that $\bar{B} \neq \bar{B}^{w^*}$ and let $x_0' \in \bar{B}^{w^*} \setminus \bar{B}$. Then there exist a sequence $\{x_n'\}_{n \geqq 1}$ in B and $x_0 \in E$ which satisfy the following conditions:*

(1) $\lim_{n\to\infty} x_n'(x_0) = x_0'(x_0)$ *and* $|x_0'(x_0)| \geqq \|x_0'\|/2$.

(2) $\{x_n' - x_0'\}_{n \geqq 1}$ *is a basic sequence.*

(3) *If* $x_0' \neq 0$, *then* $x_0' \notin [\{x_n' - x_0' : n = 1,2,\dots\}]$.

Proof. The proof is similar to the archimedean case (e.g., [1]). Choose a sequence $\{r_n\}_{n \geqq 1}$ in $\mathbb{R}$ with the following properties:

(i) $0 < r_n < 1$ for all $n \geqq 1$.

(ii) Whenever $1 \leqq p < q < \infty$, $\prod_{i=p}^{q-1}(1-r_i) > 1 - r_0$.

Take any $x_0 \in E$ such that $|x_0'(x_0)| \geqq \|x_0'\|/2$. Since $x_0' \notin \bar{B}$, there exists a real number $\delta > 0$ such that for every $z' \in B \|z' - x_0'\| \geqq \delta$. By hypothesis, there is a $y_1' \in B$ such that

$$|y_1'(x_0) - x_0'(x_0)| < 1. \tag{2.39}$$

Set $F_1 = [\{y_1' - x_0'\}]$. Then F_1 is the one-dimensional subspace of E'. Since K is locally compact, the subset

$$B_1 = \{x' \in F_1 : |\pi| \leqq \|x'\| \leqq 1\} \tag{2.40}$$

is compact, so we can pick a $r_1|\pi|/3$ net $a_1', a_2', \dots, a_{N(1)}'$ for B_1. Take $x_1, x_2, \dots, x_{N(1)}$ in E such that for all i $|\pi| \leqq \|x_i\| \leqq 1$ and

$$\frac{|a_i'(x_i)|}{\|x_i\|} \geqq \left(1 - \frac{r_1}{3}\right)\|a_i'\| \quad (i = 1,2,\dots,N(1)). \tag{2.41}$$

By hypothesis, there is a $y_2' \in B$ such that

$$\begin{gathered} |y_2'(x_0) - x_0'(x_0)| < \frac{1}{2}, \\ |y_2'(x_1) - x_0'(x_1)| < \frac{\delta r_1 |\pi|^2}{3}, \ \dots, \ |y_2'(x_{N(1)}) - x_0'(x_{N(1)})| < \frac{\delta r_1|\pi|^2}{3}. \end{gathered} \tag{2.42}$$

We wish to show that for every $x' \in E'$ and for every $\alpha \in K$,

$$\|x' + \alpha(y_2' - x_0')\| > (1 - r_1)\|x'\|. \tag{2.43}$$

We may assume that $|\pi| \leqq \|x'\| \leqq 1$. Suppose that $|\alpha| \leqq 1/\delta$. There exists an a_i' such that

$$\|x' - a_i'\| < \frac{r_1|\pi|}{3}. \tag{2.44}$$

This means that $\|x'\| = \|a'_i\|$. We have

$$\begin{aligned}||x'+\alpha(y'_2-x'_0)|| &\geqq \frac{1}{||x_i||}\,|\,(x'+\alpha(y'_2-x'_0))\,(x_i)\,| \\ &\geqq \frac{1}{||x_i||}\{\,|\,a'_i(x_i)\,|-|\,\alpha(y'_2-x'_0)\,(x_i)\,|-||x_i||\,||x'-a'_i||\} \\ &> \frac{1}{||x_i||}\left\{\left(1-\frac{r_1}{3}\right)||a'_i||\,||x_i||-\frac{1}{\delta}\frac{\delta r_1|\pi|^2}{3}-\frac{r_1|\pi|}{3}||x_i||\right\} \\ &\geqq \left(1-\frac{r_1}{3}\right)||a'_i||-\frac{r_1|\pi|}{3}-\frac{r_1|\pi|}{3}\geqq(1-r_1)||a'_i||=(1-r_1)||x'||.\end{aligned} \tag{2.45}$$

If $|\alpha| > 1/\delta$, then $|\alpha|\|y'_2 - x'_0\| > 1$. Since $\|x'\| \leqq 1$, we have

$$||x'+\alpha(y'_2-x'_0)|| = |\alpha|||y'_2-x'_0|| > ||x'|| > (1-r_1)||x'||. \tag{2.46}$$

Hence for every $x' \in E'$ and for every $\alpha \in K$ the inequality (2.43) holds. Set

$$F_2 = [\{y'_1-x'_0, y'_2-x'_0\}] \tag{2.47}$$

and

$$B_2 = \{x' \in F_2 : |\pi| \leqq ||x'|| \leqq 1\}. \tag{2.48}$$

Then B_2 is compact. Hence we can pick a $r_2|\pi|/3$ net $b'_1, b'_2, \dots, b'_{N(2)}$ for B_2. Take $y_1, y_2, \dots, y_{N(2)}$ in E such that $|\pi| \leqq \|y_i\| \leqq 1$ and

$$\frac{r_2|\pi|}{3}\frac{|\,b'_1(y_i)\,|}{||y_i||} \geqq \left(1-\frac{r_2}{3}\right)||b'_i|| \quad (i = 1,2,\dots,N(2)). \tag{2.49}$$

By hypothesis, there is a $y'_3 \in B$ such that

$$\begin{aligned}&|\,y'_3(x_0)-x'_0(x_0)\,| < \frac{1}{3}, \\ &|\,y'_3(y_1)-x'_0(y_1)\,| < \frac{\delta r_2|\pi|^2}{3},\ \dots,\ |\,y'_3(y_{N(2)})-x'_0(y_{N(2)})\,| < \frac{\delta r_2|\pi|^2}{3}.\end{aligned} \tag{2.50}$$

In a similar fashion to the above proof, for every $x' \in E'$ and for every $\alpha \in K$ we can obtain the following inequality:

$$||x'+\alpha(y'_3-x'_0)|| > (1-r_2)||x'||. \tag{2.51}$$

Thus, proceeding this way, we find a sequence $\{y'_n\}_{n\geqq 1}$ in B such that for all $n \geqq 1$,

$$|\,y'_n(x_0)-x'_0(x_0)\,| < \frac{1}{n} \tag{2.52}$$

and for which given $1 \leqq p < q < \infty$ and $\alpha_1, \alpha_2, \dots, \alpha_q$ in K,

$$\left\|\sum_{i=1}^{p}\alpha_i(y'_i-x'_0)\right\| < \frac{1}{1-r_0}\left\|\sum_{i=1}^{q}\alpha_i(y'_i-x'_0)\right\|. \tag{2.53}$$

In a similar argument to the proof of Theorem 2.6, we can conclude that there exists a sequence $\{x'_n\}_{n\geqq 1}$ in B which satisfies the given conditions. □

The next corollary follows immediately from Theorem 2.7.

Corollary 2.8. *Let A be a subset of E. If $x_0'' \in E''$ is a point of $\bar{A}^{w^*}$ in E'' such that $x_0'' \notin \bar{A}$, then there exists a sequence $\{x_n\}_{n\geqq 1}$ in A and $x_0' \in E'$ which satisfy the following conditions:*

(1) $\lim_{n\to\infty} x_0'(x_n) = x_0''(x_0')$ *and* $|x_0''(x_0')| \geqq \|x_0''\|/2$.

(2) $\{x_n - x_0''\}_{n\geqq 1}$ *is a basic sequence in* E''.

(3) *If* $x_0'' \neq 0$, *then* $x_0'' \notin [\{x_n - x_0'' : n = 1,2,\dots\}]$.

Further we have the following corollary.

Corollary 2.9. *Let $\{x_n\}_{n\geqq 1}$ be a sequence in E such that*

$$0 < \inf_n \|x_n\| \leqq \sup_n \|x_n\| < \infty. \tag{2.54}$$

Then the following statements are equivalent.

(1) $\{x_n\}_{n\geqq 1}$ *contains a basic subsequence.*

(2) $\{x_n\}_{n\geqq 1}$ *contains a basic subsequence of type* l^+.

(3) $\overline{\{x_n\}}^{w}_{n\geqq 1}$ *is not weakly compact.*

Proof. (1)⇒(3). Let $\{x_{n_i}\}_{i\geqq 1}$ be a basic subsequence of $\{x_n\}_{n\geqq 1}$. Since $0 < \inf_n \|x_n\|$, $\{x_{n_i}\}_{i\geqq 1}$ does not contain a norm-convergent subsequence, so does not contain a weakly convergent subsequence. Hence by [5], $\overline{\{x_{n_i}\}}^{w}_{i\geqq 1}$ is not weakly compact, so $\overline{\{x_n\}}^{w}_{n\geqq 1}$ is not. The implication (2)⇒(1) is trivial.

(3)⇒(2). Since $\{J_E(x_n) : n=1,2,\dots\}$ is a bounded subset in E'', $\overline{\{J_E(x_n) : n=1,2,\dots\}}^{w^*}$ is weak* compact. Hence by hypothesis, there exists a weak* limit point p_0 of $\{J_E(x_n) : n = 1,2,\dots\}$ such that $p_0 \in E''\backslash J_E(E) \subset E''\backslash[\{J_E(x_n) : n = 1,2,\dots\}]$. Hence by Corollary 2.8, there exist a subsequence $\{J_E(x_{n_k})\}_{k\geqq 1}$ of $\{J_E(x_n)\}_{n\geqq 1}$ and $x_0' \in E'$ with the following conditions:

(i) $\lim_{k\to\infty} J_E(x_{n_k})(x_0') = p_0(x_0')$ and $|p_0(x_0')| \geqq \|p_0\|/2$.

(ii) $\{J_E(x_{n_k}) - p_0\}_{k\geqq 1}$ is a basic sequence.

(iii) $p_0 \notin [\{J_E(x_{n_k}) - p_0 : k = 1,2,\dots\}]$.

Put

$$\begin{aligned} Z &= [\{p_0, J_E(x_{n_1}), J_E(x_{n_2}), \dots\}], \\ Z_1 &= [\{J_E(x_{n_k}) : k = 1,2,\dots\}], \\ Z_2 &= [\{J_E(x_{n_k}) - p_0 : k = 1,2,\dots\}]. \end{aligned} \tag{2.55}$$

Then

$$Z = Z_1 \oplus [p_0] = Z_2 \oplus [p_0]. \tag{2.56}$$

Let P and Q be projections from Z onto Z_1 and Z_2, respectively. Then $P|_{z_2}$ is an isomorphism from Z_2 onto Z_1, so it is homeomorphic. By virtue of $P(J_E(x_{n_k}) - p_0) = J_E(x_{n_k})$ $(k = 1,2,\dots)$, $\{J_E(x_{n_k}) : k = 1,2,\dots\}$ is a basis for Z_1. Since it holds that

$$|p_0(x_0')| = \left|\lim_{k\to\infty} x_0'(x_{n_k})\right| \geqq \frac{\|p_0\|}{2}, \tag{2.57}$$

$\{x_{n_k}\}_{k\geqq 1}$ contains a basic sequence of type l^+. Thus we complete the proof. □

3. Tauberian operators. Now we give characterizations of Tauberian operators and related operators by using a basic sequence. Let $B_E = B_1(0)$ and $B_{E''} = \{x'' \in E'' : \|x''\| \leqq 1\}$. Before we proceed to our main results, we show the following proposition.

Proposition 3.1. *Let $T \in L(E,F)$. Let $\{x_n\}_{n\geqq 1}$ be a bounded basic sequence such that $\{Tx_n\}_{n\geqq 1}$ is basic. If $\{Tx_n\}_{n\geqq 1}$ converges, then so does $\{x_n\}_{n\geqq 1}$.*

Proof. At first we note that if a basic sequence converges, then its convergent point is zero. Suppose that $\{x_n\}_{n\geqq 1}$ does not converge. Then we may assume that there exists a real number $\delta > 0$ such that for every n $(n = 1,2,\dots)$ $\|x_n\| \geqq \delta$. Since $\{x_n\}_{n\geqq 1}$ does not converge weakly to zero, there exist a real number $\varepsilon > 0$, $x_0' \in E'$ and a subsequence $\{x_{n_k}\}_{k\geqq 1}$ of $\{x_n\}_{n\geqq 1}$ such that $|x_0'(x_{n_k})| \geqq \varepsilon > 0$ $(k = 1,2,\dots)$. Set

$$y_k = \left(x_0'(x_{n_k})\right)^{-1} x_{n_k} \quad (k = 1,2,\dots). \tag{3.1}$$

Then $\{y_k\}_{k\geqq 1}$ is a basic sequence and

$$\frac{\delta}{\|x_0'\|} \leqq \|y_k\| \leqq \frac{1}{\varepsilon} \quad (k = 1,2,\dots). \tag{3.2}$$

Hence $\{Ty_k\}_{k\geqq 1}$ is a bounded basic sequence in F. Put $E_1 = [\{y_k : k = 1,2,\dots\}]$. Let $\{f_k : k = 1,2,\dots\} \subset E_1'$ be a biorthogonal functional sequence to $\{y_k\}_{k\geqq 1}$. Since there exists a real number t, $0 < t \leqq 1$ such that $\{y_k\}_{k\geqq 1}$ is t-orthogonal, it holds that

$$\|f_k\| \leqq \frac{\|x_0'\|}{t\delta} \quad (k = 1,2,\dots). \tag{3.3}$$

Therefore, $\{f_k : k = 1,2,\dots\}$ is bounded. Put $T_1 = T|_{E_1}$, $z_k = T_1(y_k)$ $(k = 1,2,\dots)$ and $F_1 = [\{z_k : k = 1,2,\dots\}]$. Then $\{z_k\}_{k\geqq 1}$ is a basis for F_1. Let $\{g_k : k = 1,2,\dots\} \subset F_1'$ be a biorthogonal functional sequence to $\{z_k\}_{k\geqq 1}$ and let $\overline{g_k}$ $(k = 1,2,\dots)$ denote an extension of g_k to F. Then it follows that $T_1'\overline{g_k} = f_k$. Let x' be any element of E_1'. Put $x'(y_k) = \beta_k$ $(k = 1,2,\dots)$ and $x_n' = \sum_{k=1}^n \beta_k f_k$. Then $x' = w^* - \lim_n x_n'$. Set $z_n' = \sum_{k=1}^n f_k$. Since $x_0'(y_k) = 1$ $(k = 1,2,\dots)$, $\{z_n'\}_{n\geqq 1}$ is weak* convergent to x_0' in E_1. Hence x_0' is weak* limit point of $\{T_1'(\sum_{k=1}^n \overline{g_k})\}_{n\geqq 1}$. Let C be a closed absolutely convex hull subset of $\{f_k : k = 1,2,\dots\}$ in E_1'. Then C is a bounded closed subset such that $x_0' \in \bar{C}^{w^*}$. Hence $\bar{C}^{w^*} = \bar{C} \subset \overline{T_1'(F')} \subset E'$. Now take $g_0 \in F'$ with $\|T_1'(g_0) - x_0\| < \varepsilon/2$. Then

$$\sup_k \frac{|(T_1'g_0)(y_k) - x_0'(y_k)|}{\|y_k\|} < \frac{\varepsilon}{2}. \tag{3.4}$$

Therefore

$$\sup_k \frac{|g_0(z_k) - 1|}{\|y_k\|} < \frac{\varepsilon}{2}. \tag{3.5}$$

Since for all k $(k = 1,2,\dots)$,

$$\frac{1}{\|y_k\|} > \varepsilon, \tag{3.6}$$

it holds that

$$|g_0(z_k)-1| < \frac{1}{2}. \tag{3.7}$$

While, it holds that

$$\begin{aligned}\lim_{k\to\infty} |g_0(z_k)| &\leqq \lim_{k\to\infty} ||g_0|| ||Ty_k|| \\ &= ||g_0|| \lim_{k\to\infty} |x_0'(x_{n_k})|^{-1} ||Tx_{n_k}|| \\ &\leqq \varepsilon^{-1} ||g_0|| \lim_{k\to\infty} ||Tx_{n_k}|| = 0.\end{aligned} \tag{3.8}$$

This is a contradiction to the above, and so $\{x_n\}_{n\geqq 1}$ converges to zero. □

Corollary 3.2. *Let $T \in L(E,F)$. Let $\{x_n\}_{n\geqq 1}$ be a basic sequence such that $0 < \inf_n \|x_n\| \leqq \sup_n \|x_n\| < \infty$. If $\{Tx_n\}_{n\geqq 1}$ is basic, then $\{Tx_n\}_{n\geqq 1}$ does not contain a convergent subsequence.*

The next proposition is obtained by [4] in the case of archimedean analysis.

Proposition 3.3. *Let $T \in L(E,F)$. The following statements are equivalent.*
(1) *T is Tauberian.*
(2) *T has property N and $T(B_E)$ is closed.*
(3) *T has property N and $\overline{T(B_E)} \subset R(T)$.*

Proof. (1)⇒(2). Let $y \in \overline{T(B_E)}$. Then there exists a sequence $\{x_n\}_{n\geqq 1}$ in B_E such that $\{Tx_n\}_{n\geqq 1}$ converges to y. Since $\overline{B_E}^{w^*}$ is weak* compact subset, $\{x_n\}_{n\geqq 1}$ contains a weak* convergent subsequence, call it $\{x_n\}_{n\geqq 1}$ again. Let $x = w^*-\lim_n x_n$. Then

$$T''x = w^*-\lim_{n\to\infty} T''x_n = w^*-\lim_{n\to\infty} Tx_n. \tag{3.9}$$

Hence $T''x = y \in E$. By hypothesis, we have $x \in E$. It follows that $Tx = y$. And by Theorem 2.1,

$$x = w-\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n. \tag{3.10}$$

Hence $\|x\| \leqq 1$. This means that $y \in T(B_E)$, so $T(B_E)$ is closed. Obviously (2)⇒(3). We now show that (3)⇒(1). Suppose that $T''z \in F$ and $z \in E''$. We may assume $\|z\| \leqq 1$. Then there exists a net $S \subset B_E$ such that S is weak* convergent to z. Then a net $\{T''(S)\}$, equal to $\{T(S)\}$, is weak* convergent to $T''z$. Hence $z \in \overline{T(B_E)}^{w^*}$. Since $\overline{T(B_E)}^{w^*} = \overline{T(B_E)}$ in F'' and $T''z \in F$, $z \in \overline{T(B_E)}$ in F. So by hypothesis, it follows that $T''z \in T(E)$. Hence there exists a w in E such that $T''(z-w) = 0$. Since T has property N, it follows that $z \in E$, and so T is Tauberian. □

Combing [6, Theorem 6] and Proposition 3.3, we obtain the following corollary.

Corollary 3.4. *Let $T \in L(E,F)$. The following statements are equivalent.*
(1) *T is Tauberian.*
(2) *T has property N and $R(T)$ is closed.*
(3) *T is semi-Fredholm.*

And combing Proposition 3.3 with Corollary 3.4, we obtain the following corollary.

Corollary 3.5. *If $T \in L(E,F)$ has property N, then $T(B_E)$ is closed if and only if $R(T)$ is closed.*

Theorem 3.6. *Let $T \in L(E,F)$. Then the following statements are equivalent.*

(1) *T has property N.*

(2) *Let $\{x_n\}_{n\geqq 1}$ be a bounded sequence in E. If $\{Tx_n\}_{n\geqq 1}$ converges to zero, then $\{x_n\}_{n\geqq 1}$ contains a convergent subsequence.*

Proof. Suppose that T has property N. If $z \in E''$ is a weak* limit point of $\{x_n\}_{n\geqq 1}$, then

$$T''z = \lim_{n\to\infty} T''x_n = \lim_{n\to\infty} Tx_n = 0. \tag{3.11}$$

Hence $z \in E$. This implies that $\overline{\{x_n : n = 1,2,\dots\}}^{w^*} \subset E$. Therefore, $\{x_n : n = 1,2,\dots\}$ is weakly relatively compact in E, so, by [5], it is weakly relatively sequentially compact in E. Thus $\{x_n\}_{n\geqq 1}$ obtains a convergent subsequence. Conversely, let $z \in E''$, $T''z = 0$. We may assume that $\|z\| \leqq 1$. Since K is locally compact, by the same argument used in archimedean analysis (see [1] and [19]), we have $B_{E''} = \overline{B_E}^{w^*}$ in E'', where $B_{E''} = \{x'' \in E'' : \|x''\| \leqq 1\}$. By Theorem 2.2, it holds that $B_{E''} = \overline{B_E}$ in E''. Hence there exists a sequence $\{x_n\}_{n\geqq 1}$ in B_E such that $z = \lim_n x_n$ in E''. Hence we have in F''

$$0 = T''z = \lim_{n\to\infty} T''x_n = \lim_{n\to\infty} Tx_n, \tag{3.12}$$

so $0 = w - \lim_n Tx_n$ in F. By Theorem 2.1, $\{Tx_n\}_{n\geqq 1}$ converges to zero. Therefore, by assumption, $\{x_n\}_{n\geqq 1}$ contains a convergent subsequence, say it $\{x_{n_k}\}_{k\geqq 1}$. Let $a = \lim_k x_{n_k}$. Then we have $a = z$, and so z is in E. Hence T has property N. □

Corollary 3.7. *Let $T \in L(E,F)$. The following properties are equivalent.*

(1) *T has property N.*

(2) *For every basic sequence $\{x_n\}_{n\geqq 1}$ with $0 < \inf_n \|x_n\| \leqq \sup_n \|x_n\| < \infty$, $\{Tx_n\}_{n\geqq 1}$ does not converge to zero.*

Proof. (1)⇒(2). If $\{Tx_n\}_{n\geqq 1}$ converges to zero, then by Theorem 3.6, $\{x_n\}_{n\geqq 1}$ contains a convergent subsequence. Since $\{x_n\}_{n\geqq 1}$ is basic, so is the subsequence. Hence it converges to zero, but this contradicts to the condition $0 < \inf_n \|x_n\|$.

(2)⇒(1). Let $\{z_n\}_{n\geqq 1}$ be a bounded sequence for which $\{Tz_n\}_{n\geqq 1}$ converges to zero. Suppose that all subsequences of $\{z_n\}_{n\geqq 1}$ do not converge, so they do not converge weakly and we can assume that $0 < \inf_n \|z_n\| \leqq \sup_n \|z_n\| < \infty$. Hence by [5, Theorem 2.3], $\overline{\{z_n : n = 1,2,\dots\}}^{w}$ is not weakly compact. From Corollary 2.9, it follows that $\{z_n\}_{n\geqq 1}$ contains a basic subsequence $\{z_{n_k}\}_{k\geqq 1}$ of type l^+. Hence by hypothesis, $\{Tz_{n_k}\}_{k\geqq 1}$ does not converge to zero. But this is a contradiction to our assumption. Thus we conclude that $\{z_n\}_{n\geqq 1}$ contains a convergent subsequence. By virtue of Theorem 3.6, T has property N. □

Finally, by using Corollary 3.4 and [7], we can obtain a characterization of Tauberian operators in terms of basic sequences and precompact subsets. (We recall that for locally compact K, a compactoid subset means the same as a precompact subset.) This corollary is analogous to Corollary 3.7 which is a characterization of operators having property N.

Corollary 3.8. *Let $T \in L(E,F)$. The following statements are equivalent.*

(1) *T is Tauberian.*

(2) *For every basic sequence $\{x_n\}_{n\geqq 1}$ with $0 < \inf_n \|x_n\|$, $\{Tx_n\}_{n\geqq 1}$ does not converge to zero.*

(3) *Let D be a bounded subset of E. $T(D)$ is precompact if and only if D is precompact.*

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