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ON WHITEHEAD'S INEQUALITY, $\operatorname{nil}[X,G] \le \operatorname{cat} X$

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ABSTRACT. A new proof of Whitehead's inequality, $\operatorname{nil}[X,G] \le \operatorname{cat} X$, is given.

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One of the beautiful theorems of elementary homotopy theory is the result that $\operatorname{nil}[X,G] \le \operatorname{cat} X$. We begin by explaining the notation. Let X and G be based, connected topological spaces and let G be group-like. Thus there is a multiplication $G \times G \to G$ on G which satisfies the group axioms up to homotopy [7, page 118]. Then the set $[X,G]$ of based homotopy classes of maps from X to G inherits a group structure from G. For a nilpotent group π, $\operatorname{nil}\pi$ is the nilpotency class of π. In particular, $\operatorname{nil}\pi = 0$ means that π is the trivial group and $\operatorname{nil}\pi \le 1$ means that π is abelian. Finally, $\operatorname{cat} X$ denotes the Lusternik-Schnirelmann category of X, normalized so that contractible spaces have cat = 0.

THEOREM 1 [7, page 464]. *With the above assumptions,* $\operatorname{nil}[X,G] \le \operatorname{cat} X$.

The proof given in [7, pages 462–464] uses the following definition of category [7, page 458]: $\operatorname{cat} X$ is the smallest nonnegative integer l such that the diagonal map $X \to X^{l+1}$ factors up to homotopy through the subspace of X^{l+1} with at least one coordinate equal to the base point. Recently, another equivalent definition of category given by the existence of cross-sections to certain fibrations, called Ganea fibrations, has been widely used.

The purpose of this paper is to give a new proof of Whitehead's theorem using this latter definition of category.

For a space X, we define the Ganea fibrations

$$F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X \tag{1}$$

inductively [3]: for $n = 0$ the fibration is just $\Omega X \to EX \to X$, the standard path-space fibration. Assume $F_{n-1}(X) \xrightarrow{i_{n-1}} G_{n-1}(X) \xrightarrow{p_{n-1}} X$ is defined and let $G'_n(X) = G_{n-1}(X) \cup_{i_{n-1}} CF_{n-1}(X)$ be the mapping cone of i_{n-1}. Define $p'_n : G'_n(X) \to X$ as p_{n-1} on $G_{n-1}(X)$ and trivial on the cone $CF_{n-1}(X)$. Replacing p'_n by an equivalent fibre map, we obtain the fibre sequence $F_n(X) \xrightarrow{i_n} G_n(X) \xrightarrow{p_n} X$. The connection of the Ganea fibrations to category is as follows (see [2, 4]): $\operatorname{cat} X \le n$ if and only if p_n admits a cross-section.

We now start the proof of the theorem. We begin in Lemma 2 with a general result which is probably known (see [5, page 22] and [6]). Let $f : A \to B$ be any map and consider the mapping cone sequence of f,

$$A \xrightarrow{f} B \xrightarrow{j} C_f \xrightarrow{q} \Sigma A, \tag{2}$$

where C_f is the mapping cone of f and ΣA is the suspension of A. If G is any group-like space, we obtain a homomorphism $q^* : [\Sigma A, G] \to [C_f, G]$.

Lemma 2. *The image of q^* is contained in the center of $[C_f, G]$.*

Proof. We sketch the proof which is based on the operation of $[\Sigma A, G]$ on $[C_f, G]$ [7, page 136]. We denote this operation by "$\cdot$" and the group operation in $[\Sigma A, G]$ and $[C_f, G]$ by "+". Then for $a, b \in [\Sigma A, G]$ and $x, y \in [C_f, G]$, it is easily seen (see [1] and also [5, page 5]) that

$$(a+b)\cdot(x+y) = (a\cdot x)+(b\cdot y). \tag{3}$$

Let e denote the homotopy class of the constant map. By taking $b = e$ and $x = e$, we obtain

$$a\cdot y = q^*(a)+y. \tag{4}$$

By taking $a = e$ and $y = e$, we obtain $b\cdot x = x+q^*(b)$ which we write as

$$a\cdot y = y+q^*(a). \tag{5}$$

Thus Image q^* is in the center of $[C_f, G]$. □

Lemma 3. *For any space X and group-like space G,* $\operatorname{nil}[G_k(X), G] \le k$.

Proof. This is proved by induction on k. Clearly, $\operatorname{nil}[G_0(X), G] = 0$ since $G_0(X)$ is contractible. Suppose the result is true for $k-1$. It suffices to show that $\operatorname{nil}[G'_k(X), G] \le k$. Consider the mapping cone sequence

$$F_{k-1}(X) \xrightarrow{i_{k-1}} G_{k-1}(X) \xrightarrow{j_{k-1}} G'_k(X) \xrightarrow{q_k} \Sigma F_{k-1}(X), \tag{6}$$

where $G'_k(X)$ is the mapping cone of i_{k-1}, j_{k-1} is the inclusion, and q_k is the projection. This gives an exact sequence of groups

$$[\Sigma F_{k-1}(X), G] \xrightarrow{q_k^*} [G'_k(X), G] \xrightarrow{j_{k-1}^*} [G_{k-1}(X), G]. \tag{7}$$

By Lemma 2, Image q_k^* is contained in the center of $[G'_k(X), G]$. By induction, $\operatorname{nil}[G_{k-1}(X), G] \le k-1$. Therefore, $\operatorname{nil}[G_k(X), G] \le k$. □

Now we complete the proof of the theorem. Suppose $\operatorname{cat} X = n$. Thus there is a section $s : X \to G_n(X)$, that is, $p_n s$ is homotopic to the identity map. Hence $s^* : [G_n(X), G] \to [X, G]$ is onto. Since $\operatorname{nil}[G_n(X), G] \le n$ by Lemma 3, it follows that $\operatorname{nil}[X, G] \le n$.

Remark 4. By dualizing the Ganea fibrations we obtain the Ganea cofibrations $X \to C_n(X) \to Q_n(X)$ [2, Section 4]. Then the cocategory of X is defined to be the smallest integer n such that the cofibre map $X \to C_n(X)$ has a retraction. If C is a co-H-group, then an argument dual to the one above yields $\operatorname{nil}[C, Y] \le \operatorname{cocat} Y$.

References

[1] M. Arkowitz, *Actions and coactions in a category*, in preparation.

[2] ______, *Equivalent definitions of the Ganea fibrations and cofibrations*, Manuscripta Math. **100** (1999), no. 2, 221–229. CMP 1 721 634. Zbl 940.55008.

[3] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427. MR 37#4814. Zbl 149.40703.

[4] W. J. Gilbert, *Some examples for weak category and conilpotency*, Illinois J. Math. **12** (1968), 421–432. MR 37#6930. Zbl 157.54203.

[5] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Publishers, New York, London, Paris, 1965. MR 33#6624.

[6] L. L. Larmore and E. Thomas, *Mappings into loop spaces and central group extensions*, Math. Z. **128** (1972), 277–296. MR 47#5865. Zbl 254.55010.

[7] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, Berlin, 1978. MR 80b:55001. Zbl 406.55001.

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