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# ON THE REIDEMEISTER TORSION OF RATIONAL HOMOLOGY SPHERES

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ABSTRACT. We prove that the mod $\mathbb{Z}$ reduction of the Reidemeister torsion of a rational homology 3-sphere is naturally a $\mathbb{Q}/\mathbb{Z}$-valued quadratic function uniquely determined by a $\mathbb{Q}/\mathbb{Z}$-constant and the linking form.

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**1. Introduction.** Recently, V. Turaev has proved in [3, Theorem 4.3.1] a certain identity involving the Reidemeister torsion of a rational homology sphere $M$. In this paper, we suitably interpret this identity as a second-order finite difference equation satisfied by the torsion. Roughly speaking this identity states that the finite difference Hessian of the torsion coincides with the linking form of $M$. This allows us to prove a general structure result for the mod $\mathbb{Z}$ reduction of the torsion. More precisely, in Proposition 3.3 we prove that the mod $\mathbb{Z}$ reduction of the torsion is completely determined by three data.

- a certain canonical spin$^c$-structure $\sigma_0$,
- the linking form $\mathbf{lk}$ of $M$,
- a constant $c \in \mathbb{Q}/\mathbb{Z}$.

By fixing the spin$^c$-structure $\sigma_0$, we have a natural choice of Euler structure and thus, we can identify the Reidemeister torsion with a $\mathbb{Q}$-valued function on $H := H_1(M,\mathbb{Z})$. Its mod $\mathbb{Z}$ reduction is a function $\tau : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ of the form

$$\tau(h) = c - \widehat{\mathbf{lk}}(h), \tag{1.1}$$

where $\widehat{\mathbf{lk}}$ denotes a *quadratic form* on $H$ such that

$$\widehat{\mathbf{lk}}(h_1 + h_2) - \widehat{\mathbf{lk}}(h_1) - \widehat{\mathbf{lk}}(h_2) = \mathbf{lk}(h_1, h_2). \tag{1.2}$$

As a consequence, the constant $c$ is a $\mathbb{Q}/\mathbb{Z}$-valued invariant of the rational homology sphere. Experimentations with lens spaces suggest this invariant is as powerful as the torsion itself.

**2. The Reidemeister torsion.** We review briefly a few basic facts about the Reidemeister torsion of a rational homology 3-sphere. For more details and examples we refer to [1, 3].

Suppose that $M$ is a rational homology sphere. We set $H := H_1(M,\mathbb{Z})$ and use the multiplicative notation to denote the group operation on $H$. To remove the sign ambiguities in the definition of torsion, we equip $H_*(M,\mathbb{R})$ with the canonical orientation described in [3].

Denote by $\mathrm{Spin}^c(M)$ the set of isomorphism classes of spin$^c$-structure on $M$. It is an $H$-torsor, that is, the group $H$ acts freely and transitively on $\mathrm{Spin}^c(M)$,

$$H\times \mathrm{Spin}^c(M) \ni (h,\sigma) \longmapsto h\cdot\sigma \in \mathrm{Spin}^c(M). \tag{2.1}$$

We denote by $\mathscr{F}_M$ the space of functions

$$\phi : H \longrightarrow \mathbb{Q}. \tag{2.2}$$

The group $H$ acts on $\mathscr{F}_M$ by

$$H\times \mathscr{F}_M \ni (g,\phi) \longmapsto g\cdot\phi, \tag{2.3}$$

where

$$(g\cdot\phi)(h) = \phi(hg). \tag{2.4}$$

We denote by $\int_H$ the augmentation map

$$\mathscr{F}_M \longrightarrow \mathbb{Q}, \qquad \int_H \phi := \sum_{h\in H}\phi(h). \tag{2.5}$$

According to [3], the Reidemeister torsion is an $H$-equivariant map

$$\tau : \mathrm{Spin}^c(M) \longrightarrow \mathscr{F}_M, \quad \mathrm{Spin}^c(M)\ni\sigma \longmapsto \tau_\sigma = \tau_{M,\sigma}\in\mathscr{F}_M \tag{2.6}$$

such that

$$\int_H \tau_\sigma = 0. \tag{2.7}$$

In particular, if $M$ is an integral homology sphere we have $\tau_{M,\sigma} = 0$. Denote by $\mathbf{lk}_M$ the linking form of $M$,

$$\mathbf{lk}_M : H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{2.8}$$

V. Turaev has proved in [3] that $\tau_\sigma$ satisfies the identity

$$\tau_\sigma(g_1g_2) - \tau_\sigma(g_1) - \tau_\sigma(g_2) + \tau_\sigma(1) = -\mathbf{lk}_M(g_1,g_2) \mod \mathbb{Z} \tag{2.9}$$

for all $g_1,g_2\in H,\ \sigma\in\mathrm{Spin}^c(M)$. In the above identity, we replace $\sigma$ by $h\cdot\sigma$ for an arbitrary $h\in H$ and using the $H$-equivariance of $\sigma \mapsto \tau_\sigma$, we deduce

$$\tau_\sigma(g_1g_2h) - \tau_\sigma(g_1h) - \tau_\sigma(g_2h) + \tau_\sigma(h) = -\mathbf{lk}_M(g_1,g_2) \mod \mathbb{Z} \tag{2.10}$$

for all $g_1,g_2,h\in H,\ \sigma\in\mathrm{Spin}^c(M)$.

**3. A second-order differential equation.** The identity (2.10) admits a more suggestive interpretation. To describe it, we need a few more notation.

Denote by $\mathcal{S}_M$ the space of functions $H \longrightarrow \mathbb{Q}/\mathbb{Z}$. Each $g \in H$ defines a first-order differential operator

$$\Delta_g : \mathcal{S}_M \longrightarrow \mathcal{S}_M, \quad (\Delta_g u)(h) := u(gh) - u(h), \quad \forall u \in \mathcal{S}_M,\ h \in H. \tag{3.1}$$

If $\Xi = \Xi_\sigma$ denotes the $\bmod \mathbb{Z}$ reduction of $\tau_\sigma$, then we can rewrite (2.10) as

$$(\Delta_{g_1}\Delta_{g_2}\Xi)(h) = -\mathbf{lk}_M(g_1, g_2). \tag{3.2}$$

Note that the second-order differential operator $\Delta_{g_1}\Delta_{g_2}$ can be regarded as a sort of Hessian.

We prove uniqueness and existence results for this equation. We begin with the (almost) uniqueness part.

**Lemma 3.1.** *The second-order linear differential equation (3.2) determines $\Xi$ up to an "affine" function, that is, the sum between a character of $H$ and a $\mathbb{Q}/\mathbb{Z}$-constant.*

**Proof.** Suppose that $\Xi_1$, $\Xi_2$ are two solutions of the above equation. Set $\Psi := \Xi_1 - \Xi_2$, $\Psi$ satisfies the equation

$$\Delta_{g_1}\Delta_{g_2}\Psi = 0. \tag{3.3}$$

Now, observe that any function $F \in \mathcal{S}_M$ satisfying the second-order equation

$$\Delta_u \Delta_v F = 0, \quad \forall u, v \in H \tag{3.4}$$

is affine, that is, it has the form

$$F = c + \lambda, \tag{3.5}$$

where $c \in \mathbb{Q}/\mathbb{Z}$ is a constant and $\lambda : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a character. Indeed, the condition

$$\Delta_u(\Delta_v F) = 0, \quad \forall u \tag{3.6}$$

implies $\Delta_v F$ is a constant depending on $v$, $c(v)$. Thus

$$F(vh) - F(h) = c(v), \quad \forall h. \tag{3.7}$$

The function $\lambda = F - F(1)$ satisfies the same differential equation

$$\lambda(vh) - \lambda(h) = c(v) \tag{3.8}$$

and the additional condition $\lambda(1) = 0$. If we set $h = 1$ in the above equation, we deduce

$$\lambda(v) = c(v). \tag{3.9}$$

Hence,

$$\lambda(vh) = \lambda(h) + \lambda(v), \quad \forall v, h \tag{3.10}$$

so that $\lambda$ is a character of $H$ and $F = F(1) + \lambda$. Thus, the differential equation (3.2) determines $\Xi$ up to a constant and a character. □

**LEMMA 3.2.** *Suppose that $b : H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a nonsingular, symmetric, bilinear form on $H$. Then there exists a quadratic form $q : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that*

$$\mathcal{H}q = b, \tag{3.11}$$

*where*

$$(\mathcal{H}q)(u,v) := q(uv) - q(u) - q(v). \tag{3.12}$$

**PROOF.** Let us briefly recall the terminology in this lemma. $b$ is nonsingular if the induced map $H \longrightarrow H^\sharp := \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ is an isomorphism. A quadratic form is a function $q : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that

$$q(1) = 0, \qquad q(u^k) = k^2 q(u), \quad \forall u \in H,\ k \in \mathbb{Z} \tag{3.13}$$

and $\mathcal{H}q$ is a bilinear form.

Suppose that $b$ is a nonsingular, symmetric, bilinear form $H\times H \longrightarrow \mathbb{Q}/\mathbb{Z}$. Then, according to [4, Section 7], $b$ admits a resolution. This is a nondegenerate, symmetric, bilinear form

$$B : \Lambda\times\Lambda \longrightarrow \mathbb{Z} \tag{3.14}$$

on a free abelian group $\Lambda$ such that the induced monomorphism $J_B : \Lambda \longrightarrow \Lambda^* := \mathrm{Hom}(\Lambda,\mathbb{Z})$ is a resolution of $H$,

$$0 \hookrightarrow \Lambda \xrightarrow{J_B} \Lambda^* \xrightarrow{\pi} H \longrightarrow 0 \tag{3.15}$$

and $b$ coincides with the induced bilinear form on $\Lambda^*/(J_B\Lambda)$ $(n := \#H)$,

$$b(\pi(u),\pi(v)) = \frac{1}{n^2} B(J_B^{-1}(nu), J_B^{-1}(nv)) \bmod \mathbb{Z}, \quad \forall u,v \in \Lambda^*. \tag{3.16}$$

Now, set

$$q(\pi(u)) = \frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) \bmod \mathbb{Z}. \tag{3.17}$$

This quantity is well defined, that is,

$$\frac{1}{2n^2} B(J_B^{-1}(nu), J_B^{-1}(nu)) = \frac{1}{2n^2} B(J_B^{-1}(nv), J_B^{-1}(nv)) \bmod \mathbb{Z} \tag{3.18}$$

if $v = u + J_B\lambda$, $\lambda \in \Lambda$. Clearly, $\mathcal{H}q = b$. □

Denote by $Q$ the space of solutions of the equation (3.11), that is, the space of quadratic forms $q$ on $H$ satisfying $\mathcal{H}q = -\mathbf{lk}_M$. $Q$ consists of more than one element. It is a $G$-torsor, where $G = \mathrm{Hom}(H,\mathbb{Z}_2)$ and the $G$ action is given by

$$(Q\times G) \ni (q,\mu) \longmapsto q+\mu. \tag{3.19}$$

Using the linking form on $M$ we can identify $G$ with the 2-torsion subgroup of $H$. Denote by $\Xi_\sigma$ the reduction mod $\mathbb{Z}$ of $\tau_\sigma$.

Fix a spin$^c$ structure $\sigma_0$ on $M$. We deduce that for every $q \in Q$ there exists a constant $k = k(q)$ and a character $\lambda = \lambda_q$ of $H$

$$\Xi_{\sigma_0}(h) = k(q) + \lambda_q(h) + q(h), \qquad \mathcal{H}q = -\mathbf{lk}_M. \tag{3.20}$$

In particular,

$$\begin{aligned}\Xi_{g\cdot\sigma_0}(h) :&= \Xi_\sigma(gh) = k(q) + \lambda_q(gh) + q(gh)\\ &= \underbrace{\big(k(q)+\lambda_q(g)+q(g)\big)}_{c(g,q)} + \underbrace{\big(\lambda_q(h) + (\mathcal{H}q)(g,h)\big)}_{\lambda_{q,g}(h)} + q(h)\end{aligned} \tag{3.21}$$

where $\lambda_{q,g}(\bullet) = \lambda_q(\bullet) - \mathbf{lk}_M(g,\bullet)$. Since the linking form is nondegenerate we can find a *unique* $g = g(q)$ such that $\lambda_{q,g} = 0$. We set $\vec{\sigma}(q) = g(q)\cdot\sigma_0$ and $c(q) = c(g(q), q)$. The above computation also shows that for every $\mu \in G$ we have

$$c(q+\mu) - c(q) = q(\mu), \qquad \vec{\sigma}(q+\mu) = \mu\cdot\vec{\sigma}(q). \tag{3.22}$$

We have thus proved the following result.

**Proposition 3.3.** *Suppose $M$ is a rational homology sphere. Then there exist functions*

$$c : Q \longrightarrow \mathbb{Q}/\mathbb{Z}, \qquad \vec{\sigma} : Q \longrightarrow \operatorname{Spin}^c(M) \tag{3.23}$$

*so that*

$$\tau_{\vec{\sigma}(q)}(h) := q(h) + c(q) \bmod \mathbb{Z}, \quad \forall h \in H. \tag{3.24}$$

*Moreover,*

$$c(q+\mu) - c(q) = q(\mu), \quad \vec{\sigma}(q+\mu) = \mu\cdot\vec{\sigma}(q), \quad \forall \mu \in G. \tag{3.25}$$

**Remark 3.4.** (a) Note that $q(\mu) \in (1/4)\mathbb{Z}$, $\forall q \in Q$, $\mu \in \mathbb{Z}$ so that $4c(q)$ is *independent* of $q$. It is a topological invariant of $M$!

(b) One can show that the image of the one-to-one map $\vec{\sigma}$ is $\operatorname{Spin}(M)$, the set spin$^c$ structures induced by the spin structures on $M$. We can thus regard $c$ as a map $c : \operatorname{Spin}(M) \longrightarrow \mathbb{Q}/\mathbb{Z}$.

**4. Examples.** We want to show on some simple examples that the invariant $c$ is nontrivial. First, we need some notation.

We denote by $\mathbb{Z}_n$ the cyclic group with $n$ elements. The functions $f : \mathbb{Z}_n \longrightarrow \mathbb{Q}$ can be conveniently described as polynomials $f \in \mathbb{Q}[x]$, where $x^n = 1$. Given two such polynomials $f, g$, we define the equivalence relation $\sim$ by

$$f \sim g \iff \exists m \in \mathbb{Z} : f = \pm x^m g. \tag{4.1}$$

We will not keep track of Euler structures and/or homology orientations and that is why in the sequel only the $\sim$-equivalence class of the torsion will be well defined. In particular, the map $c$ constructed in the previous section will be defined only up to a sign.

(a) Suppose that $M = L(8,3)$. Then its torsion is (see [2])

$$T_{8,3} \sim -\frac{9}{32}x^7 - \frac{3}{32}x^6 - \frac{9}{32}x^5 + \frac{5}{32}x^4 + \frac{7}{32}x^3 - \frac{3}{32}x^2 + \frac{7}{32}x + \frac{5}{32}, \tag{4.2}$$

where $x^8 = 1$ is a generator of $\mathbb{Z}_8$. Then

$$q(x^n) = \frac{-3n^2}{16}. \tag{4.3}$$

The set of possible values $(-3m^2/16) \mod \mathbb{Z}$ is

$$A := \left\{0, \frac{-3}{16}, \frac{4}{16}, \frac{5}{16}\right\}. \tag{4.4}$$

The set of possible values of $\Xi(h)$ is

$$B := \left\{-\frac{9}{32}, -\frac{3}{32}, \frac{5}{32}, \frac{7}{32}\right\}. \tag{4.5}$$

We need to find a constant $c \in \mathbb{Q}/\mathbb{Z}$ such that

$$B \pm c = A. \tag{4.6}$$

Equivalently, we need to figure out orderings $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2, b_3, b_4\}$ of $A$ and $B$ such that $b_i - a_i \mod \mathbb{Z}$ is a constant independent of $i$. A little trial and error shows that

$$\vec{A} = \left(0, -\frac{3}{16}, \frac{4}{16}, \frac{5}{16}\right), \qquad \vec{B} = \left(-\frac{3}{32}, -\frac{9}{32}, \frac{5}{32}, \frac{7}{32}\right) \tag{4.7}$$

and the constant $c = -3/32$. This is the coefficient of $x^2$. We deduce that (modulo $\mathbb{Z}$)

$$F := T_{8,3}(x) + \frac{3}{32} \sim -\frac{3}{16}x^7 - 0\cdot x^6 - \frac{3}{16}x^5 + \frac{1}{4}x^4 + \frac{1}{4}x^3 - 0\cdot x^2 + \frac{1}{4}x + \frac{1}{4}. \tag{4.8}$$

The translation of $F$ by $x^{-2}$ is

$$x^{-2}\left(T_{8,3} + \frac{3}{32}\right) = \frac{1}{4}x^7 + \frac{1}{4}x^6 - \frac{3}{16}x^5 - \frac{3}{16}x^3 + \frac{1}{4}x^2 + \frac{1}{4}x. \tag{4.9}$$

(b) Suppose that $M = L(7,2)$. Then, its torsion is (see [2])

$$T_{7,2} \sim -\frac{2}{7}x^6 + \frac{1}{7}x^5 + \frac{2}{7}x^3 + \frac{1}{7}x - \frac{2}{7}, \tag{4.10}$$

where $x^7 = 1$ is a generator of $\mathbb{Z}_7$. We see that in this form $T_{7,2}$ is symmetric, that is, the coefficient of $x^k$ is equal to the coefficient of $x^{6-k}$. The constant $c$ in this case must be the coefficient of the middle monomial $x^3$, which is $2/7$.

(c) Suppose that $M = L(7,1)$. Then

$$T_{7,1} \sim \frac{2}{7}x^6 + \frac{1}{7}x^5 - \frac{1}{7}x^4 - \frac{4}{7}x^3 - \frac{1}{7}x^2 + \frac{1}{7}x + \frac{2}{7}. \tag{4.11}$$

This is again a symmetric polynomial and the coefficient of the middle monomial is $-4/7$. We see that this invariant distinguishes the lens spaces $L(7,1)$ and $L(7,2)$. It is known that these two spaces are homotopic but nonhomeomorphic lens spaces. Thus, the invariant $c$ distinguishes their homeomorphism types, just as the torsion does.

(d) For $M = L(9,2)$, we have

$$T_{9,2} \sim -\frac{10}{27}x^8 + \frac{2}{27}x^7 - \frac{1}{27}x^6 + \frac{8}{27}x^5 + \frac{2}{27}x^4 + \frac{8}{27}x^3 - \frac{1}{27}x^2 + \frac{2}{27}x - \frac{10}{27}. \tag{4.12}$$

Again, this is a symmetric function, that is, the coefficient of $x^k$ is equal to the coefficient of $x^{8-k}$, $x^9 = 1$. The constant is the coefficient of $x^4$, which is $2/27$. We deduce that mod $\mathbb{Z}$, we have

$$T_{9,2} - \frac{2}{27} = -\frac{2}{3}x^8 - \frac{2}{9}x^7 - \frac{1}{3}x^6 - \frac{2}{9}x^7. \tag{4.13}$$

(e) Finally, when $M = L(9,7)$ we have

$$T_{9,7} \sim -\frac{8}{27}x^8 - \frac{2}{27}x^7 + \frac{10}{27}x^6 + \frac{1}{27}x^5 - \frac{2}{27}x^4 + \frac{1}{27}x^3 + \frac{10}{27}x^2 - \frac{2}{27}x - \frac{8}{27} \tag{4.14}$$

the polynomial is again symmetric so that the constant $c$ is the coefficient of $x^4$ which is $-2/27$.

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