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STRONG UNIQUE CONTINUATION OF EIGENFUNCTIONS FOR p-LAPLACIAN OPERATOR

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ABSTRACT. We show the strong unique continuation property of the eigenfunctions for p-Laplacian operator in the case $p < N$.

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1. Introduction. This paper is primarily concerned with the problem:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+V|u|^{p-2}u=0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the weight functions V is assumed to be not equivalent to zero and to lie in $L^{N/p}(\Omega)$.

Also, we know that the unique continuation property is defined by a different form. In this work, we are interested to study a family of functions which enjoys the strong unique continuation property (SUCP), that is, functions besides possibly the zero functions has a zero of infinite order.

DEFINITION 1.1. A function $u \in L^p(\Omega)$ has a zero of infinite order in p-mean at $x_0 \in \Omega$, if for each $n \in \mathbb{N}$,

$$\int_{|x-x_0|\le R} |u|^p = 0(R^n) \quad \text{as } R \longrightarrow 0. \tag{1.2}$$

There is an extensive literature on unique continuation. We refer to the work of Jerison-Kenig on the unique continuation for Shrödinger operators (cf. [3]). The same work is done by Gossez and Figueiredo, but for linear elliptic operator in the case $V \in L^{N/2}$, where $N > 2$, (cf. [1]). Also, Loulit extended this property to $N = 2$ by introducing Orlicz's space, (cf. [2, 5]). In this work, we generalize this property for the p-Laplacian in the case $V \in L^{N/p}(\Omega)$ and $p < N$.

2. Strong unique continuation theorem. In this section, we proceed to establish the strong unique continuation property of the eigenfunctions for the p-Laplacian operator in the case $V \in L^{N/p}(\Omega)$ and $p < N$.

THEOREM 2.1. *Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ solution of (1.1). If $u = 0$ on a set E of positive measure, then u has a zero of infinite order in p-mean.*

To prove this theorem we need the following lemmas.

Lemma 2.2. *Let $g \in W_0^{1,p}(\Omega)$ and $V \in L^{N/p}$. Then for each $\epsilon > 0$ there exists a positive constant k_ϵ such that*

$$\int_\Omega V|g|^p \le \epsilon \int_\Omega |\nabla g|^p + k_\epsilon \int_\Omega |g|^p. \tag{2.1}$$

Proof. Set

$$G = \left\{ x \in \frac{\Omega}{V(x)} \ge c \right\}. \tag{2.2}$$

So

$$\int_\Omega V|g|^p \le \int_G V|g|^p + k\int_\Omega |g|^p. \tag{2.3}$$

By using the Hölder and Poincaré's inequalities, we get

$$\int_\Omega V|g|^p \le c\|\chi_G V\|_L^{N/p} \int_\Omega |\nabla g|^p + k\int_\Omega |g|^p. \tag{2.4}$$

But $\|\cdot\|$ is absolutely continuous. So, given $\epsilon > 0$, there exists k such that $c\|\chi_G V\| \le \epsilon$. Which gives immediately the inequality (2.1). □

Lemma 2.3. *Let B_r and B_{2r} be two concentric balls contained in Ω. Then*

$$\int_{B_r} |\nabla u|^p \le \frac{c}{r^p}\int_{B_{2r}} |u|^p, \tag{2.5}$$

where the constant c does not depend on r.

Proof. Take $\varphi \in C_0^\infty(\Omega)$, with $\operatorname{supp}\varphi \subset B_{2r}$, $\varphi(x) = 1$ for $x \in B_r$ and $|\nabla\varphi| \le c/r$. Using $\varphi^p u$ as a test function in (1.1), we get

$$\int_{B_{2r}} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\varphi^p u + \int_{B_{2r}} V|u|^{p-2}u\,\varphi^p u = 0. \tag{2.6}$$

So

$$\int_{B_{2r}} |\nabla u|^p\varphi^p = -p\int_{B_{2r}} |\nabla u|^{p-2}\varphi^{p-2}\nabla u\cdot\nabla\varphi(\varphi u) - \int_{B_{2r}} V|\varphi u|^p. \tag{2.7}$$

Using Young's inequalities for $(((p-1)/p)+1/p = 1)$, we can estimate the first integral in the right-hand side of (2.7) by

$$(p-1)\epsilon^{p/(p-1)}\int_{B_{2r}} |\nabla u|^p\varphi^p + \epsilon^{-p}\int_{B_{2r}} |\nabla\varphi|^p|u|^p. \tag{2.8}$$

Also by the result of Lemma 2.2, we can estimate the second integral in the right-hand side of (2.7) by

$$\epsilon\int_{B_{2r}} \left|\nabla(\varphi u)\right|^p + c_\epsilon\int_{B_{2r}} |\varphi u|^p. \tag{2.9}$$

Using these estimates in (2.7), we have

$$\begin{aligned}\int_{B_{2r}} |\nabla u|^p\varphi^p \le{}& \left((p-1)\epsilon^{p/(p-1)} + \epsilon\right)\int_{B_{2r}} |\nabla u|^p|\varphi|^p \\ &+ \left(\epsilon^{-p}+\epsilon\right)\int_{B_{2r}} |u|^p|\nabla\varphi|^p + c_\epsilon\int_{B_{2r}} |u|^p|\varphi|^p.\end{aligned} \tag{2.10}$$

Using the fact that $|\nabla\varphi| \le c/r$, $|\varphi| \le c/r$, and $\varphi = 1$ in B_r, we have immediately inequality (2.5). □

Lemma 2.4. *Let $u \in W^{1,1}(B_r)$, where B_r is the ball of radius r in R^N and let $E = \{x \in B_r : u(x) = 0\}$. Then there exists a constant β depending only on N such that*

$$\int_A |u| \le \beta \frac{r^N}{|E|} |A|^{1/N} \int_{B_r} |\nabla u| \tag{2.11}$$

for all ball B_r, u as above and all measurable sets $A \subset B_r$.

To prove this lemma see [4].

Proof of Theorem 2.1. We know that almost every point of E is a point of density of E. Let $x_0 \in E$ be such a point. This means that

$$\lim_{r \to 0} \frac{|E \cap B_r|}{|B_r|} = 1, \tag{2.12}$$

where B_r denotes the ball of radius r centered at x_0 and $|S|$ denotes the Lebesgue measure of a set S. So, given $\epsilon > 0$ there is an $r_0 = r_0(\epsilon)$ such that

$$\frac{|E^c \cap B_r|}{|B_r|} < \epsilon, \qquad \frac{|E \cap B_r|}{|B_r|} > 1 - \epsilon \quad \text{for } r \le r_0, \tag{2.13}$$

where E^c denotes the complement of the set E. Taking r_0 smaller, if necessary, we can assume $B_{r_0} \subset \Omega$. Since $u = 0$ on E, by Lemma 2.4 and (2.13) we have

$$\begin{aligned} \int_{B_r} |u|^p = \int_{B_r \cap E^c} |u|^p &\le \beta \frac{r^N}{|E \cap B_r|} |E^c \cap B_r|^{1/N} \int_{B_r} |\nabla (u)^p| \\ &\le p\beta \frac{r^N}{|B_r|^{(1-1/N)}} \frac{\epsilon^{1/N}}{1-\epsilon} \int_{B_r} |u|^{p-1} |\nabla u|. \end{aligned} \tag{2.14}$$

By Hölder's inequality

$$\int_{B_r} |u|^p \le c \frac{\epsilon^{1/N}}{1-\epsilon} r \left(\int_{B_r} |\nabla u|^p \right)^{1/p} \left(\int_{B_r} |u|^p \right)^{(p-1)/p}, \tag{2.15}$$

and by using the Young's inequality, we get

$$\int_{B_r} |u|^p \le c \frac{\epsilon^{1/N}}{1-\epsilon} r \left(r^{p-1} \int_{B_r} |\nabla u|^p + \frac{p-1}{r} \int_{B_r} |u|^p \right). \tag{2.16}$$

Finally, by Lemma 2.3, we have

$$\int_{B_r} |u|^p \le c \frac{\epsilon^{1/N}}{1-\epsilon} \int_{B_{2r}} |u|^p, \tag{2.17}$$

where c is independent of ϵ and of r as $r \to 0$.

Now let us introduce the following functions:

$$f(r) = \int_{B_r} |u|^p. \tag{2.18}$$

And let us fix $n \in \mathbb{N}$, choose $\epsilon > 0$ such that $(c\epsilon^{1/N})/(1-\epsilon) \leq 2^{-n}$. Observe that consequently r_0 depends on n. Then (2.17) can be written as

$$f(r) \leq 2^{-n} f(2r) \quad \text{for } r \leq r_0. \tag{2.19}$$

Iterating (2.19), we get

$$f(\rho) \leq 2^{-kn} f(2^k \rho), \quad \text{if } 2^{k-1}\rho \leq r_0. \tag{2.20}$$

Now given $0 < r < r_0(n)$ and choose $k \in \mathbb{N}$ such that

$$2^{-k} r_0 \leq r \leq 2^{-k+1} r_0. \tag{2.21}$$

From (2.20), we obtain

$$f(r) \leq 2^{-kn} f(2^k r) \leq 2^{-kn} f(2r_0). \tag{2.22}$$

Since $2^{-k} \leq r/r_0$, we finally obtain

$$f(r) \leq \left(\frac{r}{r_0}\right)^n f(2r_0), \tag{2.23}$$

which shows that x_0 is a zero infinite order in p-mean. □

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