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WHITEHEAD GROUPS OF EXCHANGE RINGS WITH PRIMITIVE FACTORS ARTINIAN

HUANYIN CHEN and FU-AN LI

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ABSTRACT. We show that if R is an exchange ring with primitive factors artinian then $K_1(R) \cong U(R)/V(R)$, where $U(R)$ is the group of units of R and $V(R)$ is the subgroup generated by $\{(1+ab)(1+ba)^{-1} \mid a,b \in R \text{ with } 1+ab \in U(R)\}$. As a corollary, $K_1(R)$ is the abelianized group of units of R if $1/2 \in R$.

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Very recently, Ara et al. [2] showed that the natural homomorphism $\mathrm{GL}_1(R) \to K_1(R)$ is surjective provided that R is a separative exchange ring. A natural problem is the description of the kernel of the epimorphism $\mathrm{GL}_1(R) \to K_1(R)$. In [9, Theorems 1.2 and 1.6], Menal and Moncasi showed that if R satisfies unit 1-stable range or is unit-regular, then $K_1(R) \cong U(R)/V(R)$. Here $U(R)$ is the group of units of R while $V(R)$ is a subgroup described later. In [7], Goodearl and Menal remarked that if for each $x,y \in R$ there exists a unit $u \in R$ such that $x-u$ and $y-u^{-1}$ are both units, then $K_1(R) \cong U(R)^{ab}$. In this paper, we investigate the above kernel for exchange rings with primitive factors artinian.

Recall that R is called an exchange ring if for every right R-module A and two decompositions $A = M \oplus N = \oplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\oplus_{i\in I} A_i')$. It is well known that regular rings, π-regular rings, semiperfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero [1] are all exchange rings.

Many authors have studied exchange rings with primitive factors artinian. Fisher and Snider proved that every regular ring with primitive factors artinian is unit-regular (see [6, Theorem 6.10]). Moreover, Menal [8, Thereom B] proved that every π-regular ring with primitive factors artinian has stable range one. Recently, Yu [13, Thereom 1] extended these results to exchange rings and showed that every exchange ring with primitive factors artinian has stable range one. On the other hand, Pardo [10] investigated the Grothendieck group K_0 of exchange rings. In this paper, we show that the Whitehead group $K_1(R) \cong U(R)/V(R)$ for an exchange ring R with primitive factors artinian. We refer the reader to [11] for the general theory of Whitehead groups.

Throughout this paper, all rings are associative with identity. The set $U(R)$ denotes the set of all units of R, $V(R)$ denotes the subgroup generated by $\{p(a,b)p(b,a)^{-1} \mid p(a,b) \in U(R),\ a,b \in R\}$, and $J(R)$ denotes the Jacobson radical of R (see below for the definition of $p(a,b)$ and other continuant polynomials). If G is a group and G' its commutator subgroup, then G^{ab} denotes G/G'. Let $\mathrm{GL}_n(R)$ be the group of units of $M_n(R)$, the ring of all $n\times n$ matrices over R.

We start with the following new element-wise property of exchange rings with primitive factors artinian.

Lemma 1. *Let R be an exchange ring with primitive factors artinian. Then, for any $x,y\in R$, there exists a unit-regular $w\in R$ such that $1+xy-xw\in U(R)$.*

Proof. Assume that there are some $x,y\in R$ such that $1+xy-xw\notin U(R)$ for any unit-regular $w\in R$. Let Ω be the set of all two-sided ideals A of R such that $1+xy-xw$ is not a unit modulo A for any unit-regular $w+A\in R/A$. Clearly, $\Omega\neq\emptyset$.

Given any ascending chain $A_1\subseteq A_2\subseteq\cdots\subseteq A_n\subseteq\cdots$ in Ω, set $M=\cup_{1\le i\le\infty}A_i$. Then M is a two-sided ideal of R. Assume that M is not in Ω. Then there exists a unit-regular $w+M\in R/M$ such that $(1+xy-xw)+M$ is a unit of R/M. Since $w+M$ is unit-regular in R/M, we have $e+M=(e+M)^2$, $u+M\in U(R/M)$ such that $w+M=(e+M)(u+M)$. As R/M is also exchange, we may assume that $e=e^2\in R$. Thus we can find positive integers n_i $(1\le i\le 5)$ such that $(1+xy-xw)s-1\in A_{n_1}$, $s(1+xy-xw)-1\in A_{n_2}$, $w-eu\in A_{n_3}$, $ut-1\in A_{n_4}$, and $tu-1\in A_{n_5}$ for some $s,t\in R$. Let $n=\max\{n_1,n_2,n_3,n_4,n_5\}$. Then $1+xy-xw+A_n$ is a unit of R/A_n for a unit-regular $w+A_n\in R/A_n$. This contradicts the choice of A_n. So $M\in\Omega$. That is, Ω is inductive. By using Zorn's lemma, we have a two-sided ideal Q of R such that it is maximal in Ω.

Let $S=R/Q$. If $J(R/Q)\neq 0$, then $J(R/Q)=K/Q$ for some $K\supset Q$. Clearly, $S/J(S)\cong R/K$. By the maximality of Q, we can find a unit-regular $(v+Q)+J(S)$ such that $((1+xy-xv)+Q)+J(S)$ is a unit of $S/J(S)$. Since idempotents and units of $S/J(S)$ can be lifted modulo $J(S)$, we may assume that $v+Q$ is unit-regular in S. On the other hand, $(1+xy-xv)+Q=(m+Q)+(r+Q)$ for some $m+Q\in U(S)$, $r+Q\in J(S)$. Thus, $(1+xy-xv)+Q$ is a unit of S. This gives a contradiction, whence $J(R/Q)=0$.

By the maximality of Q, one easily checks that R/Q is an indecomposable ring. According to [14, Lemma 3.7], R/Q is a simple artinian ring. Clearly, $(1+xy-xy)+Q=1+Q$ is a unit of R/Q. Since R/Q is unit-regular, $y+Q$ is a unit-regular element of R/Q. This contradicts the choice of Q. Therefore the proof is complete. □

We note that the following conditions for a ring R are equivalent.

(1) For any $x,y\in R$, there exists unit-regular $w\in R$ such that $1+x(y-w)\in U(R)$.

(2) Given $aR+bR=R$, then $a+bw\in U(R)$ for some unit-regular $w\in R$.

(3) Given $ax+b=1$, then $aw+b\in U(R)$ for some unit-regular $w\in R$ (cf. [5, Theorem 2.1]).

Clearly, the conditions above are stronger than the stable range one condition and can be viewed as a generalization of rings with many idempotents [4]. Call R a ring with many unit-regular elements if the equivalent conditions above hold. In [5, Theorem 3.1], the authors showed that such rings are all GE_2. Furthermore, Chen [3] proved that if R has many unit-regular elements then so does $M_n(R)$. Now we include the fact that every exchange ring with primitive factors artinian has many unit-regular elements to make our paper self-contained.

Recall that $p(a)=a$, $p(a,b)=1+ab$, and $p(a,b,c)=a+c+abc$ for any $a,b,c\in R$. It is easy to verify that $p(a,b,c)=p(a,b)c+p(a)$, $p(a,b,c)p(b,a)=p(a,b)p(c,b,a)$.

Lemma 2. *Let $a,b,c \in R$ with $p(a,b,c) \in U(R)$. If a is unit-regular, then $p(a,b,c) \equiv p(c,b,a) \pmod{V(R)}$.*

Proof. Since a is unit-regular, there is an idempotent e and a unit u such that $a = eu$. So we have $p(a,b,c) = eu + c + eubc$, and then $p(a,b,c)u^{-1} = e + cu^{-1} + eubcu^{-1}$. Obviously, $p(e,-ub(1-e)) = p(e,-ub(1-e))(p(-ub(1-e),e))^{-1} \in V(R)$. Thus we see that

$$\begin{aligned} p(a,b,c)u^{-1} &\equiv p(e,-ub(1-e))p(a,b,c)u^{-1} \pmod{V(R)} \\ &= (1-eub(1-e))(e+cu^{-1}+eubcu^{-1}) \\ &= e+cu^{-1}+eubecu^{-1} \pmod{V(R)}. \end{aligned} \tag{1}$$

On the other hand, we can verify that

$$\begin{aligned} p(c,b,a)u^{-1} &= e+cu^{-1}+cbe \equiv (e+cu^{-1}+cu^{-1}ube)p(1-e,-ube) \pmod{V(R)} \\ &= e+cu^{-1}+cu^{-1}eube = 1+(cu^{-1}-(1-e))(1+eube) \\ &\equiv 1+(1+eube)(cu^{-1}-(1-e)) \pmod{V(R)} \\ &\equiv p(a,b,c)u^{-1} \pmod{V(R)}. \end{aligned} \tag{2}$$

Consequently, $p(a,b,c) \equiv p(c,b,a) \pmod{V(R)}$, as asserted. □

In [4, Theorem 16], Chen showed that $K_1(R) \cong U(R)/V(R)$ provided that R has idempotent 1-stable range. Now we extend this fact to exchange rings with primitive factors artinian.

Theorem 3. *Let R be an exchange ring with primitive factors artinian. Then $K_1(R) \cong U(R)/V(R)$.*

Proof. Given $a,b,c \in R$ with $p(a,b,c) \in U(R)$, we have $p(c,b,a) \in U(R)$. From Lemma 1, we can find some unit-regular $w \in R$ such that $1+b(c-w) \in U(R)$. Let $c-w = t$. Then $c = t+w$ and $1+bt \in U(R)$. We check that

$$\begin{aligned} p(a,b,c) &= a+c+abc = (a+t+abt)+(1+ab)w \\ &= (a+t+abt)+(1+ab+tb+abtb)(1+tb)^{-1}w \\ &= (a+t+abt)+(1+tb)^{-1}w+(a+t+abt)b(1+tb)^{-1}w \\ &= (1+tb)^{-1}((1+tb)(a+t+abt)+w+(1+tb)(a+t+abt)b(1+tb)^{-1}w) \\ &= (1+tb)^{-1}p((1+tb)(a+t+abt),b(1+tb)^{-1},w). \end{aligned} \tag{3}$$

In view of Lemma 2, we know that

$$\begin{aligned} p(a,b,c) &\equiv (1+tb)^{-1}p(w,b(1+tb)^{-1},(1+tb)(a+t+abt)) \pmod{V(R)} \\ &= (1+tb)^{-1}(p(w,b(1+tb)^{-1})(1+tb)(a+t+abt)+p(w)) \\ &= (1+tb)^{-1}(p(w,b(1+tb)^{-1})p(t,b)p(a,b,t)+p(w)) \\ &= (1+tb)^{-1}(p(w,b(1+tb)^{-1})p(t,b,a)p(b,t)+p(w)). \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} p(c,b,a) &= p\big(w,(1+bt)^{-1}b,(t+a+tba)(1+bt)\big)(1+bt)^{-1} \\ &= \big(p\big(w,(1+bt)^{-1}b\big)(t+a+tba)(1+bt)+p(w)\big)(1+bt)^{-1} \\ &= \big(p\big(w,(1+bt)^{-1}b\big)p(t,b,a)p(b,t)+p(w)\big)(1+bt)^{-1}. \end{aligned} \tag{5}$$

Since $b(1+tb)^{-1} = (1+bt)^{-1}b$, $(1+tb)p(a,b,c) \equiv p(c,b,a)(1+bt) \pmod{V(R)}$. Hence $p(a,b,c)(p(c,b,a))^{-1} \in V(R)$ because $U(R)/V(R)$ is abelian. By virtue of [13, Theorem 1], R has stable range one. It follows from [9, Theorem 1.2] that $K_1(R) \cong U(R)/W(R) \cong U(R)/V(R)$, where the notation $W(R)$ denotes the subgroup of $U(R)$ generated by $\{p(a,b,c)p(c,b,a)^{-1} \mid p(a,b,c) \in U(R),\ a,b,c \in R\}$. So the proof is complete. □

Recall that an exchange ring is said to be of bounded index if there exists some positive integer n such that $x^n = 0$ for all nilpotent $x \in R$. We can derive the following corollary.

Corollary 4. *Let R be an exchange ring of bounded index. Then $K_1(R) \cong \mathrm{GL}_3(R)^{ab} \cong U(R)/V(R)$.*

Proof. It is easy to obtain the first isomorphism by an argument of Menal. In view of [13, Theorem 3], R is an exchange ring with primitive factors artinian. Therefore we complete the proof by Theorem 3. □

Write $L_1(R)$ for the subgroup generated by the elements in $U(R)' \cup L$, where L is the subgroup generated by all $1+er(1-e)$ with $e=e^2$, $r \in R$. Clearly, $U(R)' \subseteq L_1(R) \subseteq V(R)$.

Lemma 5. *Let $a,b \in R$ with $1+ab \in U(R)$. If a is unit-regular, then $p(a,b) \equiv p(b,a) \pmod{L_1(R)}$.*

Proof. Since a is unit-regular, there is an idempotent e and a unit u such that $a = ue$. So we have

$$\begin{aligned} p(a,b) &= 1+ueb = u(1+ebu)u^{-1} \\ &= u(1+ebu)u^{-1}(1+ebu)^{-1}(1+ebu) \\ &\equiv (1+ebu) \pmod{L_1(R)} \\ &\equiv (1-ebu(1-e))(1+ebu) \pmod{L_1(R)} \\ &= 1+ebue = (1+bue)(1-(1-e)bue) \\ &\equiv (1+bue) \pmod{L_1(R)} = p(b,a), \quad \text{as required.} \end{aligned} \tag{6}$$

□

Lemma 6. *Let R be an exchange ring with primitive factors artinian. If R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, then, for any $x,y \in R$, there exists a $w \in U(R)$ such that $1+x(y-w) \in U(R)$ and $y-w$ is unit-regular.*

Proof. Assume that there exist some $x,y \in R$ such that $1+x(y-w) \notin U(R)$ or $y-w$ is not unit-regular for any unit $w \in R$. Let Ω be the set of all two-sided ideals A of R such that $1+x(y-w)$ is not a unit or $y-w$ is not unit-regular modulo A for any unit $w+A \in R/A$. Obviously, $\Omega \neq \varnothing$.

Analogously to the discussion in Lemma 1, we can find a two-sided ideal Q of R such that it is maximal in Ω. Set $S = R/Q$. If $J(S) \neq 0$, then $J(S) = K/Q$ for some $K \supset Q$. Obviously, $S/J(S) \cong R/K$. By the maximality of Q, we have a unit $(v+Q)+J(S)$ such that $((1+x(y-v))+Q)+J(S)$ is a unit of $S/J(S)$ and $((y-v)+Q)+J(S)$ is unit-regular in $S/J(S)$. Since units of $S/J(S)$ can be lifted modulo $J(S)$ because S is an exchange ring, we may assume that $v+Q$ is unit of S. Similarly to Lemma 1, we see that $(1+x(y-v))+Q = (m+Q)+(r+Q)$ for some $m+Q \in U(S)$, $r+Q \in J(S)$. Thus, $(1+x(y-v))+Q$ is a unit of S. On the other hand, idempotents of $S/J(S)$ can be lifted modulo $J(S)$ because S is an exchange ring. So we may assume that $((y-v)+Q)+J(S) = ((f+Q)+J(S))((u+Q)+J(S))$ with $f+Q = (f+Q)^2 \in R/Q$, $u+Q \in U(S)$. Thus we can find some $t \in R$ with $t+Q \in J(S)$ such that $(y-(v-t))+Q = (f+Q)(u+Q)$ is unit-regular in S. From $(1+x(y-(v-t)))+Q = ((1+x(y-v))+Q)+(x+Q)(t+Q)$ with $(1+x(y-v))+Q \in U(S)$ and $t+Q \in J(S)$, one easily checks that $(1+x(y-(v-t)))+Q$ is a unit of S. This contradicts the choice of Q. Therefore $J(S) = 0$. The maximality of Q implies that S is indecomposable as a ring. By virtue of [14, Lemma 3.7], S is a simple artinian ring. Assume that $R/Q \cong M_n(D)$, where n is a positive integer and D is a division ring. Since R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, we claim that $n \geq 2$, or $n = 1$, and $D \not\cong \mathbb{Z}/2\mathbb{Z}$. Thus R/Q satisfies unit 1-stable range. From $((1+xy)+Q)S + ((-x)+Q)S = S$, we have a unit $s+Q$ of R/Q such that $(1+x(y-s))+Q = ((1+xy)+Q)+((-x)+Q)(s+Q)$ is a unit of S. Since S is simple artinian, $(y-s)+Q$ is unit-regular in S, a contradiction. Therefore the proof is complete. □

Theorem 7. *Let R be an exchange ring with primitive factors artinian. If R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image, then $K_1(R) \cong U(R)/L_1(R)$.*

Proof. Since R is an exchange ring with primitive factors artinian, by virtue of Theorem 3, we know that $K_1(R) \cong U(R)/V(R)$. Let $b, c \in R$ with $p(b,c) \in U(R)$. In view of Lemma 6, there exists some $w \in U(R)$ such that $1+b(c-w) \in U(R)$ and $c-w = s$ unit-regular. Observe that

$$\begin{aligned} p(b,c) &= 1+bc = 1+bs+bw = 1+bs+b(1+sb)(1+sb)^{-1}w \\ &= (1+bs)\big(1+b(1+sb)^{-1}w\big) = p(b,s)p\big(b(1+sb)^{-1},w\big). \end{aligned} \tag{7}$$

Likewise, we see that

$$p(c,b) = p\big(w,b(1+sb)^{-1}\big)(1+sb) = p\big(w,b(1+sb)^{-1}\big)p(s,b). \tag{8}$$

By Lemma 5 and the fact that units commute mod $L_1(R)$, we see that

$$\begin{aligned} p(b,c) &= p(b,s)p\big(b(1+sb)^{-1},w\big) \equiv p\big(b(1+sb)^{-1},w\big)p(b,s) \\ &\equiv p\big(w,b(1+sb)^{-1}\big)p(s,b) = p(c,b) \pmod{L_1(R)}. \end{aligned} \tag{9}$$

Hence, $L_1(R) = V(R)$ and we conclude that $K_1(R) \cong U(R)/L_1(R)$, as asserted. □

Corollary 8. *Let R be an exchange ring with primitive factors artinian. If $2 \in U(R)$, then $K_1(R) \cong U(R)^{ab}$.*

Proof. Let $e = e^2 \in R$. Since $2 \in U(R)$, we have $e = 1/2 + (2e-1)/2$. Obviously, $((2e-1)/2)(4e-2) = 1$. By [9, Lemma 1.5], $1+eR(1-e) \subseteq U(R)'$, and then $L_1(R) = U(R)'$.

On the other hand, $2 \in U(R)$ implies that R does not have $\mathbb{Z}/2\mathbb{Z}$ as a homomorphic image. Using Theorem 7, we conclude that $K_1(R) \cong U(R)/L_1(R) \cong U(R)/U(R)' \cong U(R)^{ab}$, as desired. □

Corollary 9. *Let R be an exchange ring of bounded index of nilpotence. If $2 \in U(R)$, then $K_1(R) \cong U(R)^{ab}$.*

Proof. By [12, Proposition 2.1], we see that the primitive factors of R are artinian. Thus the result follows from Corollary 8. □

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Huanyin Chen: Department of Mathematics, Hunan Normal University, Changsha 410006, China

E-mail address: chyzxl@sparc2.hunnu.edu.cn

Fu-an Li: Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China

E-mail address: fal@math08.math.ac.cn