SOME ALGEBRAIC UNIVERSAL SEMIGROUP COMPACTIFICATIONS

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ABSTRACT. Universal compactifications of semitopological semigroups with respect to the properties satisfying the varieties of semigroups and groups are studied through two function algebras.

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1. Introduction. In the main approach to semigroup compactification, whose considerations goes back at least to the pioneering paper of de Leeuw and Glicksberg [2], the spectra of some C^*-algebras of functions on a semitopological semigroup are employed to construct certain universal compactifications. For instance, Junghenn in his elaborate study of distal functions characterized the universal group compactification, [7]. The main goal of the present work is to construct two function algebras $\mathcal{F}_V$ and $\mathcal{F}_{\mathcal{V}}$ whose corresponding compactifications are universal with respect to the properties satisfying a variety V of semigroups and a variety $\mathcal{V}$ of groups (the structures of which in terms of subdirect products are given in [1, Section 3.3]).

2. Preliminaries. Here we highlight some required notions and notations from Berglund et al. [1], which is our ground rule. On $\mathcal{C}(S)$, the C^*-algebra of all continuous bounded complex-valued functions on a semitopological semigroup S, the left and right translations L_s and R_s are defined so that, $(L_s f)(t) = f(st) = (R_t f)(s)$. A unital C^*-subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ which is a left translation invariant (i.e., $L_s f \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$) is called m-admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ lies in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (equal to the spectrum of $\mathcal{F}$). Then the pair $(\varepsilon, S^{\mathcal{F}})$ is a (semigroup) compactification (called $\mathcal{F}$-compactification) of S, in which $S^{\mathcal{F}}$ is furnished with the Gelfand topology and the multiplication $\mu\nu = \mu \circ T_\nu$, and $\varepsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping.

The m-admissible algebras of the left multiplicatively continuous, (weakly) [strongly] almost periodic, and strongly minimal distal functions, are denoted by $\mathcal{LMC}$, $(\mathcal{WAP})$ $[\mathcal{SAP}]\mathcal{AP}$, and $\mathcal{GP}$, respectively; see [1, 7].

3. The main results. From now on, F_A is a fixed free semigroup on the countable alphabet A, on which the identities of V are expressed, and $p\,(= \prod_{i=1}^{m} a_i) = q\,(= \prod_{j=1}^{n} b_j)$ is a fixed arbitrary formal identity of V; see [6].

DEFINITION 3.1. We define $\mathcal{F}_V(S)$ as the set of those $f \in \mathcal{LMC}(S)$ such that the identities

$$\lim_\alpha \left(\left(\prod_{i=2}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) f \right) (\theta_\alpha(a_1)) = \lim_\alpha \left(\left(\prod_{j=2}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) f \right) (\theta_\alpha(b_1)), \tag{3.1}$$

$$\lim_\alpha \left(\left(\prod_{i=2}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) \left(\lim_\alpha R_{s_\alpha} f \right) \right) (\theta_\alpha(a_1)) = \lim_\alpha \left(\left(\prod_{j=2}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) \left(\lim_\alpha R_{s_\alpha} f \right) \right) (\theta_\alpha(b_1)), \tag{3.2}$$

$$\left(\prod_{i=1}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) f = \left(\prod_{j=1}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) f, \tag{3.3}$$

$$\left(\prod_{i=1}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) \left(\lim_\alpha R_{s_\alpha} f \right) = \left(\prod_{j=1}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) \left(\lim_\alpha R_{s_\alpha} f \right), \tag{3.4}$$

are valid for all identities $\prod_{i=1}^m a_i = \prod_{j=1}^n b_j$ of V, and all nets θ_α (of mappings from A into S), for which the involved pointwise limits exist. (Note that, the notation $\prod$ is being used for the iterated multiplications in different semigroups, and one should realize that $\prod_{i=2}^m \lim_\alpha R_{\theta_\alpha(a_i)}$ plays the role of identity operator whenever $m = 1$; similarly for $n = 1$.)

The next lemma presents $\mathcal{F}_V$ in terms of $S^{\mathcal{LMC}}$.

LEMMA 3.2. *A function $f \in \mathcal{LMC}$ belongs to $\mathcal{F}_V$ if and only if for every homomorphism $\Phi : F_A \to S^{\mathcal{LMC}}$, each identity $p = q$ of V, and every $\mu \in S^{\mathcal{LMC}}$, $\Phi(p)(f) = \Phi(q)(f)$, $(\Phi(p)\mu)(f) = (\Phi(q)\mu)(f)$, $T_{\Phi(p)}f = T_{\Phi(q)}f$, and $T_{(\Phi(p)\mu)}f = T_{(\Phi(q)\mu)}f$.*

PROOF. For the necessity, it is enough to show that $\Phi(p)(g) = \Phi(q)(g)$ and $T_{\Phi(p)}g = T_{\Phi(q)}g$, for all $g \in X_f \cup \{f\}$, where X_f is the pointwise closure of $R_S f$ in $\mathcal{C}(S)$. There exists a net $\theta_\alpha : A \to S$ such that, for each i and j, $\lim_\alpha \varepsilon(\theta_\alpha(a_i)) = \Phi(a_i)$ and $\lim_\alpha \varepsilon(\theta_\alpha(b_j)) = \Phi(b_j)$. Now using (3.1) and (3.2), we have

$$\begin{aligned}
\Phi(p)(g) &= \Phi(a_1) \left(\prod_{i=2}^m T_{\Phi(a_i)} \right) g \\
&= \lim_\alpha \left(\left(\prod_{i=2}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) g \right) (\theta_\alpha(a_1)) \\
&= \lim_\alpha \left(\left(\prod_{j=2}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) g \right) (\theta_\alpha(b_1)) \\
&= \Phi(q)(g).
\end{aligned} \tag{3.5}$$

A similar argument, using (3.3) and (3.4), shows that $T_{\Phi(p)}g = T_{\Phi(q)}g$. Conversely, for any net θ_α, of mappings from A into S, consider the mapping $\theta : A \to S^{\mathcal{LMC}}$ defined by $\theta(a) = \lim_\alpha \varepsilon(\theta_\alpha(a))$. Using the crucial property of free semigroups, θ can be extended to a homomorphism $\Phi : F_A \to S^{\mathcal{LMC}}$. Therefore, for every $g \in X_f \cup \{f\}$,

$$\left(\prod_{i=1}^m \lim_\alpha R_{\theta_\alpha(a_i)} \right) g = \left(\prod_{i=1}^m T_{\Phi(a_i)} \right) g = T_{\Phi(p)}g = T_{\Phi(q)}g = \left(\prod_{j=1}^n \lim_\alpha R_{\theta_\alpha(b_j)} \right) g; \tag{3.6}$$

which is equivalent to (3.3) and (3.4), taken together. A similar argument can be used to obtain the other required identity, which is equivalent to (3.1) and (3.2) taken together.
□

Now we have the next result which describes the main property of F_V.

THEOREM 3.3. $\mathscr{F}_V$ *is* m-*admissible and* $\mathscr{F}_V$-*compactification is universal with respect to the property satisfying* V.

PROOF. Using Lemma 3.2, the m-admissibility of $\mathscr{F}_V$ may be readily verified. Again Lemma 3.2 implies that, for each homomorphism $\Phi : F_A \to S^{\mathscr{F}_V}$ and $f \in \mathscr{F}_V$, $\Phi(p)(f) = \Phi(q)(f)$; which means $S^{\mathscr{F}_V} \in V$. It is enough to show that for every other compactification (ψ, X) of S, with $X \in V$, $\psi^*(\mathscr{C}(X)) \subseteq \mathscr{F}_V(S)$. Let $\pi : S^{\mathscr{L}\mathscr{M}\mathscr{C}} \to X$ be the homomorphism for which $\pi \circ \varepsilon = \psi$. This implies that $\psi^*(\mathscr{C}(X)) \subseteq \mathscr{L}\mathscr{M}\mathscr{C}(S)$. Furthermore, if $h \in \mathscr{C}(X)$, then for each homomorphism $\Phi : F_A \to S^{\mathscr{L}\mathscr{M}\mathscr{C}}$ and each $\mu \in S^{\mathscr{L}\mathscr{M}\mathscr{C}}$,

$$(\Phi(p)\mu)(\psi^*(h)) = h((\pi \circ \Phi)(p)\pi(\mu)) = h((\pi \circ \Phi)(q)\pi(\mu)) = (\Phi(q)\mu)(\psi^*(h)). \quad (3.7)$$

Similar arguments show that

$$\begin{aligned}
\Phi(p)(\psi^*(h)) &= \Phi(q)(\psi^*(h)), \\
T_{\Phi(p)}\psi^*(h) &= T_{\Phi(q)}\psi^*(h), \\
T_{(\Phi(p)\mu)}\psi^*(h) &= T_{(\Phi(q)\mu)}\psi^*(h).
\end{aligned} \quad (3.8)$$

Now Lemma 3.2 implies that $\psi^*(h) \in \mathscr{F}_V(S)$, as claimed.
□

Let $\mathscr{F}_{V_c}$ consist of those $f \in \mathscr{C}(S)$ such that $f(\Phi(p)) = f(\Phi(q))$, $f(\Phi(p)s) = f(\Phi(q)s)$, $f(s\Phi(p)) = f(s\Phi(q))$, and $f(s\Phi(p)t) = f(s\Phi(q)t)$, for all $s, t \in S$, for every homomorphism $\Phi : F_A \to S$, and all identities $p = q$ of V. Trivially, $\mathscr{F}_V \subseteq \mathscr{F}_{V_c}$ (with the equality holding in the compact setting). Due to joint continuity of the multiplication of $S^{\mathscr{A}\mathscr{P}}$, we get the next simplification of $\mathscr{F}_V \cap \mathscr{A}\mathscr{P}$.

PROPOSITION 3.4. $\mathscr{F}_V \cap \mathscr{A}\mathscr{P} = \mathscr{F}_{V_c} \cap \mathscr{A}\mathscr{P}$.

PROOF. According to Lemma 3.2 it is enough to show that if $f \in \mathscr{F}_{V_c} \cap \mathscr{A}\mathscr{P}$, then for each homomorphism $\Phi : F_A \to S^{\mathscr{A}\mathscr{P}}$, and all $\mu \in S^{\mathscr{A}\mathscr{P}}$, $\Phi(p)(f) = \Phi(q)(f)$, $(\Phi(p)\mu)(f) = (\Phi(q)\mu)(f)$, $T_{\Phi(p)}f = T_{\Phi(q)}f$, and $T_{(\Phi(p)\mu)}f = T_{(\Phi(q)\mu)}f$. Imitating the methods of the proof of Lemma 3.2, let $\Phi_\alpha : F_A \to S$ be that net (of homomorphisms) for which $\lim_\alpha \varepsilon(\Phi_\alpha(a_i)) = \Phi(a_i)$ and $\lim_\alpha \varepsilon(\Phi_\alpha(b_j)) = \Phi(b_j)$; and also t_α be a net in S such that $\lim_\alpha \varepsilon(t_\alpha) = \mu$. Now, for all $s \in S$;

$$\begin{aligned}
(T_{\Phi(p)\mu}f)(s) &= \left(\left(\prod_{i=1}^m \Phi(a_i)\right)\mu\right)(L_s f) \\
&= \left(\left(\prod_{i=1}^m \lim_\alpha \varepsilon(\Phi_\alpha(a_i))\right)\left(\lim_\alpha \varepsilon(t_\alpha)\right)\right)(L_s f) \\
&= \lim_\alpha \varepsilon(\Phi_\alpha(p)t_\alpha)(L_s f) = \lim_\alpha f(s\Phi_\alpha(p)t_\alpha) \\
&= \lim_\alpha f(s\Phi_\alpha(q)t_\alpha) = (T_{\Phi(q)\mu}f)(s).
\end{aligned} \quad (3.9)$$

Similar arguments may apply to the other required equalities.
□

REMARKS. (a) The present results are a generalization of what we have described in [4], for the familiar varieties of abelian semigroups, AB, bands, BD, and semilattices, SL.

By Definition 3.1, $\mathscr{F}_{AB}$ consists of those $f \in \mathscr{LMC}$ such that the identities

$$
\begin{aligned}
&\lim_\alpha \left(\lim_\alpha R_{s_\alpha} f \right)(t_\alpha) = \lim_\alpha \left(\lim_\alpha R_{t_\alpha} f \right)(s_\alpha), \\
&\lim_\alpha \left(\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{u_\alpha} f \right) \right)(t_\alpha) = \lim_\alpha \left(\lim_\alpha R_{t_\alpha} \left(\lim_\alpha R_{u_\alpha} f \right) \right)(s_\alpha), \\
&\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{t_\alpha} f \right) = \lim_\alpha R_{t_\alpha} \left(\lim_\alpha R_{s_\alpha} f \right), \\
&\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{t_\alpha} \left(\lim_\alpha R_{u_\alpha} f \right) \right) = \lim_\alpha R_{t_\alpha} \left(\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{u_\alpha} f \right) \right)
\end{aligned}
\tag{3.10}
$$

hold for all nets s_α, t_α, and u_α in S, for which the limits exist. Since $S^{\mathscr{F}_{AB}}$ is semitopological, one realizes that the latter seemingly complicated limit process is preparatory, and can be condensed so that it presents $\mathscr{F}_{AB}$ in the simple form

$$
\{ f \in \mathscr{WAP} : f(st) = f(ts), \text{ and } f(stu) = f(sut) \, \forall s, t, u \in S \}.
\tag{3.11}
$$

In other words, $\mathscr{F}_{AB} = \mathscr{F}_{AB_c} \cap \mathscr{WAP}$.

Again, Definition 3.1 implies that $\mathscr{F}_{BD}$ is the set of all $f \in \mathscr{LMC}$, for which

$$
\begin{aligned}
&\lim_\alpha \left(\lim_\alpha R_{s_\alpha} f \right)(s_\alpha) = \lim_\alpha f(s_\alpha), \\
&\lim_\alpha \left(\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{t_\alpha} f \right) \right)(s_\alpha) = \lim_\alpha \left(\lim_\alpha R_{t_\alpha} f \right)(s_\alpha), \\
&\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{s_\alpha} f \right) = \lim_\alpha R_{s_\alpha} f, \\
&\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{t_\alpha} f \right) \right) = \lim_\alpha R_{s_\alpha} \left(\lim_\alpha R_{t_\alpha} f \right);
\end{aligned}
\tag{3.12}
$$

for all nets s_α and t_α in S, for which the limits exist. In contrast to the situation for $\mathscr{F}_{AB}$, in general $\mathscr{F}_{BD} \nsubseteq \mathscr{WAP}$ (e.g., a direct verification shows that the characteristic function of even numbers on $\mathbb{N}$, with its maximum multiplication, lies in $\mathscr{F}_{BD}$ but not in $\mathscr{WAP}$). Furthermore, as it is seen by easy examples (e.g., consider $[0, 1] \times \mathbb{N}$ under its rectangular multiplication) $\mathscr{F}_{BD} \neq \mathscr{F}_{BD_c}$, in general. It would be desirable to investigate the equality of $\mathscr{F}_{BD} = \mathscr{F}_{BD_c} \cap \mathscr{LMC}$, the left and right introversion of $\mathscr{F}_{BD}$, the relation between $\mathscr{F}_{BD}$ and the space similarly defined in terms of the left translations, and to characterize the topological center of $S^{\mathscr{F}_{BD}}$. We believe that there are close connections among these problems.

By the joint continuity theorem of Lawson [8], $\mathscr{F}_{SL} \subseteq \mathscr{AP}$ and so $\mathscr{F}_{SL} = \mathscr{F}_{SL_c} \cap \mathscr{AP}$.

It would be desirable to examine $\mathscr{F}_V$ for the variety of simple semigroups.

(b) It might be readily verified that the conditions (3.3) and (3.4) in the definition of $\mathscr{F}_V$, taken together, are equivalent to the fact that the enveloping semigroup $\sum(S, X_f \cup \{f\})$, of the natural flow $(S, X_f \cup \{f\})$, lies in V; and so the latter is satisfied whenever $f \in \mathscr{F}_V$. A natural question that arises is whether the converse is also true. As a helpful answer, one can verify that; a function $f \in \mathscr{GP}$ lies in $\mathscr{F}_V$ if and only if $\sum(S, X_f)$ lies in V; that is, $\mathscr{F}_V \cap \mathscr{GP} = \{ f \in \mathscr{GP} : \sum(S, X_f) \in V \}$.

Now, for a variety $\mathscr{V}$ of groups, defined by the set of laws Ω (see [9]), we define $\mathscr{F}_{\mathscr{V}}(S)$ as the set of all $f \in \mathscr{GP}(S)$ such that $(\prod_{i=1}^m (\lim_\alpha R_{s_{i_\alpha}})^{l_i})(\lim_\alpha R_{s_\alpha} f) = \lim_\alpha R_{s_\alpha} f$, for all laws $\prod_{i=1}^m x_i^{l_i}$ in Ω, and all nets s_{i_α} and s_α in S for which the required limits exists.

It is easy to verify that, a function $f \in \mathscr{GP}$ lies in $\mathscr{F}_{\mathscr{V}}$ if and only if $T_{\omega\mu}f = T_\mu f$ for all $\mu \in S^{\mathscr{GP}}$ and all values in $S^{\mathscr{GP}}$ of the laws $\omega \in \Omega$; which is also equivalent to the fact that $\sum(S, X_f) \in \mathscr{V}$. Using these, one may obtain the next result, which is the group version of Theorem 3.3.

THEOREM 3.5. *$\mathscr{F}_{\mathscr{V}}$ is m-admissible and $\mathscr{F}_{\mathscr{V}}$-compactification is universal with respect to the property satisfying $\mathscr{V}$.*

It should be mentioned that $\mathscr{F}_{\mathscr{V}}$ is not in $\mathscr{WAP}$ in general (e.g., for the variety of all groups we get $\mathscr{GP}$ which is not always in $\mathscr{WAP}$). Using the (joint continuity) theorem of Ellis [5], we have

$$\mathscr{F}_{\mathscr{V}} \cap \mathscr{WAP} = \mathscr{F}_{\mathscr{V}} \cap \mathscr{AP} = \mathscr{F}_{\mathscr{V}} \cap \mathscr{SAP}. \tag{3.13}$$

More precisely, each side of (3.13) (and hence $\mathscr{F}_{\mathscr{V}}$, when S is compact) consists of those $f \in \mathscr{SAP}$ such that, $(\prod_{i=1}^{m}(R_{s_i})^{l_i})(R_s f) = R_s f$, for all laws $\prod_{i=1}^{m} x_i{}^{l_i}$ in Ω, and all $s_1, s_2, \ldots, s_m$, and s in S. (For instance, for the variety of abelian groups, $\mathscr{F}_{\mathscr{V}}$ consists of those $f \in \mathscr{SAP}$ such that $f(stu) = f(sut)$ for all s, t and u in S, [3]; which, of course, is equal to $\mathscr{F}_{AB} \cap \mathscr{GP}$, see the previous remarks (b)). Hence, for every element of $\mathscr{V}$, each side of (3.13) is equal to $\mathscr{SAP}$, and so for a compact element S of $\mathscr{V}$, $\mathscr{F}_{\mathscr{V}}(S) = \mathscr{C}(S)$.

As we have shown in [3], for the variety of nilpotent groups, in the definition of $\mathscr{F}_{\mathscr{V}}$, $\lim_\alpha R_{s_\alpha} f$ can be replaced by f. However this seems not to be possible in general.

REFERENCES

[1]   J. F. Berglund, H. D. Junghenn, and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, John Wiley and Sons, New York, 1989. MR 91b:43001. Zbl 727.22001.

[2]   K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97. MR 24#A1632. Zbl 104.05501.

[3]   H. R. Ebrahimi-Vishki and M. A. Pourabdollah, *The universal nilpotent group compactification of a semigroup*, Proc. Amer. Math. Soc. **125** (1997), 2171–2174. MR 97i:43002.

[4]   _____, *The universal semilattice compactification of a semigroup*, Int. J. Math. Math. Sci. **22** (1999), no. 1, 85–89. MR 2000h:22004. Zbl 916.22004.

[5]   R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24** (1957), 119–125. MR 19,561b. Zbl 079.16602.

[6]   J. M. Howie, *An Introduction to Semigroup Theory*, no. 7, Academic Press, London, 1976, L.M.S. Monographs. MR 57#6235. Zbl 355.20056.

[7]   H. D. Junghenn, *Distal compactifications of semigroups*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 379–397. MR 84h:43017. Zbl 514.22003.

[8]   J. D. Lawson, *Joint continuity in semitopological semigroups*, Illinois J. Math. **18** (1974), 275–285. MR 49#454. Zbl 278.22002.

[9]   D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1982. MR 84k:20001. Zbl 483.20001.

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