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THE NUMBER OF CONNECTED COMPONENTS OF CERTAIN REAL ALGEBRAIC CURVES

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ABSTRACT. For an integer $n \ge 2$, let $p(z) = \prod_{k=1}^{n}(z-\alpha_k)$ and $q(z) = \prod_{k=1}^{n}(z-\beta_k)$, where α_k, β_k are real. We find the number of connected components of the real algebraic curve $\{(x,y) \in \mathbb{R}^2 : |p(x+iy)| - |q(x+iy)| = 0\}$ for some α_k and β_k. Moreover, in these cases, we show that each connected component contains zeros of $p(z)+q(z)$, and we investigate the locus of zeros of $p(z)+q(z)$.

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1. Introduction. Throughout the paper, n is an integer ≥ 2. Let $f(x,y)$ be an integral polynomial of degree n. Let A be the real algebraic curve defined by $A = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$. It is known that A consists of at most finitely many connected components. More precisely, when the curve is real nonsingular, each unbounded component is homeomorphic to a line and each bounded component is homeomorphic to a circle. We will call a bounded component an oval, and an unbounded component an ∞-component. Also, we will write "component" instead of "connected component" for convenience. Let $p(z) = \prod_{k=1}^{n}(z-\alpha_k)$ and $q(z) = \prod_{k=1}^{n}(z-\beta_k)$, where α_k, β_k are real. The zeros of $g(z) := p(z)+q(z)$ are clearly contained in the locus of the real algebraic curve

$$C := \{(x,y) \in \mathbb{R}^2 : |p(x+iy)| - |q(x+iy)| = 0\}. \tag{1.1}$$

In fact, in their study of "cylindrical algebraic decomposition," Arnon, Collins, and McCallum [1, 2] provide an algorithm for calculating the number of components given a specific example. However, we do not know the answer in the general case. We provide a different idea in this paper from that in [1, 2]. With the above terminology, here are some general questions.

(a) Given $P(x,y) = 0$ for real variables x and y, how many components are there? It is still unclear how to describe all possibilities for the topological nature of all components of an arbitrary $P(x,y) = 0$; this is the essence of the Hilbert's 16th problem. On the other hand, one of the most significant theorems of real algebraic geometry (Harnack (see [3, pages 257–258]), 1876) tells us that the number of components is at most one more than the genus.

(b) The curve C has finitely many components. Must each component have zeros of $g(z) = 0$?

We answer the questions (a) and (b) for some real algebraic curves of the form (1.1). Define, for real variables x and y,

$$P(x,y) := |p(x+iy)|^2 - |q(x+iy)|^2, \tag{1.2}$$

where $\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\} \subseteq \{1,2,\dots,2n\}$. The simplest case for the questions (a) and (b) is $\{\alpha_k\} = \{1,2,3,\dots,n\}$ and $\{\beta_k\} = \{n+1,n+2,\dots,2n\}$. Then all zeros of $P(x,y)$ obviously lie on the vertical line $x = n+1/2$, so $P(x,y)$ has only one component. We will study the case $\{\alpha_k\} = \{2,2,\dots,2\}$ and $\{\beta_k\} = \{1,n+1,n+1,\dots,n+1\}$ in Section 3. Moreover, in Section 2, we will investigate the locus of zeros of the more general polynomial equation

$$g(x,t) := (x-2)^n + (x-1)(x-t)^{n-1} = 0, \quad t \geq 3. \tag{1.3}$$

2. The zeros of $g(x,t) = 0$. We need the following two lemmas. First, Lemma 2.1 easily follows from the theorems of Hurwitz (see [4, page 4]) and Rouché (see [4, page 2]).

Lemma 2.1. *Let $n > m > 0$ be integers. Let A, B, and C be real numbers with $C \neq 0$. If a trinomial equation*

$$Az^n + Bz^m + C = 0 \quad \text{with } |B| \geq |A| + |C| \tag{2.1}$$

has no zeros on $|z| = 1$, then it has exactly m zeros strictly inside $|z| = 1$.

Lemma 2.2. *The zeros of $g(x,t)$ are $(2+a_{n,t})/(1+a_{n,t})$, where each $a_{n,t}^{-1/(n-1)}$ is a zero of the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$.*

Proof. From $g(x,t) = 0$, we obtain $-(x-2)/(x-1) = ((x-t)/(x-2))^{n-1}$. Let

$$-\frac{x-2}{x-1} = \left(\frac{x-t}{x-2}\right)^{n-1} = a, \tag{2.2}$$

where $a := a_{n,t}$ is a complex number. From $-(x-2)/(x-1) = a$, we find that $x = (2+a)/(1+a)$, and it easily follows from $((x-t)/(x-2))^{n-1} = a$ that $x = (2a^{1/(n-1)} - t)/(a^{1/(n-1)} - 1)$. Equating these two formulae for x leads to $a^{n/n-1} + (1-t)a + 2 - t = 0$. The result follows by multiplying each side by $a^{-n/(n-1)}$. □

Now we find a relation between x (a zero of $g(x,t) = 0$) and z (a zero of $(2-t)z^n + (1-t)z + 1 = 0$) as follows:

$$x = \frac{2z^{n-1}+1}{z^{n-1}+1} = 1 + \frac{1}{1+1/z^{n-1}}. \tag{2.3}$$

So

$$z^{n-1} = \frac{x-1}{2-x}, \quad \text{that is, } z = \left(\frac{x-1}{2-x}\right)^{1/(n-1)}. \tag{2.4}$$

Using Lemmas 2.1 and 2.2, we have the following proposition.

Proposition 2.3. *The function $g(x,t)$ has only one zero x_0 in $\Re x < 3/2$, and has no zeros in $3/2 \leq \Re x \leq (t+2)/2$.*

Proof. Observe that the strip $3/2 \le \Re x \le (t+2)/2$ is zero-free, since, for such x, $|x-2| \le |x-t|$ and $|x-2| < |x-1|$. Now we consider the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$. It has no zero on $|z| = 1$, since, if there were such a zero z, then by (2.4), $1 = |z^{n-1}| = |(x-1)/(2-x)|$, that is, $x = 3/2 + i\beta$ for some real number β. This is a contradiction. Hence, by Lemma 2.1, the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$ has exactly one zero z_0 interior to $|z| = 1$. Then $|z_0| = |((x_0-1)/(2-x_0))^{1/(n-1)}| < 1$, that is, $|x_0 - 1| < |2 - x_0|$ for some real number x_0. Hence $\Re x_0 < 3/2$ which proves the proposition. □

Next, we study further the unique zero x_0 given by Proposition 2.3.

Proposition 2.4. *Let n be an integer ≥ 3 and $t \ge 3$. Then the only zero x_0 of $g(x,t)$ in $\Re x \le (t+2)/2$ is real and*

$$\frac{1+2(-\epsilon+1/n)^{n-1}}{1+(-\epsilon+1/n)^{n-1}} < x_0 < \frac{1+2(\epsilon+1/n)^{n-1}}{1+(\epsilon+1/n)^{n-1}}, \tag{2.5}$$

where $\epsilon = \epsilon(n,t) = 2^n(t-2)/(t-1)^{n+1}$.

Proof. For n an integer ≥ 3, let $\epsilon = \epsilon(n,t) = 2^n(t-2)/(t-1)^{n+1}$. Then $0 < \epsilon \le 1/(t-1)$, since $(2/(t-1))^n < 1/(t-2)$ and $n \ge 3$. Then the trinomial equation $(2-t)z^n + (1-t)z + 1 = 0$ has at least one real zero z_0 in $(1/(t-1)-\epsilon, 1/(t-1)+\epsilon)$. In fact, by algebra, we can see that the left side of the trinomial equation is

$$-(2^n + (1+2^n(-2+t)(-1+t)^{-n})^n)(-2+t)(-1+t)^{-n} < 0 \tag{2.6}$$

at $z = 1/(t-1)+\epsilon$, and

$$-(-2^n + (1-2^n(-2+t)(-1+t)^{-n})^n)(-2+t)(-1+t)^{-n} > 0 \tag{2.7}$$

at $z = 1/(t-1)-\epsilon$. Set $z_0 = ((x_0-1)/(2-x_0))^{1/(n-1)}$. Since z_0 is real, so is x_0. Now we obtain the inequality $|((x_0-1)/(2-x_0))^{1/(n-1)} - 1/(t-1)| < \epsilon$, and from this we have the inequality (2.5). A simple calculation yields that $(1+2A)/(1+A) < (t+2)/2$ for $A > 0$. This proves the result. □

Remark 2.5. (a) For $n = 2$ and $t \ge 3$, we can easily check that $g(x,t)$ has two real zeros. Here the smaller zero is $\le (t+2)/2$, but it does not satisfy (2.5).

(b) In Lemma 2.2, we encountered a trinomial equation $(t-2)z^n + (t-1)z - 1 = 0$ ($t \ge 3$). Here we define a more general polynomial

$$h(z) = (t-2)z^n + (t-1)z - s \quad (s \ge 0). \tag{2.8}$$

Then we have the following zero distributions. The function $h(z)$ has

$$\begin{cases} \text{all its zeros with modulus} > 1 & \text{if } s > 2t-3, \\ \text{one (real) zero with modulus} = 1 \text{ and all others} > 1 & \text{if } s = 2t-3, \\ \text{one (real) zero with modulus} < 1 \text{ and all others} > 1 & \text{if } 0 \le s \le 1. \end{cases} \tag{2.9}$$

This can be proved by elementary calculation, Lemma 2.1, and Eneström-Kakeya theorem (see [4, page 136]). However, we did not consider the case $1 < s < 2t-3$. We conjecture that, for $1 < s < 2t-3$, $h(z)$ has one (real) zero with modulus < 1 and all others > 1, as the case $0 \le s \le 1$, but it remains an open problem.

3. The number of components of $|(z-2)^n| = |(z-1)(z-(n+1))^{n-1}|$. Let

$$g(z) := (z-2)^n + (z-1)\left(z-(n+1)\right)^{n-1}. \tag{3.1}$$

If $g(z) = 0$, then $|(z-1)(z-(n+1))^{n-1}/(z-2)^n|^2 = 1$. This motivates, for real variables x and y, the introduction of

$$G(x,y) := \frac{\left((x-1)^2+y^2\right)\left(\left(x-(n+1)\right)^2+y^2\right)^{n-1}}{\left((x-2)^2+y^2\right)^n} - 1. \tag{3.2}$$

Here $G(x,y)$ is obviously symmetric about the x-axis. In this section, we find the number of components of $G(x,y) = 0$ and show that each component has zeros of $g(z) = 0$. First, using Proposition 2.3, we find that the number of components of $G(x,y) = 0$ is at least two.

Proposition 3.1. *The locus of*

$$|(z-2)^n| = |(z-1)(z-t)^{n-1}|, \quad t \geq 3 \tag{3.3}$$

has at least two components.

Proof. We showed in Proposition 2.3 that $g(x,t)$ has one real zero < 2 and $n-1$ zeros with real part $> (t+2)/2 > 2$. So it suffices to show that, on $z = 2+is$ (s real), the two absolute values are never equal. On $z = 2+is$ (s real),

$$|(z-1)(z-t)^{n-1}|^2 - |(z-2)^n|^2 = (1+s^2)\left((t-2)^2+s^2\right)^{n-1} - s^{2n} \geq (t-2)^2 > 0. \tag{3.4}$$

□

Next, we show that the points where the locus of $G(x,y) = 0$ has vertical tangents lie on the real axis. We use this later to show that the locus consists of either one oval, one ∞-component or three ∞-components. In order to prove this, we need the following lemma.

Lemma 3.2. *Let n be an integer ≥ 3. Define*

$$f(x) := \left(\frac{-2x+3}{(n-1)(-2x+n+2)}\right)^{n-1} - \frac{-2x+n+2}{(n-1)(-2x+n+3)}. \tag{3.5}$$

Then all real zeros of $f(x)$ are

$$\begin{cases} \dfrac{n^2+n-5}{2n-4}, & n \text{ even}, \\[2ex] \dfrac{n^2+n-5}{2n-4}, r(n), & n \text{ odd}, \end{cases} \tag{3.6}$$

where $(n^2+n-5)/(2n-4)$ is a double zero in each case and $3/2 < r = r(n) < (n^2+n+1)/2n$.

Proof. From $f(x) = 0$, we find that

$$\left(\frac{-2x+3}{(n-1)(-2x+n+2)}\right)^{n-1} = \frac{-2x+n+2}{(n-1)(-2x+n+3)} = a, \tag{3.7}$$

where $a := a_n$ is a complex number. From $(-2x+3)/(n-1)(-2x+n+2) = a^{1/(n-1)}$, we get

$$x = -\frac{3 - a^{1/(n-1)}(n-1)(n+2)}{-2+2a^{1/(n-1)}(n-1)}, \tag{3.8}$$

and also

$$x = -\frac{n+2-a(n-1)(n+3)}{-2+2a(n-1)} \tag{3.9}$$

from $(-2x+n+2)/(n-1)(-2x+n+3) = a$. Equating these two formulae for x leads to $(n-1)a^{n/(n-1)} - na + 1 = 0$, and so $a^{1/(n-1)}$ is a zero of the trinomial equation $w(y) := (n-1)y^n - ny^{n-1} + 1 = 0$. Now, we have

$$\frac{w(y)}{(y-1)^2} = (n-1)y^{n-2} + (n-2)y^{n-3} + (n-3)y^{n-4} + \cdots + 2y + 1. \tag{3.10}$$

Since $a^{1/(n-1)}$ is real if and only if the corresponding x in (3.7) is real, the number of real zeros of $f(x)$ is equal to that of $w(y)$. By (3.10), $w(y)$ has a real double zero at 1, and its corresponding x is $(n^2+n-5)/(2n-4)$, since $(-2x+3)/(n-1)(-2x+n+2) = 1$. On the other hand, it follows from Eneström-Kakeya theorem that $w(y)/(y-1)^2$ has no zero for $|y| > 1$. Also it is obvious that $w(y)/(y-1)^2$ has no real zero ≥ 0. In order to find the real zeros of $f(x)$, we first need to determine whether $w(y)$ has a real zero on $(-1,0)$ or not. We see that $w'(y) = n(n-1)y^{n-2}(y-1)$. So if n is even, then $w'(y) < 0$ for $-1 < y < 0$. Moreover, $w(0) = 1 > 0$, which implies there are no real zeros of $w(y)$ other than 1. Hence $f(x)$ has only one (double) real zero $(n^2+n-5)/(2n-4)$. Suppose that n is odd. Then $w'(y) > 0$ on $-1 < y < 0$, $w(-1) = 2(1-n) < 0$, and $w(0) > 0$. This implies that there must be exactly one real zero on $(-1,0)$. Say x_0 is its corresponding real number. Then by (3.7)

$$-1 < \frac{-2x_0+3}{(n-1)(-2x_0+n+2)} < 0. \tag{3.11}$$

Simple calculations yield that $3/2 < x_0 < (n^2+n+1)/2n$. This completes the proof. □

Now we have the following Proposition.

Proposition 3.3. *The points where the locus of $G(x,y) = 0$ has vertical tangents lie on the real axis.*

Proof. It suffices to show that $\langle 0,1\rangle \cdot \nabla G(x,y) = 0$ and $G(x,y) = 0$ implies $y = 0$. A calculation shows that $\langle 0,1\rangle \cdot \nabla G(x,y) = \partial G/\partial y = 0$ if and only if $y = 0$ or $y^2 = A(x)$, where

$$A(x) = \frac{2(n-2)x^3 - (n^2+5n-17)x^2 + 2(n^2+n-12)x - (n^2-2n-11)}{-2(n-2)x + n^2 + n - 5}. \tag{3.12}$$

Suppose that $y^2 = A(x)$. Then

$$f(x) := G(x,y) = \begin{cases} \dfrac{1}{4x^2-16x+15}, & n = 2, \\ \left(\dfrac{-2x+3}{(n-1)(-2x+n+2)}\right)^{n-1} - \dfrac{-2x+n+2}{(n-1)(-2x+n+3)}, & n \geq 3, \end{cases} \tag{3.13}$$

by simplifying the equations. So it is clear that there are no zeros of $f(x)$ in the case of $n=2$. Suppose that $n\geq 3$. By Lemma 3.2, $(n^2+n-5)/(2n-4)$ is a (double) real zero of $f(x)$ and, in particular, if n is even, such a real zero is unique. But $A((n^2+n-5)/(2n-4))$ is not defined. So this is a contradiction. Suppose that n is odd. Then by Lemma 3.2, all zeros of $f(x)$ are $(n^2+n-5)/(2n-4)$ and $r(n)$, where $3/2<r(n)<(n^2+n+1)/2n$. As above, $A((n^2+n-5)/(2n-4))$ is not defined. So it is enough to consider $r(n)$. Now, we have that $A(3/2)=-1/4<0$ and $A((n^2+n+1)/2n)=-(n^4-2n^3+5n^2-4n+1)/4n^2<0$. So if we show that $A'(x)<0$ on $3/2<x<(n^2+n+1)/2n$, then $y^2=A(x)<0$, which is a contradiction. We see that

$$A'(x)=-\frac{2s(x)}{(-2(n-2)x+n^2+n-5)^2}, \tag{3.14}$$

where $s(x)=4(n-2)^2x^3-4(n-2)^2(n+4)x^2+(n^2+5n-17)(n^2+n-5)x-n^4-n^3+12n^2+10n-38$. So it is enough to show that $s(x)>0$ on $3/2<x<(n^2+n+1)/2n$. Now

$$\begin{aligned} s\left(\frac{3}{2}\right)&=\frac{1}{2}(n-1)^3(n+1)>0,\\ s\left(\frac{n^2+n+1}{2n}\right)&=\frac{(n-1)^3(2n-1)(n^2-2n+2)}{n^3}>0,\\ s'(x)&=(6(2-n)x+n^2+5n-17)(2(2-n)x+n^2+n-5). \end{aligned} \tag{3.15}$$

Hence, $(n^2+5n-17)/6(n-2)$ and $(n^2+n-5)/2(n-2)$ are the zeros of $s'(x)$, and we can check that

$$\begin{cases} \dfrac{n^2+5n-17}{6(n-2)}<\dfrac{3}{2}<\dfrac{n^2+n+1}{2n}<\dfrac{n^2+n-5}{2(n-2)}, & n=3,\\[2ex] \dfrac{3}{2}<\dfrac{n^2+5n-17}{6(n-2)}<\dfrac{n^2+n+1}{2n}<\dfrac{n^2+n-5}{2(n-2)}, & n\geq 4. \end{cases} \tag{3.16}$$

This proves the result, since $s(3/2)>0$ and $s((n^2+n+1)/2n)>0$. □

Next we establish the following Proposition.

Proposition 3.4. *For fixed $y_0\neq 0$,*

(a) $\lim_{x\to\pm\infty}G(x,y_0)=0$,

(b) *for $|x|$ large, the limit is approached from above for $x\to-\infty$ and the limit is approached from below for $x\to+\infty$,*

(c) *$G(x,0)$ has exactly three real zeros. Moreover, $(\partial G/\partial x)(x,y_0)$ has at most four real zeros,*

(d)

$$\frac{\partial^2 G}{\partial x^2}(x,y_0)\begin{cases}\geq 0 & \text{as } x\longrightarrow-\infty,\\ \leq 0 & \text{as } x\longrightarrow\infty.\end{cases} \tag{3.17}$$

Proof. Let y_0 be nonzero and fixed. It is obvious that $\lim_{x\to\pm\infty}G(x,y_0)=0$. By a calculation, we have

$$\frac{\partial G}{\partial x}(x,y_0)=\frac{-2n((x-n-1)^2+y_0^2)^nB(x,y_0)}{((x-2)^2+y_0^2)^{n+1}((x-n-1)^2+y_0^2)^2}, \tag{3.18}$$

where

$$B(x)=B(x,y_0)$$
$$=(n-2)y_0^4+(n^2-n+1)(x-2)y_0^2-(x-1)(x-2)(x-(n+1))((n-2)x-n+3) \tag{3.19}$$

is a polynomial in x of degree 4 whose leading coefficient is $2-n$. So it follows from the positivity of the leading coefficient of the numerator of the right side of (3.18) that, for $|x|$ large, $(\partial G/\partial x)(x,y_0)>0$, that is, $G(x,y_0)$ is increasing on (x_1,∞) and $(-\infty,-x_1)$ for x_1 is sufficiently large. On the other hand, by (a), $\lim_{x\to\pm\infty}G(x,y_0)=0$. Hence (b) holds. For (c), we observe that $(\partial G/\partial x)(x,0)$ has the three real zeros $1,n+1,(n-3)/(n-2)$, and we can check that $G(1,0)=G(n+1,0)=-1<0$ and $G((n-3)/(n-2),0)>0$. So $G(x,0)$ has exactly three real zeros. The second assertion of (c) is easily seen from $\deg B(x)=4$, since $(x,y)\neq(n+1,0)$. Finally, we see that

$$\frac{\partial^2 G}{\partial x^2}(x,y_0)=\frac{2n\left((x-n-1)^2+y_0^2\right)^n C(x)}{\left((x-2)^2+y_0^2\right)^{n+2}\left((x-n-1)^2+y_0^2\right)^3}, \tag{3.20}$$

where $C(x)$ is a polynomial in x of degree 7 whose leading coefficient is $2(2-n)$. So it follows from the negativity of the leading coefficient of the numerator of the right side of (3.20) that (d) holds. □

By Proposition 3.4(c), $G(x,0)$ has exactly three real zeros, and for fixed $y\neq 0$ the graph of $G(x,y)$ indicates that the value 0 can be taken on at most three times. Thus, by Propositions 3.1 and 3.3, the locus consists of

$$\{\text{one oval, one }\infty\text{-component}\}\text{ or }\{\text{three }\infty\text{-components}\}. \tag{3.21}$$

Next we examine the number of real zeros of $(\partial G/\partial x)(x,y)$ for $|y|$ sufficiently large.

Lemma 3.5. *For $|y_0|$ sufficiently large, $(\partial G/\partial x)(x,y_0)$ has exactly two real zeros.*

Proof. Let y_0 be sufficiently large and fixed. From (3.18),

$$\frac{\partial G}{\partial x}(x,y_0)=\frac{-2n\left((x-n-1)^2+y_0^2\right)^n B(x,y_0)}{\left((x-2)^2+y_0^2\right)^{n+1}\left((x-n-1)^2+y_0^2\right)^2}. \tag{3.22}$$

Since $(x-n-1)^2+y_0^2\neq 0$, $(\partial G/\partial x)(x,y_0)=0$ is equivalent to $B(x,y_0)=0$. Then

$$B(x)=B(x,y_0)=(ux+v)-(x-1)(x-2)(x-(n+1))((n-2)x-n+3), \tag{3.23}$$

where u and v are positive numbers with v/u large. Observe that the zeros of $ux+v$ and $-(x-1)(x-2)(x-(n+1))((n-2)x-n+3)$ are $-v/u$, $(n-3)/(n-2)$, 1, 2, $n+1$. By sign changes, we observe that there are no real zeros of $B(x)$ on $(-\infty,-v/u)\cup((n-3)/(n-2),1)\cup(2,n+1)$, and there is at least one real zero of $B(x)$ on $(-v/u,(n-3)/(n-2))$. Also there are no real zeros of $B(x)$ on $[0,(n-3)/(n-2)]\cup(1,2)$, since v/u is large. On the other hand, we can check that $B(-x)$ has only one sign change in its coefficients. Hence, by Descartes' rule of signs and the above, there is only one real zero of $B(x)$ on $(-v/u,0)$. But the degree of $B(x)$ is four, so the number of real zeros on $(n+1,\infty)$ is either one or three. It is obvious that more than two real zeros are not on $(n+1,\infty)$. Hence $(\partial G/\partial x)(x,y_0)$ has exactly two real zeros. □

By Proposition 3.4(a), (b) and Lemma 3.5, there is only one real x with $G(x,y)=0$ for $|y|$ sufficiently large. This shows that originally there could have been at most one ∞-component. Hence, by the above, equation (3.21), Proposition 2.3, and the proof of Proposition 3.1, we have the following theorem.

Theorem 3.6. *The locus of*

$$|(z-2)^n| = \left|(z-1)(z-(n+1))^{n-1}\right| \tag{3.24}$$

has exactly two components; one oval and one ∞-component. Each component has zeros of $(z-2)^n+(z-1)(z-(n+1))^{n-1}=0$.

Here Figure 3.1 $(n=3)$ is enlightening.

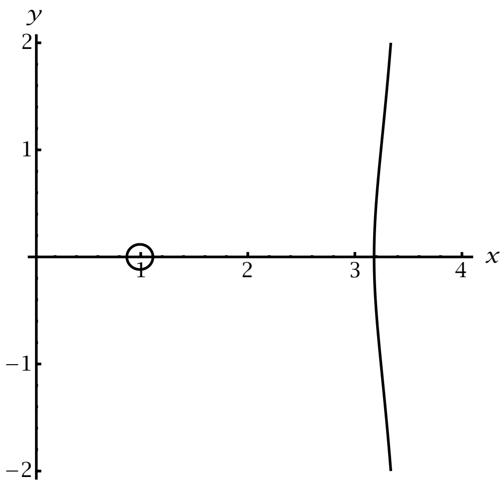

Figure 3.1. $|(z-2)^3| = |(z-1)(z-4)^2|$.

Remark 3.7. Let n and m be positive integers with $1 \le k < n$. If we choose $\{\alpha_k\} = \{1,2,\dots,m,n+m+1,n+m+2,\dots,2n\}$ and $\{\beta_k\} = \{m+1,m+2,\dots,m+n\}$ in (1.2), we can show that the locus of $P(x,y)=0$ has at least two components.

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References

[1] D. S. Arnon, G. E. Collins, and S. McCallum, *Cylindrical algebraic decomposition. I: the basic algorithm*, SIAM J. Comput. **13** (1984), no. 4, 865–877. MR 86h:68067a. Zbl 562.14001.

[2] ———, *Cylindrical algebraic decomposition. II: an adjacency algorithm for the plane*, SIAM J. Comput. **13** (1984), no. 4, 878–889. MR 86h:68067b. Zbl 562.14001.

[3] S. Lang, *Introduction to Algebraic Geometry*, Interscience Tracts in Pure and Applied Mathematics, no. 5, Interscience Publishers, New York, 1958. MR 20#7021. Zbl 095.15301.

[4] M. Marden, *Geometry of Polynomials*, 2nd ed., Mathematical Surveys, no. 3, American Mathematical Society, Providence, 1966. MR 37#1562. Zbl 162.37101.

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