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ON n-FOLD IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

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Abstract. We introduce the notion of n-fold implicative filters and n-fold implicative lattice implication algebras. We give characterizations of n-fold implicative filters and n-fold implicative lattice implication algebras. Finally, we construct an extension property for n-fold implicative filter.

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1. Introduction. In order to research the logical system whose propositional value is given in a lattice, Xu [2] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [3] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. The author of this paper [1] gave an equivalent condition of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In this paper, we discuss the foldness of implicative filters in lattice implication algebras.

2. Preliminaries

Definition 2.1 (see [2]). By a *lattice implication algebra* we mean a bounded lattice $(L,\vee,\wedge,0,1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

$$x \longrightarrow (y \longrightarrow z) = y \longrightarrow (x \longrightarrow z), \tag{2.1}$$

$$x \longrightarrow x = 1, \tag{2.2}$$

$$x \longrightarrow y = y' \longrightarrow x', \tag{2.3}$$

$$x \longrightarrow y = y \longrightarrow x = 1 \Longrightarrow x = y, \tag{2.4}$$

$$(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x, \tag{2.5}$$

$$(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z), \tag{2.6}$$

$$(x \wedge y) \longrightarrow z = (x \longrightarrow z) \vee (y \longrightarrow z), \tag{2.7}$$

for all $x,y,z \in L$.

Example 2.2 (see [3]). Let $L := \{0,a,b,c,1\}$. Define the partially-ordered relation on L as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x,y\}, \qquad x \vee y := \max\{x,y\}, \tag{2.8}$$

Table 2.1.

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	a	c	1	1
1	0	a	b	c	1

for all $x,y \in L$ and "$'$" and "$\to$" as in Table 2.1. Then $(L,\vee,\wedge,',\to)$ is a lattice implication algebra.

In what follows, the binary operation "$\to$" will be denoted by juxtaposition. We can define a partial ordering "$\le$" on a lattice implication algebra L by $x \le y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [2]):

$$0x = 1, \qquad 1x = x, \qquad x1 = 1, \tag{2.9}$$

$$xy \le (yz)(xz), \tag{2.10}$$

$$x \le y \text{ implies } yz \le xz,\ zx \le zy. \tag{2.11}$$

In what follows, L will denote a lattice implication algebra, unless otherwise specified.

Definition 2.3 (see [3]). A subset F of L is called a *filter* of L if it satisfies for all $x,y \in L$ the following:

$$1 \in F, \tag{2.12}$$

$$x \in F,\ xy \in F \text{ imply } y \in F. \tag{2.13}$$

Definition 2.4 (see [3]). A subset F of L is called an *implicative filter* of L if it satisfies (2.12) and

$$x(yz) \in F,\ xy \in F \text{ imply } xz \in F, \quad \forall x,y,z \in L. \tag{2.14}$$

Proposition 2.5 (see [1, Proposition 3.2]). *Every filter F of L has the property*

$$x \le y,\ x \in F \textit{ imply } y \in F. \tag{2.15}$$

3. n-fold implicative filters. For any elements x and y of L and any positive integer n, let $x^n y$ denote $x(\cdots(x(xy))\cdots)$ in which x occurs n times, and $x^0 y = y$.

Definition 3.1. Let n be a positive integer. A subset F of L is called an *n-fold implicative filter* of L, if it satisfies (2.12) and

$$x^n(yz) \in F,\ x^n y \in F \text{ imply } x^n z \in F, \quad \forall x,y,z \in L. \tag{3.1}$$

Note that the 1-fold implicative filter is an implicative filter.

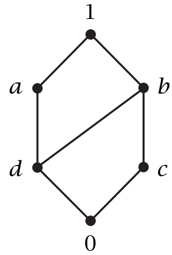

FIGURE 3.1.

TABLE 3.1.

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

(a)

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

(b)

EXAMPLE 3.2. Let $L := \{0,a,b,c,d,1\}$ be a set with Figure 3.1 as a partial ordering. Define a unary operation "$'$" and a binary operation denoted by juxtaposition on L as in Tables 3.1a and 3.1b, respectively.

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y, \quad x \wedge y := ((x'y')y')', \quad \forall x,y \in L. \tag{3.2}$$

Then L is a lattice implication algebra. It is easy to check that $F := \{b,c,1\}$ is an n-fold implicative filter of L.

THEOREM 3.3. *Every n-fold implicative filter of L is a filter of L.*

PROOF. Let F be an n-fold implicative filter of L. Taking $x = 1$ in (3.1) and using (2.9), we conclude that $yz \in F$ and $y \in F$ imply $z \in F$, that is, (2.13) holds. Hence F is a filter of L. □

The converse of Theorem 3.3 is not true. For example, let L be a lattice implication algebra in Example 3.2. Then $\{1\}$ is a filter of L, but $\{1\}$ is not a 1-fold implicative filter of L because $d^1(bc) = db = 1$ and $d^1b = 1$, but $d^1c = b \neq 1$.

We give conditions for a filter to be an n-fold implicative filter.

THEOREM 3.4. *Let F be a filter of L. Then the following statements are equivalent:*

(i) *F is an n-fold implicative filter of L.*

(ii) *$x^{n+1}y \in F$ implies $x^n y \in F$.*

(iii) *$x^n(yz) \in F$ implies $(x^n y)(x^n z) \in F$.*

Proof. (i)⇒(ii). Assume that F is an n-fold implicative filter of L and let $x, y \in L$ be such that $x^{n+1}y \in F$. Then $x^n(xy) \in F$, and since $x^n x = 1 \in F$, it follows from (3.1) that $x^n y \in F$.

(ii)⇒(iii). Suppose (ii) holds and let $x, y, z \in L$ be such that $x^n(yz) \in F$. Since $x^n(yz) \le x^n((x^n y)(x^n z))$, we have

$$x^{n+1}(x^{n-1}((x^n y)z)) = x^n(x^n((x^n y)z)) = x^n((x^n y)(x^n z)) \in F. \tag{3.3}$$

It follows from (ii) that $x^{n+1}(x^{n-2}((x^n y)z)) = x^n(x^{n-1}((x^n y)z)) \in F$. Using (ii) again, we get

$$x^{n+1}(x^{n-3}((x^n y)z)) = x^n(x^{n-2}((x^n y)z)) \in F. \tag{3.4}$$

Repeating this process, we conclude that $(x^n y)(x^n z) = x^n((x^n y)z) \in F$.

(iii)⇒(i). Let $x, y, z \in L$ be such that $x^n(yz) \in F$ and $x^n y \in F$. It follows from (iii) that $(x^n y)(x^n z) \in F$ and $x^n y \in F$, so from (2.13), we have $x^n z \in F$. Hence F is an n-fold implicative filter of L. □

Definition 3.5. Let n be a positive integer. A lattice implication algebra L is said to be *n-fold implicative* if it satisfies the equality $x^{n+1}y = x^n y$ for all $x, y \in L$.

Corollary 3.6. *In an n-fold implicative lattice implication algebra, the notion of filters and n-fold implicative filters coincide.*

We give a characterization of an n-fold implicative lattice implication algebra.

Theorem 3.7. *A lattice implication algebra L is n-fold implicative if and only if the filter $\{1\}$ of L is n-fold implicative.*

Proof. Necessity is by Corollary 3.6. Assume that the filter $\{1\}$ of L is n-fold implicative. Noticing that $x^n((xy)y) = 1$, and applying Theorem 3.4, we have

$$(x^{n+1}y)(x^n y) = (x^n(xy))(x^n y) = 1. \tag{3.5}$$

On the other hand, it is clear that $(x^n y)(x^{n+1}y) = 1$. Hence $x^{n+1}y = x^n y$, as desired. □

The following is a characterization of an n-fold implicative filter.

Theorem 3.8. *A nonempty subset F of L is an n-fold implicative filter of L if and only if it satisfies (2.12) and*

$$x(y^{n+1}z) \in F,\ x \in F \text{ imply } y^n z \in F, \quad \forall x, y, z \in L. \tag{3.6}$$

Proof. Suppose that F is an n-fold implicative filter of L and let $x, y, z \in L$ be such that $x(y^{n+1}z) \in F$ and $x \in F$. Since F is a filter of L (see Theorem 3.3), it follows that $y^{n+1}z \in F$. Using Theorem 3.4, we know that $y^n z \in F$.

Conversely, assume that F satisfies (2.12) and (3.6). Let $x, y \in L$ be such that $xy \in F$ and $x \in F$. Then $x(1^{n+1}y) = xy \in F$ and $x \in F$. Thus, by (3.6), we have $y = 1^n y \in F$. Hence F is a filter of L. Now, if $x^{n+1}y \in F$ for all $x, y \in L$, then $1(x^{n+1}y) = x^{n+1}y \in F$ and $1 \in F$. It follows from (3.6) that $x^n y \in F$. Hence F is an n-fold implicative filter of L by Theorem 3.4. This completes the proof. □

Theorem 3.9 (extension property for n-fold implicative filters)**.** *Let F and G be filters of L such that $F \subseteq G$. If F is n-fold implicative, then so is G.*

Proof. Let $x, y \in L$ be such that $x^{n+1}y \in G$. Since $x^{n+1}((x^{n+1}y)y) = 1 \in F$, it follows from (2.1) and Theorem 3.4(ii) that

$$(x^{n+1}y)(x^n y) = x^n((x^{n+1}y)y) \in F \subseteq G, \tag{3.7}$$

so that $x^n y \in G$ since G is a filter. Using Theorem 3.4, we conclude that G is an n-fold implicative filter of L. □

Using Theorems 3.7 and 3.9, we have the following theorem.

Theorem 3.10. *A lattice implication algebra is n-fold implicative if and only if every filter is n-fold implicative.*

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