THE METHOD OF AVERAGING AND FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. We present a natural extension of the method of averaging to fast oscillating functional differential equations with delay. Unlike the usual approach where the analysis is kept in an infinite-dimensional Banach space, our analysis is achieved in $\mathbb{R}^n$. Our results are formulated in classical mathematics. They are proved within Internal Set Theory which is an axiomatic description of nonstandard analysis.

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1. Introduction. An important tool in the rigorous study of differential equations with a small parameter is the method of averaging, which is well known for ordinary differential equations [1, 9, 13, 14] and for functional differential equations with small delay [6, 7, 17]. In both cases, the corresponding averaged equations are ordinary differential equations. However, for fast oscillating functional differential equations with large bounded delay, the method of averaging is not nearly so developed as in the two previous cases. Among recent works devoted to this last case, we will cite the paper of Hale and Verduyn Lunel [8]. Without going into details, we will emphasize, in this work, that the authors introduce an extension of the method of averaging to abstract evolutionary equations in Banach spaces. In particular, they rewrite a functional differential equation with delay as an ordinary differential equation in an infinite dimensional Banach space and proceed formally from there.

In this paper, we develop an improved theory of averaging for functional differential equations with delay under smoothness hypotheses that are less restrictive than those of [8]. Also all our analysis is kept in $\mathbb{R}^n$. This is performed in Section 2. There, we state closeness of solutions of the averaged and original equations on finite time intervals (Theorem 2.1). We also investigate the long time behaviour of the solutions of the original equation (Theorem 2.5). The proofs of Theorems 2.1 and 2.5 are established within an axiomatic description of Robinson's *Nonstandard Analysis* (NSA) [12], namely *Internal Set Theory* (IST), proposed by Nelson [11]. Section 3 starts with a short tutorial on IST. Then we present the nonstandard translates (Theorems 3.5 and 3.7) in the language of IST of Theorems 2.1 and 2.5. We end this section with an external characterization of the uniform asymptotic stability which is the main assumption for the validity of the result of Theorem 3.7. Finally, in Section 4 we give the proofs of Theorems 3.5 and 3.7. As ordinary differential equations and functional differential equations with small delay are special cases of functional differential equations with delay, the proofs developed in this section provide alternative

proofs to the techniques of averaging on these equations found, for example, in [13, 14].

2. The method of averaging. In this section, we present the main results on averaging for functional differential equations with delay.

Suppose $f : \mathbb{R} \times U \to \mathbb{R}^n$ is a continuous function, where U is an open subset of $\mathbb{R}^n$. Let $\phi : [-r, 0] \to U$ be a continuous function, where $r > 0$ is a constant. Let $\varepsilon > 0$ be a real parameter. Along with the functional differential equation with delay

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t-r)\right), \quad \text{for } t > 0, \qquad x(t) = \phi(t), \quad \text{for } t \in [-r, 0], \qquad (2.1)$$

we consider the averaged equation

$$\dot{y}(t) = f^o(y(t-r)), \quad \text{for } t > 0, \qquad y(t) = \phi(t), \quad \text{for } t \in [-r, 0], \qquad (2.2)$$

where

$$f^o(x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t, x) \, dt. \qquad (2.3)$$

As a first result, we give a comparison of the solutions of the averaged and the original equations on finite time intervals.

THEOREM 2.1. *Assume that*
(H.0) *The function f is bounded on $\mathbb{R} \times U$.*
(H.1) *The continuity of f in $x \in U$ is uniform with respect to $t \in \mathbb{R}_+$.*
(H.2) *For all $x \in U$ there exists a limit*

$$f^o(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t, x) \, dt. \qquad (2.4)$$

(H.3) *Equation (2.2) has a unique solution.*

Let y be the solution of (2.2) and let $J = [-r, \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. For any $L > 0$, $L \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ any solution x of (2.1) is defined at least on $[0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, L]$.

REMARK 2.2. Assume that the initial time $t_0 \neq 0$. Let $y(\cdot; t_0)$ be the solution of (2.2) and let $J = [t_0 - r, t_0 + \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. The conclusions of Theorem 2.1 become for any $L > 0$, $L + t_0 \in J$, and any $\delta > 0$ there exists an $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ any solution $x(\cdot; t_0)$ of (2.1) is defined at least on $[t_0, t_0 + L]$ and satisfies $\|x(t; t_0) - y(t; t_0)\| < \delta$ for all $t \in [t_0, t_0 + L]$.

One can also precise the long time behaviour of a solution of (2.1) provided that more is known about the solution of (2.2). To give estimate for all time, we assume that the solution of (2.2) tends toward an equilibrium. Before this, we first recall the concept of *uniform asymptotic stability* of equilibrium points of autonomous functional differential equations with delay.

Consider the autonomous functional differential equation with delay

$$\dot{y}(t) = f^o(y(t-r)), \qquad (2.5)$$

where $f^o : U \to \mathbb{R}^n$ is a continuous function, and U is an open subset of $\mathbb{R}^n$, $r > 0$ is a constant. For $t_0 \in \mathbb{R}$ and $\phi : [-r, 0] \to \mathbb{R}^n$ a continuous function, let $y(\cdot; t_0, \phi)$ denotes the solution of (2.5) for the initial function $y(t; t_0, \phi) = \phi(t)$ for $t \in [-r, 0]$.

Since (2.5) is autonomous, the concepts of *asymptotic stability* and *uniform asymptotic stability* of equilibrium points of (2.5) coincide. Then, it is sufficient to deal directly with *uniform notions*.

DEFINITION 2.3. The equilibrium point y_e of (2.5) is said to be

(1) Uniformly stable (in the sense of Liapunov) if for any $\mu > 0$, there exists $\eta = \eta(\mu) > 0$ with the property that for all $t_0 \in \mathbb{R}$ any solution $y(\cdot; t_0, \phi)$ of (2.5) for which $\|\phi(t) - y_e\| < \eta$ for $t \in [-r, 0]$, can be continued for all $t > t_0$ and satisfies $\|y(t; t_0, \phi) - y_e\| < \mu$.

(2) Uniformly attractive if there exists $b_0 > 0$ with the respective properties:

(a) For all $t_0 \in \mathbb{R}$ any solution $y(\cdot; t_0, \phi)$ of (2.5) for which $\|\phi(t) - y_e\| < b_0$ for $t \in [-r, 0]$, can be continued for all $t > t_0$.

(b) For every $\delta > 0$, there exists $T = T(\delta) > 0$ (T depends on δ but not on t_0) such that $\|y(t; t_0, \phi) - y_e\| < \delta$ for $t > t_0 + T(\delta)$, that is, $\lim_{t \to \infty} y(t; t_0, \phi) = y_e$ uniformly in $t_0 \in \mathbb{R}$.

(3) Uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

REMARK 2.4. The ball $\mathcal{B}$ of center y_e and radius b_0 where the attraction is uniform will be called the basin of attraction of y_e.

We now return to the averaged equation (2.2). We assume that y_e is an equilibrium point of (2.2), that is, $f^o(y_e) = 0$. As a next result of this section, we prove the validity of the approximation of a solution x of (2.1) by the solution y of (2.2) for all (future) time, under the additional conditions about the equilibrium point y_e and the initial function ϕ.

THEOREM 2.5. *Let the hypotheses (H.0)–(H.3) of Theorem 2.1 be true, and assume that*

(H.4) *The point y_e is uniformly asymptotically stable.*

(H.5) *The initial function ϕ in (2.2) lies in the basin of attraction of y_e.*

Let y be the solution of (2.2). Then for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$ any solution x of (2.1) is defined for all $t \geq 0$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \geq 0$.

3. Nonstandard results

3.1. A short tutorial on Internal Set Theory. *Internal Set Theory* (IST) is an axiomatic description of nonstandard analysis proposed by Nelson [11]. We complete the ordinary mathematical language (say ZFC) with a new undefined monadic predicate symbol st (read standard). We call *internal*, the formulas of IST without any occurrence of the predicate st in them; otherwise, we call them *external*. Thus internal formulas are the formulas of ZFC. The axioms of IST are all axioms of ZFC, restricted to internal formulas (in other words, IST is an extension of ZFC), plus three others which govern the use of the new predicate. Thus *all theorems* of ZFC *remain valid* in IST. IST is

a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. There is an algorithm (a well-known *reduction algorithm*) to reduce any external formula $F(x_1,\ldots,x_n)$ of IST without other free variables than $x_1,\ldots,x_n$, to an internal formula $F'(x_1,\ldots,x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \Leftrightarrow F'$ for all standard values of the free variables. In other words, any result which may be formalized within IST by a formula $F(x_1,\ldots,x_n)$ is equivalent to the classical property $F'(x_1,\ldots,x_n)$, provided the parameters $x_1,\ldots,x_n$ are restricted to standard values. We give the reduction of the frequently occurring formula $\forall x \ (\forall^{st} y A \Rightarrow \forall^{st} z \ B)$ where A and B are internal formulas

$$\forall x (\forall^{st} y A \Longrightarrow \forall^{st} z B) \equiv \forall z \ \exists^{fin} y' \ \forall x (\forall y \in y' A \Longrightarrow B). \tag{3.1}$$

A real number x is called *infinitesimal*, denoted by $x \simeq 0$, if its absolute value $|x|$ is smaller than any standard strictly positive real number, *limited* if its absolute value $|x|$ is smaller than some standard real number, *unlimited*, denoted by $x \simeq \pm\infty$, if it is not limited, and *appreciable* if it is neither unlimited nor infinitesimal. Two real numbers x and y are *infinitely close*, denoted by $x \simeq y$, if their difference $x - y$ is infinitesimal.

For x and y in a standard metric space E, the notation $x \simeq y$ means that the distance from x to y is infinitesimal. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *nearstandard* in E and the standard point x_0 is called the *standard part* of x (it is unique) and is also denoted by ${}^o x$. The *halo* of x, denoted by hal(x), is the set, usually external, of all y such that $x \simeq y$. For any subset X of E, a point $x \in E$ is called nearstandard in X if there exists a standard point $x_0 \in X$ such that $x \simeq x_0$. We recall that, if X is standard, X is open if and only if it contains the halo of all its standard elements. The *shadow* of a subset X of E, denoted by ${}^o X$, is the unique standard set whose standard elements are precisely those whose halo intersects X.

Let E and F be standard metric spaces, and g be an internal function defined on $\mathcal{D}(g) \subset E$ and with values in F. The function g is called *S-continuous* at $x_0 \in \mathcal{D}(g)$ if for all $x \simeq x_0$, $g(x) \simeq g(x_0)$ holds, *S*-continuous in $E \times F$ if it is *S*-continuous at each point $x \in \mathcal{D}(g)$ such that $(x, g(x))$ is nearstandard in $E \times F$. For g standard, the continuity and the *S*-continuity in $\mathcal{D}(g) \times \mathcal{B}(g)$, where $\mathcal{B}(g)$ is a target of g, coincide. The shadow in $E \times F$ of the graph of an *S*-continuous function g is the graph of a standard continuous function g_0, called the shadow of g, and denoted by ${}^o g$.

In ZFC in principle all sets are defined using the only nonlogical symbol $\in$. In IST there is also the possibility to define collections with the nonlogical symbol st. Those collections which fall outside the range of ZFC are called *external sets*. External sets are often easily recognized: mostly some elementary classical property fails to hold. For instance, the set of infinitesimal real numbers hal(0) must be external, for it constitutes a bounded subset of $\mathbb{R}$ without lower upper bound. It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is noticed that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a large domain. In nonstandard analysis, statements which affirm that the validity of a property exceeds the domain where it was established in a direct way are called *permanence principles*. Many permanence results used in nonstandard analysis are based upon

the self-evident statement *"no external set is internal."* This statement is called the *Cauchy principle.* It has the following frequently used application.

LEMMA 3.1 (Robinson's lemma)**.** *If r is an internal real function such that $r(t) \simeq 0$ for all limited $t \geq 0$, then there exists $v \simeq +\infty$ such that $r(t) \simeq 0$ for all $t \in [0,v]$.*

PROOF.   Indeed, $\{l \in \mathbb{R} \mid l \geq 1, l \text{ limited}\} \subset \{l \in \mathbb{R} \mid \forall t \in [0,l] | r(t)| < 1/l\}$. The first set is external and the second set is internal. By the Cauchy principle the inclusion is strict.   □

Together with one among its corollaries, we conclude this section with another application of Cauchy's principle which will be used later.

LEMMA 3.2. *Let I be an internal set and $h : I \to \mathbb{R}$ be an internal function such that $h(t) \simeq 0$ for all $t \in I$. Then $\sup_{t \in I} \{h(t)\} \simeq 0$.*

PROOF.   We have $\{l \in \mathbb{R}_+ \mid l \notin \mathrm{hal}(0)\} \subset \{l \in \mathbb{R} \mid \forall t \in I \ |h(t)| < l\}$. The first set is external, otherwise $\mathrm{hal}(0)$ would be internal, and the second set is internal. By the Cauchy principle the inclusion is strict.   □

LEMMA 3.3 (corollary of Lemma 3.2)**.** *Let $a < b$, $b - a$ limited, and let $g, \tilde{g} : [a,b] \to \mathbb{R}^n$ be internal integrable functions such that $g(t) \simeq \tilde{g}(t)$ for all $t \in [a,b]$. Then*

$$\int_a^b g(t)\,dt \simeq \int_a^b \tilde{g}(t)\,dt. \tag{3.2}$$

REMARK 3.4.   The interested reader is referred to [2, 3, 4, 5, 10, 11, 12, 15, 16] for more informations on nonstandard analysis and its applications.

3.2. The averaging results.   First we give nonstandard formulations of Theorem 2.1, Remark 2.2, and Theorem 2.5. Then, by use of the reduction algorithm, we show that the reduction of Theorem 3.5, Remark 3.6, and Theorem 3.7 are Theorem 2.1, Remark 2.2 and Theorem 2.5, respectively.

THEOREM 3.5. *Let U be a standard open subset of $\mathbb{R}^n$. Let $f : \mathbb{R} \times U \to \mathbb{R}^n$ and $\phi : [-r,0] \to U$ be standard continuous functions. Assume that hypotheses (H.0)–(H.3) in Theorem 2.1 hold. Let y be the solution of (2.2) and let $J = [-r,\omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $L > 0$, $L \in J$, any solution x of (2.1) is defined at least on $[0,L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0,L]$.*

REMARK 3.6.   Assume that the initial time $t_0 \neq 0$. Let $y(\cdot;t_0)$ be the solution of (2.2) and let $J = [t_0 - r, t_0 + \omega)$, $0 < \omega \leq +\infty$, be its maximal interval of definition. The conclusions of Theorem 3.5 become: let $\varepsilon > 0$ be infinitesimal, then for any standard $L > 0$, $L + t_0 \in J$, any solution x of (2.1) is defined at least on $[t_0, t_0 + L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [t_0, t_0 + L]$.

THEOREM 3.7. *Let U be a standard open subset of $\mathbb{R}^n$. Let $f : \mathbb{R} \times U \to \mathbb{R}^n$ and $\phi : [-r,0] \to U$ be standard continuous functions. Let y_e be a standard equilibrium point of (2.2). Assume that hypotheses (H.0)–(H.5) in Theorem 2.5 hold. Let y be the*

solution of (2.2). Let $\varepsilon > 0$ be infinitesimal. Then any solution x of (2.1) is defined for all $t \geq 0$ and satisfies $x(t) \simeq y(t)$ for all $t \geq 0$.

The proofs of Theorems 3.5 and 3.7 are postponed to Section 4. Theorem 3.5, Remark 3.6, and Theorem 3.7 are external statements. We show that the reduction of Theorem 3.5 (resp., Remark 3.6 and Theorem 3.7) is Theorem 2.1 (resp., Remark 2.2 and Theorem 2.5).

REDUCTION OF THEOREM 3.5. Let B be the formula "if $\delta > 0$ then any solution x of (2.1) is defined at least on $[0, L]$ and satisfies $\|x(t) - y(t)\| < \delta$ for all $t \in [0, L]$". To say that "any solution x of (2.1) is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$" is the same as saying $\forall^{\mathrm{st}} \delta \, B$. Then Theorem 3.5 asserts that

$$\forall \varepsilon (\forall^{\mathrm{st}} \eta \, \varepsilon < \eta \implies \forall^{\mathrm{st}} \delta B). \tag{3.3}$$

In this formula L is standard and ε, η, and δ range over the strictly positive real numbers. By (3.1), formula (3.3) is equivalent to

$$\forall \delta \, \exists^{\mathrm{fin}} \eta' \, \forall \varepsilon (\forall \eta \in \eta' \varepsilon < \eta \Rightarrow B). \tag{3.4}$$

For η' a finite set, for all $\eta \in \eta' \varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min \eta'$, and so formula (3.4) is equivalent to

$$\forall \delta \, \exists \varepsilon_0 \, \forall \varepsilon (\varepsilon < \varepsilon_0 \Rightarrow B). \tag{3.5}$$

This shows that for any standard $L > 0$, $L \in J$, the statement of Theorem 2.1 holds, thus by transfer, it holds for any $L > 0$, $L \in J$.   □

The reduction of Remark 3.6 (resp., Theorem 3.7) to Remark 2.2 (resp., Theorem 2.5) follows almost verbatim the reduction of Theorem 3.5 to Theorem 2.1 and is left to the reader.

3.3. Uniform asymptotic stability. As the condition (H.4) will be used in its external form, we give the external characterizations of the notion of uniform stability and uniform attractiveness of the equilibrium point y_e of (2.5), given in Definition 2.3.

By transfer, we may assume that f^o, r, and y_e are standard.

LEMMA 3.8. *The equilibrium point y_e of (2.5) is*

(1) Uniformly stable if and only if for all $t_0 \in \mathbb{R}$ any solution $y(\cdot; t_0, \phi)$ of (2.5) for which $\phi(t) \simeq y_e$ for $t \in [-r, 0]$, can be continued for all $t > t_0$ and satisfies $y(t; t_0, \phi) \simeq y_e$.

(2) Uniformly attractive if and only if it admits a standard basin of attraction, that is, there exists a standard $b_0 > 0$ with the property that for all $t_0 \in \mathbb{R}$ any solution $y(\cdot; t_0, \phi)$ of (2.5) for which $\|\phi(t) - y_e\| < b_0$ for $t \in [-r, 0]$, ϕ standard, can be continued for all $t > t_0$ and satisfies $y(t; t_0, \phi) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$.

PROOF. (1) Let A be the formula "$\|\phi(t) - y_e\| < \eta$ for $t \in [-r, 0]$" and let C be the formula "Any solution $y(\cdot; t_0, \phi)$ of (2.5) can be continued for all $t > t_0$ and satisfies the inequality $\|y(t; t_0, \phi) - y_e\| < \mu$." The characterization of uniform stability in the lemma is

$$\forall t_0 \forall \phi (\forall^{\mathrm{st}} \eta A \implies \forall^{\mathrm{st}} \mu B). \tag{3.6}$$

In this formula $y(\cdot;\cdot,\cdot)$, r, and y_e are standard parameters and η, μ range over the strictly positive real numbers. By (3.1), formula (3.6) is equivalent to

$$\forall \mu \; \exists^{\text{fin}} \eta' \; \forall t_0 \forall \phi \, (\forall \eta \in \eta' A \Longrightarrow B). \tag{3.7}$$

For η' a finite set, for all $\eta \in \eta'$ A is the same as A for $\eta = \min \eta'$, and so formula (3.7) is equivalent to

$$\forall \mu \; \exists \eta \; \forall t_0 \forall \phi (A \Longrightarrow B). \tag{3.8}$$

This is the usual definition of uniform stability.

(2) By transfer, the uniform attractiveness of y_e is equivalent to the existence of a standard basin of attraction, that is, b_0 in Remark 2.4 is standard. The characterization of standard basin of attraction in the lemma is that for all standard continuous function $\phi : [-r, 0] \to U$ such that $\|\phi(t) - y_e\| < b_0$ for $t \in [-r, 0]$, we have the property that for all $t_0 \in \mathbb{R}$ any solution $y(\cdot;t_0, \phi)$ of (2.5) can be continued for all $t > t_0$ and satisfies

$$\forall t_0 \forall t \, (\forall^{\text{st}} T \; t - t_0 > T \Longrightarrow \forall^{\text{st}} \delta \|y(t;t_0, \phi) - y_e\| < \delta). \tag{3.9}$$

In this formula $y(\cdot;\cdot, \phi)$ and y_e are standard parameters and T, δ range over the strictly positive real numbers. By (3.1), formula (3.9) is equivalent to

$$\forall \delta \; \exists^{\text{fin}} T' \; \forall t_0 \forall t \, (\forall T \in T' \; t - t_0 > T \Longrightarrow \; \|y(t;t_0, \phi) - y_e\| < \delta). \tag{3.10}$$

For T' a finite set for all $T \in T'$ $t - t_0 > T$ is the same as $t - t_0 > T$ for $T = \max T'$, and so formula (3.10) is equivalent to

$$\forall \delta \; \exists T \; \forall t_0 \forall t \; (t - t_0 > T \Longrightarrow \|y(t;t_0, \phi) - y_e\| < \delta). \tag{3.11}$$

We have shown that for all standard continuous function ϕ in the basin of attraction of y_e (and consequently, by transfer, for all continuous function ϕ in the basin of attraction of y_e), for all $t_0 \in \mathbb{R}$ any solution $y(\cdot;t_0, \phi)$ of (2.5) can be continued for all $t > t_0$ and satisfies $\lim_{t \to \infty} y(t;t_0, \phi) = y_e$, the limit being uniform in t_0. $\qquad \square$

Assume that (2.5) has the uniqueness of the solutions with prescribed initial functions. Let $\phi : [-r, 0] \to U$ be continuous. For $t_0 \in \mathbb{R}$, let $y(\cdot;t_0, \phi)$ be the solution of (2.5) for initial function $y(t;t_0, \phi) = \phi(t)$ for $t \in [-r, 0]$. This solution is defined on the interval $I(t_0, \phi) = [t_0 - r, t_0 + \beta)$. It is well known that the function y is continuous with respect to the initial function ϕ. The external formulation of this result is as follows.

LEMMA 3.9. *Let ϕ and $\phi_0 : [-r, 0] \to U$ be continuous, with ϕ_0 standard. If $\phi(t) \simeq \phi_0(t)$ on $[-r, 0]$, then for all $t \in I(t_0, \phi_0)$, $t > t_0$, such that $t - t_0$ is standard, we have $t \in I(t_0, \phi)$ and $y(t;t_0, \phi) \simeq y(t;t_0, \phi_0)$.*

PROOF. The reduction of Lemma 3.9 is the usual continuity of the solutions with respect to the initial functions. $\qquad \square$

LEMMA 3.10. *Assume that (2.5) has the uniqueness of the solutions with the prescribed initial functions. The equilibrium point y_e of (2.5) is uniformly asymptotically*

stable if and only if there exists a standard $a > 0$ with the property that for all $t_0 \in \mathbb{R}$ any solution $y(\cdot; t_0, \phi)$ of (2.5) for which $\|\phi(t) - y_e\| < a$ for $t \in [-r, 0]$, can be continued for all $t > t_0$ and satisfies $y(t; t_0, \phi) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$.

PROOF. Assume that y_e is uniformly asymptotically stable. Then it is uniformly attractive, and so it admits a ball $\mathcal{B}$ of center y_e and radius $b_0 > 0$, b_0 standard, as a standard basin of attraction. Let $a > 0$ be standard such that the closure of the ball $\mathcal{B}'$ of center y_e and radius a is included in $\mathcal{B}$. Let ϕ and $\phi_0 : [-r, 0] \to U$ be continuous, ϕ_0 standard, with $\phi(t) \in \mathcal{B}'$ and $\phi_0(t) \in \mathcal{B}$ for all $t \in [-r, 0]$. Assume that $\phi(t) \simeq \phi_0(t)$ for all $t \in [-r, 0]$. For $t_0 \in \mathbb{R}$, let $y = y(\cdot; t_0, \phi)$ and $y_0 = y_0(\cdot; t_0, \phi_0)$ be the solutions of (2.5) with the initial functions ϕ and ϕ_0, respectively. By the uniform attractiveness of y_e, the solution y_0 is defined for all $t > t_0$ and satisfies $y_0(t) \simeq y_e$ for all $t - t_0 \simeq +\infty$. By Lemma 3.9, $y(t) \simeq y_0(t)$ on $[t_0, t_0 + L]$ for all limited $L > 0$. By Robinson's lemma, there exists $\nu \simeq +\infty$ such that $y(t) \simeq y_0(t)$ on $[t_0, t_0 + \nu]$. Thus $y(t) \simeq y_e$ for all $t \leq t_0 + \nu$, $t - t_0 \simeq +\infty$, and in particular we have $y(t) \simeq y_e$ for all $t \in [t_0 + \nu - r, t_0 + \nu]$. By the uniform stability of y_e we have $y(t) \simeq y_e$ for all $t > t_0 + \nu$. Hence $y(t) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$. Conversely, assume y_e satisfies the property in the lemma. By Lemma 3.8(2), the ball $\mathcal{B}'$ is a standard basin of attraction of y_e. Hence y_e is uniformly attractive. Let $\phi : [-r, 0] \to U$ be continuous with $\phi(t) \simeq y_e$ for all $t \in [-r, 0]$. For $t_0 \in \mathbb{R}$, let $y(\cdot; t_0, \phi)$ be the solution of (2.5). By hypothesis we have $y(t; t_0, \phi) \simeq y_e$ for all t such that $t - t_0 \simeq +\infty$, and by Lemma 3.9, $y(t; t_0, \phi) \simeq y(t; t_0, y_e) = y_e$ for all t such that $t - t_0$ is limited. By Lemma 3.8(1), y_e is uniformly stable. Thus y_e is uniformly asymptotically stable. $\qquad\square$

4. Proofs of Theorems 3.5 and 3.7

4.1. Preliminary lemmas. Hereafter we give some results we need for the proof of Theorem 3.5. We assume throughout this section that f and ϕ are standard. We suppose also that f satisfies conditions (H.0), (H.1), and (H.2) of Theorem 3.5. The conditions (H.1) and (H.2) will be used in their following external forms

(H.1′) $\forall^{\mathrm{st}} x_0 \in U \ \forall t > 0 \ \forall x \in U \ (x \simeq x_0 \Rightarrow f(t, x) \simeq f(t, x_0))$.

(H.2′) There is a standard function $f^o : U \to \mathbb{R}^n$ such that

$$\forall^{\mathrm{st}} x_0 \in U \quad \forall T \simeq +\infty \quad \left(f^o(x_0) \simeq \frac{1}{T} \int_0^T f(t, x_0) \, dt \right). \tag{4.1}$$

LEMMA 4.1. *The function f^o is continuous and we have*

$$f^o(x) \simeq \frac{1}{T} \int_0^T f(t, x) \, dt \tag{4.2}$$

for all x nearstandard in U and all $T \simeq +\infty$.

PROOF. See [14, Lemma 4, page 106]. $\qquad\square$

LEMMA 4.2. *For all limited $t > 0$ and all x nearstandard in U, there is $\alpha > 0$, $\alpha \simeq 0$, such that*

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x) \, d\tau \simeq f^o(x). \tag{4.3}$$

PROOF. See [14, Lemma 5, page 107], with $T = 1$, $S = \alpha/\varepsilon$ and $s = t/\varepsilon$. $\square$

LEMMA 4.3. *Let $L_1 > 0$ be standard and let $\tilde{x}$ be a function defined on $[-r, L_1]$. We assume that $\tilde{x}$ is continuous on $[-r, 0]$, $\tilde{x}(t)$ is nearstandard in U for all $t \in [0, L_1]$, and satisfies $\tilde{x}(t) \simeq \tilde{x}(t_n)$ for all $t \in [t_n, t_{n+1}]$ with $0 = t_0 < \cdots < t_n < t_{n+1} < \cdots < t_\omega \leq L_1 < t_{\omega+1}$ and $t_{n+1} - t_n = \alpha_n \simeq 0$, where α_n is determined by* Lemma 4.2. *Then*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau \simeq \int_0^t f^o(\tilde{x}(\tau - r)) d\tau \quad \forall t \in [0, L_1]. \tag{4.4}$$

PROOF. Let $t \in [0, L_1]$ and let N be a positive integer such that $t_N \leq t < t_{N+1}$. We have

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau - r)) d\tau$$

$$= \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r))\right) d\tau$$

$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r))\right) d\tau \tag{4.5}$$

$$+ \int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r))\right) d\tau$$

$$\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r))\right) d\tau,$$

since

$$\left\|\int_{t_N}^t \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) - f^o(\tilde{x}(\tau - r))\right) d\tau\right\|$$

$$\leq \int_{t_N}^t \left(\left\|f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right)\right\| + \|f^o(\tilde{x}(\tau - r))\|\right) d\tau \tag{4.6}$$

$$\leq 2M(t - t_N) \leq 2M(t_{N+1} - t_N) \leq 2M\alpha \simeq 0,$$

where $\alpha = \max\{\alpha_n\} \simeq 0$ (see Lemma 3.2) and M is a bound for f and then for f^o too, M is standard.

On the other hand, it is easy to verify that $\tilde{x}(\tau - r) \simeq \tilde{x}(t_n - r) = cte := \tilde{x}_n$ for $\tau \in [t_n, t_{n+1}]$. By the continuity of f, condition (H.1'), and Lemma 4.1 (the continuity of f^o) it follows, respectively, that

$$f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) \simeq f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right), \qquad f^o(\tilde{x}(\tau - r)) \simeq f^o(\tilde{x}_n), \tag{4.7}$$

or

$$f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau - r)\right) = f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) + \gamma_n(\tau), \qquad f^o(\tilde{x}(\tau - r)) = f^o(\tilde{x}_n) + \delta_n(\tau), \tag{4.8}$$

with

$$\gamma_n(\tau) \simeq 0 \simeq \delta_n(\tau). \tag{4.9}$$

Hence, from (4.5), we obtain

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau-r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau-r)) d\tau$$

$$\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n) + \gamma_n(\tau) + \delta_n(\tau)\right) d\tau$$

$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (\gamma_n(\tau) + \delta_n(\tau)) d\tau$$

$$\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau + \sum_{n=0}^{N-1} (\gamma_n + \delta_n) \int_{t_n}^{t_{n+1}} d\tau \tag{4.10}$$

$$\simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau + (\gamma + \delta) \sum_{n=0}^{N-1} (t_{n+1} - t_n)$$

$$= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau + (\gamma + \delta) \cdot t_N,$$

with $\gamma_n(\tau) \simeq \delta_n(\tau) \simeq 0$ for $\tau \in [t_n, t_{n+1}]$ and where $\gamma_n + \delta_n = \sup_{\tau \in [t_n,t_{n+1}]}\{\gamma(\tau)\} + \sup_{\tau \in [t_n,t_{n+1}]}\{\delta(\tau)\} \simeq 0$ and $\gamma + \delta = \max\{\gamma_n\} + \max\{\delta_n\} \simeq 0$ (see Lemma 3.2).

Since L_1 is standard and $t_N \in [0, L_1]$, t_N is limited and then $(\gamma + \delta) \cdot t_N \simeq 0$. Therefore,

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau-r)\right) d\tau - \int_0^t f^o(\tilde{x}(\tau-r)) d\tau \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau. \tag{4.11}$$

By Lemma 4.2, we have

$$\int_{t_n}^{t_{n+1}} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau = \int_{t_n}^{t_n + \alpha_n} \left(f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) - f^o(\tilde{x}_n)\right) d\tau$$

$$= \int_{t_n}^{t_n + \alpha_n} f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) d\tau - \int_{t_n}^{t_n + \alpha_n} f^o(\tilde{x}_n) d\tau$$

$$= \int_{t_n}^{t_n + \alpha_n} f\left(\frac{\tau}{\varepsilon}, \tilde{x}_n\right) d\tau - \alpha_n \cdot f^o(\tilde{x}_n)$$

$$= \varepsilon \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(s, \tilde{x}_n) ds - \alpha_n \cdot f^o(\tilde{x}_n), \quad \text{where } s = \frac{\tau}{\varepsilon}$$

$$= \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon + \alpha_n/\varepsilon} f(s, \tilde{x}_n) ds - f^o(\tilde{x}_n)\right)$$

$$= \alpha_n \cdot \beta_n, \quad \text{with } \beta_n \simeq 0. \tag{4.12}$$

Therefore

$$\int_0^t f\left(\frac{\tau}{\varepsilon},\tilde{x}(\tau-r)\right)d\tau - \int_0^t f^o(\tilde{x}(\tau-r))\,d\tau \simeq \sum_{n=0}^{N-1}\alpha_n\cdot\beta_n$$

$$\simeq \beta\sum_{n=0}^{N-1}\alpha_n, \quad\text{where } \beta=\max\{\beta_n\} \quad(4.13)$$

$$= \beta\sum_{n=0}^{N-1}(t_{n+1}-t_n) = \beta\cdot t_N.$$

By Lemma 3.2, $\beta\simeq 0$ and then $\beta\cdot t_N\simeq 0$. This implies the lemma. $\qquad\square$

LEMMA 4.4. *Let $L_1 > 0$ be standard and let x be a solution of (2.1). Assume that $[0,L_1]\subset I$ and $x(t)$ is nearstandard for all $t\in[0,L_1]$. Then x is S-continuous on $[0,L_1]$ and its shadow on $[0,L_1]$ coincides with the solution y of (2.2) on this interval, and satisfies $x(t)\simeq y(t)$ for all $t\in[0,L_1]$.*

PROOF. The solution x is given as

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon},x(\tau-r)\right)d\tau, \quad\text{for } t\in[0,L_1]. \quad(4.14)$$

As f is bounded on $\mathbb{R}\times U$, it is clear that x is S-continuous on $[0,L_1]$.

Let $\tilde{x}$ be a function which satisfies the hypotheses of Lemma 4.3 and such that

$$x(t)\simeq \tilde{x}(t), \quad\forall t\in[-r,L_1]. \quad(4.15)$$

Consider now the following equality which is always true

$$\int_0^t f\left(\frac{\tau}{\varepsilon},x(\tau-r)\right)d\tau - \int_0^t f^o(x(\tau-r))\,d\tau$$

$$= \int_0^t \left(f\left(\frac{\tau}{\varepsilon},x(\tau-r)\right)-f\left(\frac{\tau}{\varepsilon},\tilde{x}(\tau-r)\right)\right)d\tau$$

$$+\int_0^t \left(f\left(\frac{\tau}{\varepsilon},\tilde{x}(\tau-r)\right)-f^o(\tilde{x}(\tau-r))\right)d\tau \quad(4.16)$$

$$+\int_0^t \left(f^o(\tilde{x}(\tau-r))-f^o(x(\tau-r))\right)d\tau.$$

As $x(t)$ is nearstandard in U for any $t\in[-r,L_1]$, by the continuity of f, condition (H.1′) and (4.15), we have

$$f\left(\frac{t}{\varepsilon},x(t-r)\right)\simeq f\left(\frac{t}{\varepsilon},\tilde{x}(t-r)\right) \quad(4.17)$$

and by Lemma 3.3,

$$\int_0^t f\left(\frac{\tau}{\varepsilon},x(\tau-r)\right)d\tau \simeq \int_0^t f\left(\frac{\tau}{\varepsilon},\tilde{x}(\tau-r)\right)d\tau. \quad(4.18)$$

On the other hand, by Lemma 4.1 (the continuity of f^o) and by (4.15), we have

$$f^o(\tilde{x}(t-r)) \simeq f^o(x(t-r)), \tag{4.19}$$

and by Lemma 3.3, we obtain

$$\int_0^t f^o(\tilde{x}(\tau-r))\,d\tau \simeq \int_0^t f^o(x(\tau-r))\,d\tau. \tag{4.20}$$

By Lemma 4.3, for any $t \in [0,L_1]$, we have

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, \tilde{x}(\tau-r)\right)d\tau \simeq \int_0^t f^o\left(\tilde{x}(\tau-r)\right)d\tau. \tag{4.21}$$

Hence, for $t \in [0,L_1]$, (4.16) implies

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau-r)\right)d\tau \simeq \int_0^t f^o(x(\tau-r))\,d\tau. \tag{4.22}$$

Using (4.14) and (4.22), we obtain

$$\begin{aligned}
x(t) &= \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau-r)\right)d\tau \\
&\simeq \phi(0) + \int_0^t f^o(x(\tau-r))\,d\tau.
\end{aligned} \tag{4.23}$$

Let $^o x$ be the shadow of x on $[0,L_1]$. It is easy to see that the function z, where

$$z(t) = \begin{cases} ^o x(t), & \text{for } t \in [0,L_1], \\ \phi(t), & \text{for } t \in [-r,0], \end{cases} \tag{4.24}$$

is a solution of (2.2). The hypothesis (H.3) insures that $z = y$ on $[-r,L_1]$. Hence, we have $x(t) \simeq y(t)$ for $t \in [0,L_1]$. □

PROOF OF THEOREM 3.5. Let $L > 0$ be standard in J. Since $\Gamma = y([0,L])$ is a standard compact subset of U, there exists $\rho > 0$, ρ standard, and K, a standard compact neighborhood of Γ included in U, such that dist$(\Gamma,K) = \inf\{\|y-z\|/y \in \Gamma,\ z \in \mathbb{R}^n - K\} > \rho$. Let $x : I \to U$ be a solution of (2.1). Define the set $A = \{L_1 \in I \cap [0,L] \mid x([0,L_1]) \subset K\}$. The set A is nonempty ($0 \in A$) and bounded above by L. Let L_0 be a lower upper bound of A. There is $L_1 \in A$ such that $L_0 - \varepsilon < L_1 \le L_0$. Thus $x([0,L_1]) \subset K$. Hence on $[0,L_1]$ the function x is nearstandard in U. By Lemma 4.4, we have $x(t) \simeq y(t)$ for $t \in [0,L_1]$. Consider the interval $[0,L_1+\varepsilon]$. Let $t \in [0,L_1+\varepsilon]$. As $t - r$ is in $[-r,L_1+\varepsilon-r] \subset [-r,L_1]$, $x(t-r)$ is defined and so is

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x(\tau-r)\right)d\tau. \tag{4.25}$$

On the other hand, we have, for $t \in [L_1,L_1+\varepsilon]$, $x(t) \simeq x(L_1) \simeq y(L_1)$ with $y(L_1)$ nearstandard in U. Hence, on $[L_1,L_1+\varepsilon]$, x is nearstandard in U. Thus, on $[0,L_1+\varepsilon]$, x is defined and nearstandard in U. By Lemma 4.4, we have $x(t) \simeq y(t)$ for $t \in [0,L_1+\varepsilon]$. Hence $[0,L_1+\varepsilon] \subset I$ and $x([0,L_1+\varepsilon]) \subset K$. Suppose $L_1+\varepsilon \le L$, then $L_1+\varepsilon \in A$ which is a contradiction. Thus $L_1+\varepsilon > L$, that is, we have $x(t) \simeq y(t)$ for all $t \in [0,L] \subset [0,L_1+\varepsilon]$. □

PROOF OF THEOREM 3.7. By condition (H.5) and the uniform attractiveness of y_e (see Lemma 3.8(2)), the solution y is defined for all $t > 0$ and satisfies $y(t) \simeq y_e$ for all $t \simeq +\infty$. Let $x : I \to U$ be a solution of (2.1). By Theorem 3.5, for all limited $L > 0$, x is defined on $[0, L]$ and the approximation $x(t) \simeq y(t)$ holds for all $t \in [0, L]$. By Robinson's lemma, there exists $t_1 \simeq +\infty$ such that $x(t) \simeq y(t)$ on $[0, t_1]$. And then we have

$$x(t) \simeq y(t) \simeq y_e, \quad \forall t \leq t_1, \ t \simeq +\infty. \tag{4.26}$$

It remains to prove that x is defined for all $t \geq t_1$ and satisfies $x(t) \simeq y(t)$ for all $t > t_1$. Assume that this is false. Then there exists $s > t_1$ such that $x(s) \not\simeq y(s)$, that is,

$$\|x(s) - y(s)\| = \frac{2}{3}\kappa \tag{4.27}$$

is appreciable. Since $y(t) \simeq y_e$ for all $t \simeq +\infty$, we have

$$\|x(s) - y_e\| \leq \|x(s) - y(s)\| + \|y(s) - y_e\| \leq \frac{2}{3}\kappa + \frac{\kappa}{3} = \kappa. \tag{4.28}$$

Let $\mathcal{B}$, the ball of center y_e and radius $b_0 > 0$, b_0 standard, be the basin of attraction of y_e. We can choose s in (4.28) so that the ball $\mathcal{B}'$ of center y_e and radius κ is included in $\mathcal{B}$, with $b_0 - \kappa$ appreciable. Let t_2 be the first instant in time such that equality (4.27) holds. Clearly $t_2 > t_1$.

CASE 1 $(t_{1,2} = t_2 - t_1 \simeq +\infty)$**.** Redefine in (2.2) the initial time $r = t_0$. Let $z_1(\cdot; r, x, t_2)$ denote the solution of (2.2) such that $z_1(t; r, x, t_2) = x(t_2 - t)$ for $t \in [0, r]$. By Theorem 3.5 and Remark 3.6, for all limited $L > 0$, $z_1(\cdot; r, x, t_2)$ is defined on $[r, r + L]$ and satisfies $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ for $t \in [r, r + L]$. By Robinson's lemma, there exists $\bar{t}_{1,2} \simeq +\infty$, which one can choose such that $r + \bar{t}_{1,2} \leq t_{1,2}$, with the property that $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ on $[r, r + \bar{t}_{1,2}]$. In particular, $z_1(t; r, x, t_2) \simeq x(t_2 - t)$ on $[\bar{t}_{1,2}, r + \bar{t}_{1,2}] \subset [0, t_{1,2}]$. Since $x(t)$ belongs to $\mathcal{B}'$ for all $t \in [t_1, t_2]$, $x(t_2 - t)$ lies in $\mathcal{B}'$ for all $t \in [0, t_{1,2}]$. This implies that $z_1(t; r, x, t_2)$ is in $\mathcal{B}$ for all $t \in [\bar{t}_{1,2}, r + \bar{t}_{1,2}]$. By the uniform attractiveness of y_e (see Lemma 3.10), through the transformation $t \mapsto -t$, the solution of (2.2) with the initial function $z_1(-t; r, x, t_2)$ for $t \in [-r - \bar{t}_{1,2}, -\bar{t}_{1,2}]$ which coincides with $z_1(\cdot; r, x, t_2)$ (by uniqueness; hypothesis (H.3)) is defined for all $t > -\bar{t}_{1,2}$ and satisfies $z_1(-t; r, x, t_2) \simeq y_e$ for $t + \bar{t}_{1,2} \simeq +\infty$. Take $t = 0$, then $x(t_2) \simeq z_1(0; r, x, t_2) \simeq y_e$. Since $y(t_2) \simeq y_e$, this implies that $x(t_2) \simeq y(t_2)$. Which is a contradiction with $\|x(t_2) - y(t_2)\|$ being appreciable.

CASE 2 $(t_{1,2} = t_2 - t_1$ is limited$)$**.** By the continuity of the function $\|x(t) - y(t)\|$, there exists at least $t \in (t_1, t_2)$ such that

$$\|x(t) - y(t)\| = \frac{\kappa}{2}. \tag{4.29}$$

Let t_3 and t_4, respectively, be the first and the last instants in time such that equality (4.29) holds. We have $t_1 < t_3 \leq t_4 < t_2$. It is clear that

$$\|x(t) - y(t)\| < \frac{\kappa}{2}, \quad \forall t \in [t_1, t_3). \tag{4.30}$$

It is also clear that

$$0 < \frac{\kappa}{2} \leq \|x(t) - y(t)\| \leq \frac{2}{3}\kappa, \quad \forall t \in [t_4, t_2]. \tag{4.31}$$

Redefine in (2.2) the initial time $t_1 = t_0$. Let $z(\cdot; t_1, x)$ denote the solution of (2.2) such that $z(t; t_1, x) = x(t)$ for $t \in [t_1 - r, t_1]$. By (4.26) we have

$$z(t; t_1, x) = x(t) \simeq y(t) \simeq y_e, \quad \text{for } t \in [t_1 - r, t_1]. \tag{4.32}$$

Since $t_4 - t_1$ is limited, according to Theorems 3.5 and Remark 3.6

$$z(t; t_1, x) \simeq x(t), \quad \text{on } [t_1, t_4 + L] = [t_1, t_1 + ((t_4 - t_1) + L)], \quad \forall \text{ limited } L > 0. \tag{4.33}$$

By Robinson's lemma, there exists $\omega \simeq +\infty$ such that $z(t; t_1, x) \simeq x(t)$ on $[t_1, t_4 + \omega]$. Thus we have

$$z(t; t_1, x) \simeq x(t), \quad \text{on } [t_1, t_5], \text{ where } t_5 = t_4 + \omega. \tag{4.34}$$

By (4.32) and the uniform stability of y_e (see Lemma 3.8(1)) we deduce that

$$z(t; t_1, x) \simeq y(t) \, (\simeq y_e), \quad \forall t \geq t_1. \tag{4.35}$$

Thus, by (4.34) and (4.35)

$$x(t) \simeq y(t), \quad \forall t \in [t_1, t_5]. \tag{4.36}$$

Therefore $t_5 < t_2$ since $\|x(t_2) - y(t_2)\|$ is appreciable.

Take $t = t_5$. By (4.36) we have $x(t_5) \simeq y(t_5)$. This contradicts (4.31) since $t_5 \in [t_4, t_2]$. $\qquad\qquad \square$

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