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SPECTRAL GEOMETRY OF HARMONIC MAPS INTO WARPED PRODUCT MANIFOLDS II

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ABSTRACT. Let (M^n, g) be a closed Riemannian manifold and N a warped product manifold of two space forms. We investigate geometric properties by the spectra of the Jacobi operator of a harmonic map $\phi : M \to N$. In particular, we show if N is a warped product manifold of Euclidean space with a space form and $\phi, \psi : M \to N$ are two projectively harmonic maps, then the energy of ϕ and ψ are equal up to constant if ϕ and ψ are isospectral. Besides, we recover and improve some results by Kang, Ki, and Pak (1997) and Urakawa (1989).

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1. Introduction. In this paper, we deal with the inverse spectral problem of the Jacobi operator of a harmonic map from a compact manifold into warped product manifold.

The relationship between the geometry of a smooth manifold and the spectrum of the Laplacian has been studied by many authors (cf. [1, 5, 6]). In [6], Gilkey computed some spectral invariants concerning the asymptotic expansion of the trace of the heat kernel for an elliptic differential operator acting on the space of sections of a vector bundle (see also [5]). Urakawa applied the Gilkey's results to the Jacobi operator of a harmonic map from a closed (compact without boundary) manifold, M^n, into a space form of constant curvature, $N^m(c)$, and proved that if the Jacobi operators of two harmonic maps from M into N have the same spectrum, then these harmonic maps have the same energy. The Jacobi operator of a harmonic map arises in the second variational formula of the energy functional and several people studied in this field (see [9, 10, 11, 12]). In the case of Jacobi operator of a harmonic map, the spectral invariants computed by Gilkey can be expressed explicitly by the integration of geometric notions like curvature.

We will consider the Jacobi operator of a harmonic map from a closed manifold into a warped product manifold of two space forms which may be different. We generalize the results in [12] and prove some similar results about warped product manifolds. Warped product manifolds give us various examples and the structure of those are simple in some sense other than space forms (see [2]). Recently, Cheeger and Colding studied warped product manifolds and proved several remarkable results (see [4]). Also in [8], Ivanov and Petrova classified 4-dimensional Riemannian manifolds of positive constant curvature eigenvalues and showed that a warped product manifold is one of those manifolds and Gilkey, Leahy, and Sadofsky generalized this result for dimensions $n = 5,\ 6$, or $n \geq 9$ (see [7]).

2. Preliminaries. In this section, we describe, briefly, some results due to Gilkey and Urakawa about the asymptotic expansion of the trace of the heat kernel for the Jacobi operator of a harmonic map.

Let (M,g) be an n-dimensional compact Riemannian manifold without boundary and (N,h) an m-dimensional Riemannian manifold. A smooth map $\phi : M \to N$ is said to be *harmonic* if it is a critical point of the energy functional E defined by

$$E(\phi) = \int_M e(\phi)\, dv_g, \tag{2.1}$$

where $e(\phi) = (1/2)\sum h(\phi_* e_i, \phi_* e_i)$ called the *energy density*, ϕ_* is the differential of ϕ, and $\{e_i\}$ is a local orthonormal frame of M. In other words, for any vector field V along ϕ,

$$\frac{d}{dt}\bigg|_{t=0} E(\phi_t) = 0, \tag{2.2}$$

where $\phi_t : M \to N$ is a one parameter family of smooth maps with $\phi_0 = \phi$ and

$$\frac{d}{dt}\bigg|_{t=0} \phi_t = V_x \in T_{\phi(x)}N \tag{2.3}$$

for every point x in M.

The second variational formula of the energy E for a harmonic map ϕ is given by

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\phi_t) = \int_M h(V, J_\phi V)\, dv_g. \tag{2.4}$$

Here J_ϕ is a differential operator (called the *Jacobi operator*) acting on the space $\Gamma(\phi^{-1}TN)$ of sections of the induced bundle $\phi^{-1}TN$. The operator J_ϕ is of the form

$$J_\phi V = \tilde{\nabla}^* \tilde{\nabla} V - \sum_{i=1}^{n} R^N(\phi_* e_i, V)\phi_* e_i, \quad V \in \Gamma(\phi^{-1}TN), \tag{2.5}$$

where $\tilde{\nabla}$ is the connection of $\phi^{-1}TN$ which is induced by

$$\tilde{\nabla}_X V = \nabla^h_{\phi_* X} V, \tag{2.6}$$

where $V \in \phi^{-1}TN$, X is a tangent vector of M, ∇^h is the Levi-Civita connection of (N,h), and R^N is the curvature tensor of (N,h). Since J_ϕ is a selfadjoint, second-order elliptic operator, and M is compact, J_ϕ has a discrete spectrum of eigenvalues with finite multiplicities. We denote the spectrum of the Jacobi operator J_ϕ of the harmonic map ϕ by

$$\operatorname{Spec}(J_\phi) = \{\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots \uparrow \infty\}. \tag{2.7}$$

The operator e^{-tJ_ϕ} is defined by

$$e^{-tJ_\phi} V(x) = \int_M K(t,x,y,J_\phi) V(y)\, dv_g(y), \tag{2.8}$$

where $K(t,x,y,J_\phi)$ is an endomorphism from the fiber of $\phi^{-1}TN$ at y to the fiber at x, called the kernel function. Then one has an asymptotic expansion for the L^2-trace

$$\operatorname{Tr}(e^{-tJ_\phi}) = \sum_{j=1}^{\infty} \exp(-t\lambda_j) \sim (4\pi t)^{-n/2} \sum_{m=0}^{\infty} a_m(J_\phi)t^m \quad (\text{as } t \longrightarrow 0^+), \tag{2.9}$$

where $a_m(J_\phi)$ is the spectral invariant of J_ϕ which depends only on the spectrum, $\operatorname{Spec}(J_\phi)$. Moreover, since M is compact and without boundary, the odd terms of a_m vanish. For more detail, see [5, 6].

Finally, define the endomorphism L for $\phi^{-1}TN$ by

$$L(V) = \sum_{i=1}^{n} R^N(\phi_* e_i, V)\phi_* e_i, \quad V \in \phi^{-1}TN. \tag{2.10}$$

Then we have

$$\operatorname{Tr}_g(L) = \operatorname{Tr}_g(\phi^* \operatorname{Ric}^N), \tag{2.11}$$

where Ric^N denotes the Ricci curvature tensor of (N,h).

Now applying Gilkey's results to the Jacobi operator of a harmonic map, one has the following theorem.

Theorem 2.1 (see [5, 6, 12]). *For a harmonic map $\phi : (M,g) \to (N,h)$,*

$$\begin{aligned} a_0(J_\phi) &= m\operatorname{Vol}(M,g), \\ a_2(J_\phi) &= \frac{m}{6}\int_M s_M\, dv_g + \int_M \operatorname{Tr}_g(\phi^* \operatorname{Ric}^N)\, dv_g, \\ a_4(J_\phi) &= \frac{m}{360}\int_M \left\{5s_M^2 - 2||\operatorname{Ric}^M||^2 + 2||R^M||^2\right\} \\ &\quad + \frac{1}{360}\int_M \left\{-30||\phi^* R^N||^2 + 60 s_M \operatorname{Tr}_g(\phi^* \operatorname{Ric}^N) + 180\|L\|^2\right\} dv_g, \end{aligned} \tag{2.12}$$

where for tangent vectors $X, Y \in T_xM$, $(\phi^* R^N)(X,Y)$ is the endomorphism of $T_{\phi(x)}N$ given by $(\phi^* R^N)(X,Y) = R^N(\phi_* X, \phi_* Y)$ and s_M is the scalar curvature of (M,g).

3. Spectral invariants for warped product manifolds. We now assume that the target manifold (N,h) is a warped product manifold of the form $N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$, where $N^{m_i}(c_i)$ is a space form of constant curvature c_i $(i = 1,2)$, and f is a positive smooth function defined on $N^{m_1}(c_1)$. Furthermore, the Riemannian metric h is of the form $h = h_1 + f^2 h_2$, where h_i is the standard metric on $N^{m_i}(c_i)$ with constant curvature c_i.

We use the following convention $R(X,Y) = -[D_X, D_Y] + D_{[X,Y]}$ for the Riemannian curvature tensor, and so denoting $h = \langle\, ,\, \rangle$ we have in the space form of curvature c,

$$R(X,Y)Z = c\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\}. \tag{3.1}$$

Let $\{e_i\}_{i=1}^n$ be a local frame on M and $\{E_1, E_2, \dots, E_{m_1}, F_1, \dots, F_{m_2}\}$ be a local frame on N such that $E_1, E_2, \dots, E_{m_1}$ are tangent to $N^{m_1}(c_1)$ and $F_1, \dots, F_{m_2}$ are tangent to $N^{m_2}(c_2)$. Then $\phi_* e_i$ splits into the horizontal part

$$(\phi_* e_i)^T = \sum_{k=1}^{m_1} \langle \phi_* e_i, E_k \rangle E_k \tag{3.2}$$

and the vertical part

$$(\phi_* e_i)^\perp = \sum_{k=1}^{m_2} \langle \phi_* e_i, F_k \rangle F_k. \tag{3.3}$$

So

$$\langle \phi_* e_i, \phi_* e_j \rangle = \langle \phi_* e_i^T, \phi_* e_j^T \rangle + \langle \phi_* e_i^\perp, \phi_* e_j^\perp \rangle. \tag{3.4}$$

Denoting $e(\phi)^T = (1/2)\sum_{i=1}^n \langle \phi_* e_i^T, \phi_* e_i^T \rangle$ and $e(\phi)^\perp = (1/2)\sum_{i=1}^n \langle \phi_* e_i^\perp, \phi_* e_i^\perp \rangle$, the energy density of ϕ splits as follows:

$$e(\phi) = \frac{1}{2}\sum_{i=1}^n \langle \phi_* e_i, \phi_* e_i \rangle = e(\phi)^T + e(\phi)^\perp. \tag{3.5}$$

Finally, we denote

$$\begin{aligned} \|\phi^* h\|^2 &= \sum_{i,j=1}^n \langle \phi_* e_i, \phi_* e_j \rangle^2, \\ \|\phi^* h^T\|^2 &= \sum_{i,j=1}^n \langle \phi_* e_i^T, \phi_* e_j^T \rangle^2, \\ \|\phi^* h^\perp\|^2 &= \sum_{i,j=1}^n \langle \phi_* e_i^\perp, \phi_* e_j^\perp \rangle^2. \end{aligned} \tag{3.6}$$

Then we have

$$\|\phi^* h\|^2 = \|\phi^* h^T\|^2 + \|\phi^* h^\perp\|^2 + 2\langle \phi^* h^T, \phi^* h^\perp \rangle, \tag{3.7}$$

where

$$\langle \phi^* h^T, \phi^* h^\perp \rangle = \sum_{i,j=1}^n \langle \phi_* e_i^T, \phi_* e_j^T \rangle \langle \phi_* e_i^\perp, \phi_* e_j^\perp \rangle. \tag{3.8}$$

The rest of this section is devoted to compute the terms $a_2(J_\phi)$ and $a_4(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ in the case $N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$. To compute them, we have to calculate the terms $\mathrm{Tr}_g(L) = \mathrm{Tr}_g(\phi^* \mathrm{Ric}^N)$, $\|R^{\tilde{\nabla}}\|^2 = \|\phi^* R^N\|^2$, and $\mathrm{Tr}_g(L^2) = \|L\|^2$. To do this, the following lemma is needed. From now on, M is a closed Riemannian manifold and $N = (N, h)$ is $N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$ unless otherwise stated.

Lemma 3.1. *Let X, Y, Z be vector fields on $N^{m_1}(c_1)$ and U, V, W vector fields on $N^{m_2}(c_2)$. Then the Riemannian curvature tensor $R = R^N$ of N satisfies the following:*

$$\begin{aligned}
R(U,V)W &= \frac{c_2 - |\nabla f|^2}{f^2}\{\langle U,W\rangle V - \langle V,W\rangle U\},\\
R(X,V)Y &= -\frac{1}{f}\langle D_X \nabla f, Y\rangle V,\\
R(X,Y)V &= R(V,W)X = 0,\\
R(X,V)W &= R(X,W)V = \frac{1}{f}\langle V,W\rangle D_X\nabla f,\\
R(X,Y)Z &= \frac{c_1}{f^2}\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\},
\end{aligned} \tag{3.9}$$

where D denotes the Riemannian connection on M, and ∇f denotes the gradient of f.

Proof. The proof follows from [2, Lemma 7.4] and (3.1). □

From now, we will compute norms of curvature tensors.

3.1. $\mathrm{Tr}_g(L)$**.** Note that

$$\mathrm{Tr}_g(L) = \mathrm{Tr}_g(\phi^* \mathrm{Ric}^N) = \sum_{i=1}^{n}\sum_{j=1}^{m}\langle R^N(\phi_* e_i, E_j)\phi_* e_i, E_j\rangle, \tag{3.10}$$

where $m = m_1 + m_2$, and $E_{m_1+k} = F_k$, $k = 1,\dots,m_2$.

Using $\phi_* e_i = \phi_* e_i^T + \phi_* e_i^\perp$, and Lemma 3.1, one can get

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m_1}\langle R^N(\phi_* e_i, E_j)\phi_* e_i, E_j\rangle &= \frac{2c_1(m_1-1)}{f^2}e(\phi)^T - \frac{2}{f}e(\phi)^\perp\left(\sum_{j=1}^{m_1}\langle D_{E_j}\nabla f, E_j\rangle\right),\\
\sum_{i=1}^{n}\sum_{j=1}^{m_2}\langle R^N(\phi_* e_i, F_j)\phi_* e_i, F_j\rangle &= 2(m_2-1)\left(\frac{c_2 - |\nabla f|^2}{f^2}\right)e(\phi)^\perp\\
&\quad + \frac{1}{f}\sum_{i=1}^{n}\langle \phi_* e_i^\perp, D_{\phi_* e_i^\perp}\nabla f\rangle - \frac{m_2}{f}\sum_{i=1}^{n}\langle \phi_* e_i^T, D_{\phi_* e_i^T}\nabla f\rangle.
\end{aligned} \tag{3.11}$$

On the other hand, since ∇f is a horizontal vector field, that is, the tangential component of ∇f to $N^{m_2}(c_2)$ is zero, one has

$$\begin{aligned}
\sum_{j=1}^{m_1}\langle D_{E_j}\nabla f, E_j\rangle &= \Delta f,\\
\langle \phi_* e_i^\perp, D_{\phi_* e_i^T}\nabla f\rangle &= 0,\\
\sum_{i=1}^{n}\langle \phi_* e_i^T, D_{\phi_* e_i^T}\nabla f\rangle &= \mathrm{Tr}_g(\phi^*(Ddf)),
\end{aligned} \tag{3.12}$$

where Ddf denotes the Hessian of f.

Hence using these identities, one has

$$\begin{aligned}\operatorname{Tr}_g(L) = {} & \frac{2c_1(m_1-1)}{f^2}e(\phi)^T + 2(m_2-1)\left(\frac{c_2-|\nabla f|^2}{f^2}\right)e(\phi)^\perp \\ & -\frac{2\Delta f}{f}e(\phi)^\perp - \frac{m_2}{f}\operatorname{Tr}_g(\phi^*(Ddf)).\end{aligned} \tag{3.13}$$

3.2. $\|R^{\tilde{\nabla}}\|^2$. Note that

$$||R^{\tilde{\nabla}}||^2 = ||\phi^* R^N||^2 = \sum_{i,j=1}^{n}\sum_{k=1}^{m}\langle R^N(\phi_* e_i, \phi_* e_j)E_k, R^N(\phi_* e_i, \phi_* e_j)E_k\rangle, \tag{3.14}$$

where $m = m_1 + m_2$, and $E_{m_1+k} = F_k$, $k = 1, \dots, m_2$. The similar argument as in computing $\operatorname{Tr}_g(L)$ in Section 3.1, using $\phi_* e_i = \phi_* e_i^T + \phi_* e_i^\perp$, and the fact that ∇f is a horizontal vector field, and Lemma 3.1, one can get

$$\begin{aligned}\sum_{i,j=1}^{n}\sum_{k=1}^{m_1}||R^N(\phi_* e_i, \phi_* e_j)E_k||^2 = {} & \frac{8c_1^2}{f^4}(e(\phi)^T)^2 + \frac{4}{f^2}||\phi^* Ddf||^2 e(\phi)^\perp \\ & - \frac{2c_1^2}{f^4}||\phi^* h^T||^2 - \frac{2}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle, \\ \sum_{i,j=1}^{n}\sum_{k=1}^{m_2}||R^N(\phi_* e_i, \phi_* e_j)F_k||^2 = {} & 8\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2(e(\phi)^\perp)^2 + \frac{4}{f^2}||\phi^* Ddf||^2 e(\phi)^\perp \\ & - 2\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2||\phi^* h^\perp\|^2 - \frac{2}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle,\end{aligned} \tag{3.15}$$

where

$$\langle \phi^* h^\perp, \phi^* Ddf\rangle = \sum_{i,j=1}^{n}\langle \phi_* e_i^\perp, \phi_* e_j^\perp\rangle\langle \phi^* Ddf(e_i), \phi^* Ddf(e_j)\rangle. \tag{3.16}$$

Summing up these two equations, one gets

$$\begin{aligned}||R^{\tilde{\nabla}}||^2 &= \sum_{i,j=1}^{n}\sum_{k=1}^{m_1}||R^N(\phi_* e_i, \phi_* e_j)E_k||^2 + \sum_{i,j=1}^{n}\sum_{k=1}^{m_2}||R^N(\phi_* e_i, \phi_* e_j)F_k||^2 \\ &= \frac{8c_1^2}{f^4}(e(\phi)^T)^2 + 8\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2(e(\phi)^\perp)^2 \\ &\quad + \frac{8}{f^2}||\phi^* Ddf||^2 e(\phi)^\perp - \frac{2c_1^2}{f^4}||\phi^* h^T||^2 \\ &\quad - 2\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2||\phi^* h^\perp||^2 - \frac{4}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle.\end{aligned} \tag{3.17}$$

3.3. $\mathrm{Tr}_g(L^2)$**.** Note that

$$\begin{aligned}\mathrm{Tr}_g(L^2) &= \|L\|^2 = \sum_{k=1}^{m} \|L(E_k)\|^2 \\ &= \sum_{k=1}^{m}\sum_{i,j=1}^{n} \langle R^N(\phi_* e_i, E_k)\phi_* e_i, R^N(\phi_* e_j, E_k)\phi_* e_j\rangle.\end{aligned} \tag{3.18}$$

A straightforward computation which is a little complicated, but not still hard shows

$$\begin{aligned}&\sum_{i,j=1}^{n}\sum_{k=1}^{m_1} \langle R^N(\phi_* e_i, E_k)\phi_* e_i, R^N(\phi_* e_j, E_k)\phi_* e_j\rangle \\ &\quad= \frac{4(m_1-2)c_1^2}{f^4}(e(\phi)^T)^2 - 8c_1\frac{\Delta f}{f^3}e(\phi)^T e(\phi)^\perp \\ &\qquad+ \frac{c_1^2}{f^4}\|\phi^* h^T\|^2 + \frac{4c_1}{f^3}e(\phi)^\perp \mathrm{Tr}_g(\phi^* Ddf) \\ &\qquad+ \frac{4}{f^2}(e(\phi)^\perp)^2\|Ddf\|^2 + \frac{1}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle.\end{aligned} \tag{3.19}$$

In the last term one can use the following identity:

$$\sum_{k=1}^{m_1} \langle D_{E_k}\nabla f, \phi_* e_i^T\rangle\langle D_{E_k}\nabla f, \phi_* e_j^T\rangle = \langle \phi^* Ddf(e_i), \phi^* Ddf(e_j)\rangle. \tag{3.20}$$

Similarly one has

$$\begin{aligned}&\sum_{i,j=1}^{n}\sum_{k=1}^{m_2} \langle R^N(\phi_* e_i, F_k)\phi_* e_i, R^N(\phi_* e_j, F_k)\phi_* e_j\rangle \\ &\quad= \frac{m_2}{f^2}(\mathrm{Tr}_g\,\phi^* Ddf)^2 - 4(m_2-1)\left(\frac{c_2-|\nabla f|^2}{f^3}\right)e(\phi)^\perp(\mathrm{Tr}_g\,\phi^* Ddf) \\ &\qquad+ 4(m_2-2)\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2(e(\phi)^\perp)^2 + \left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2\|\phi^* h^\perp\|^2 \\ &\qquad+ \frac{1}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle.\end{aligned} \tag{3.21}$$

Therefore,

$$\begin{aligned}\mathrm{Tr}_g(L^2) &= \frac{4(m_1-2)c_1^2}{f^4}(e(\phi)^T)^2 + 4(m_2-2)\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2(e(\phi)^\perp)^2 \\ &\quad- 8c_1\frac{\Delta f}{f^3}e(\phi)^T e(\phi)^\perp + \frac{c_1^2}{f^4}\|\phi^* h^T\|^2 + \frac{4c_1}{f^3}e(\phi)^\perp\mathrm{Tr}_g(\phi^* Ddf) \\ &\quad+ \frac{4}{f^2}(e(\phi)^\perp)^2\|Ddf\|^2 + \frac{m_2}{f^2}(\mathrm{Tr}_g\,\phi^* Ddf)^2 \\ &\quad- 4(m_2-1)\left(\frac{c_2-|\nabla f|^2}{f^3}\right)e(\phi)^\perp(\mathrm{Tr}_g\,\phi^* Ddf) \\ &\quad+ \left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2\|\phi^* h^\perp\|^2 + \frac{2}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle.\end{aligned} \tag{3.22}$$

Now for a map $\phi : M \to N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$, let $\phi = (\phi_1, \phi_2)$, where $\phi_i = \pi_i \circ \phi$ $(i = 1,2)$, and $\pi_i : N \to N^{m_1}(c_i)$ $(i = 1,2)$ be the projection. Then substituting (3.13), (3.17), and (3.22) into Theorem 2.1, one gets the following theorem.

Theorem 3.2. *Let $\phi : (M,g) \to N^{m_1}(c_1) \times_f N^{m_2}(c_2)$ be a harmonic map of an n-dimensional compact Riemannian manifold (M,g) into an $m(= m_1 + m_2)$-dimensional Riemannian warped product manifold N. Then the coefficients $a_0(J_\phi)$, $a_2(J_\phi)$, and $a_4(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are, respectively, given by*

$$\begin{aligned}
a_0(J_\phi) &= m \operatorname{Vol}(M,g), \\
a_2(J_\phi) &= \frac{m}{6}\int_M s_g\, dv_g + 2c_1(m_1-1)\int_M \left(\frac{1}{f^2}\circ\phi_1\right) e(\phi)^T\, dv_g \\
&\quad + 2\int_M \left(\left\{\frac{(m_2-1)(c_2-|\nabla f|^2)}{f^2} - \frac{\Delta f}{f}\right\}\circ\phi_1\right) e(\phi)^\perp\, dv_g \\
&\quad - m_2\int_M \left(\frac{1}{f}\circ\phi_1\right)\operatorname{Tr}_g(\phi^* Ddf)\, dv_g, \\
a_4(J_\phi) &= \frac{m}{360}\int_M \left\{5s_g^2 - 2\|\operatorname{Ric}^M\|^2 + 2\|R^M\|^2\right\} dv_g \\
&\quad - \frac{1}{12}\int_M \left\{8\left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2 (e(\phi)^\perp)^2 + 8\frac{\|\phi^* Ddf\|^2}{f^2} e(\phi)^\perp + \frac{8c_1^2}{f^4}(e(\phi)^T)^2\right\} dv_g \\
&\quad + \frac{1}{6}\int_M \left\{\frac{c_1^2}{f^4}\|\phi^* h^T\|^2 + \left(\frac{c_2-|\nabla f|^2}{f^2}\right)^2 \|\phi^* h^\perp\|^2 + \frac{2}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle\right\} dv_g \\
&\quad + \frac{1}{6}\int_M s_g\left\{2\left(\frac{(m_2-1)(c_2-|\nabla f|^2)}{f^2} - \frac{\Delta f}{f}\right) e(\phi)^\perp + \frac{2c_1(m_1-1)}{f^2} e(\phi)^T\right\} dv_g \\
&\quad - \frac{m_2}{6}\int_M \left(\frac{1}{f}\circ\phi_1\right)\operatorname{Tr}_g(\phi^* Ddf)\, dv_g + \frac{1}{2}\int_M Q\, dv_g,
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
Q &= \frac{4(m_1-2)c_1^2}{f^4}(e(\phi)^T)^2 - \frac{8c_1}{f^3} e(\phi)^T e(\phi)^\perp \Delta f + \frac{c_1^2}{f^4}\|\phi^* h^T\|^2 \\
&\quad + \frac{4c_1}{f^3} e(\phi)^\perp \operatorname{Tr}_g(\phi^* Ddf) + \frac{4}{f^2}(e(\phi)^\perp)^2 \|Ddf\|^2 \\
&\quad + 4(m_2-2)\left(\frac{c_2-|\nabla f|^2}{f^2}\right)(e(\phi)^\perp)^2 + \frac{m_2}{f^2}(\operatorname{Tr}_g(\phi^* Ddf))^2 \\
&\quad - 4(m_2-1)\left(\frac{c_2-|\nabla f|^2}{f^2}\right) e(\phi)^\perp \operatorname{Tr}_g(\phi^* Ddf) \\
&\quad + \left(\frac{c_2-|\nabla f|^2}{f^2}\right)\|\phi^* h^\perp\|^2 + \frac{2}{f^2}\langle \phi^* h^\perp, \phi^* Ddf\rangle.
\end{aligned} \tag{3.24}$$

Note that the integration of the function f over M means the integration of $f \circ \phi_1$ over M.

In the product case, that is, f is a constant function 1, Theorem 3.2 reduces to the following which is a result due to [9]. However our expression looks a little more concrete.

Corollary 3.3. *Let $\phi : (M,g) \to N = N^{m_1}(c_1) \times N^{m_2}(c_2)$ be a harmonic map of an n-dimensional compact Riemannian manifold (M,g) into an $m(= m_1 + m_2)$-dimensional Riemannian product manifold N. Then the coefficients $a_0(J_\phi)$, $a_2(J_\phi)$, and $a_4(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are, respectively, given by*

$$\begin{aligned}
a_0(J_\phi) &= m\,\mathrm{Vol}(M,g),\\
a_2(J_\phi) &= \frac{m}{6}\int_M s_g\,dv_g + 2c_1(m_1-1)\int_M e(\phi)^T\,dv_g\\
&\quad + 2c_2(m_2-1)\int_M e(\phi)^\perp\,dv_g,\\
a_4(J_\phi) &= \frac{m}{360}\int_M \left\{5s_g^2 - 2\|\mathrm{Ric}^M\|^2 + 2\|R^M\|^2\right\}dv_g\\
&\quad + \frac{2}{3}c_1^2(3m_1-7)\int_M \left(e(\phi)^T\right)^2 dv_g\\
&\quad + \frac{2}{3}c_2^2(3m_2-7)\int_M \left(e(\phi)^\perp\right)^2 dv_g\\
&\quad + \frac{2}{3}c_1^2\int_M \|\phi^* h^T\|^2\,dv_g + \frac{2}{3}c_2^2\int_M \|\phi^* h^\perp\|^2\,dv_g\\
&\quad + \frac{1}{3}c_1(m_1-1)\int_M s_g e(\phi)^T\,dv_g\\
&\quad + \frac{1}{3}c_2(m_2-1)\int_M s_g e(\phi)^\perp\,dv_g.
\end{aligned} \tag{3.25}$$

Remark 3.4. Since $\pi_i : N \to N^{m_i}(c_i)$ is totally geodesic in case $f \equiv 1$, the composition $\pi_i \circ \phi = \phi_i$ is also harmonic. So Corollary 3.3 implies that the coefficients of the asymptotic expansion for the Jacobi operator J_ϕ split as follows:

$$\begin{aligned}
a_0(J_\phi) &= a_0(J_{\phi_1}) + a_0(J_{\phi_2}),\\
a_2(J_\phi) &= a_2(J_{\phi_1}) + a_2(J_{\phi_2}),\\
a_4(J_\phi) &= a_4(J_{\phi_1}) + a_4(J_{\phi_2}).
\end{aligned} \tag{3.26}$$

Also Corollary 3.3 reproves a result of [12].

Corollary 3.5 (see [12]). *Let $\phi : (M,g) \to N^m(c)$ be a harmonic map of an n-dimensional compact Riemannian manifold (M,g) into an m-dimensional space form N. Then the coefficients $a_0(J_\phi)$, $a_2(J_\phi)$, and $a_4(J_\phi)$ of the asymptotic expansion for* *the Jacobi operator J_ϕ are, respectively, given by*

$$\begin{aligned} a_0(J_\phi) &= m\,\mathrm{Vol}(M,g),\\ a_2(J_\phi) &= \frac{m}{6}\int_M s_g\,dv_g + 2c(m-1)E(\phi),\\ a_4(J_\phi) &= \frac{m}{360}\int_M \left\{5s_g^2 - 2\|\mathrm{Ric}^M\|^2 + 2\|R^M\|^2\right\}dv_g\\ &\quad + \frac{2}{3}c^2\int_M \left\{(3m-7)e(\phi)^2 + \|\phi^* h\|^2\right\}dv_g\\ &\quad + \frac{1}{3}c(m-1)\int_M s_g e(\phi)\,dv_g. \end{aligned} \tag{3.27}$$

Proof. One can consider ϕ as a map $\phi : M \to N^m(c)\times N^m(c)$ with $\phi = (\phi, \text{const.})$ and apply Corollary 3.3. □

As a special case of Corollary 3.3, when $m_1 = m_2 = m$ and $c_1 = c_2 = c$, one gets the following corollary.

Corollary 3.6. *Let $\phi : (M,g) \to N^m(c)\times N^m(c)$ be a harmonic map of an n-dimensional compact Riemannian manifold (M,g) into a $2m$-dimensional Riemannian product manifold N. Then the coefficients $a_0(J_\phi)$, $a_2(J_\phi)$, and $a_4(J_\phi)$ of the asymptotic expansion for the Jacobi operator J_ϕ are, respectively, given by*

$$\begin{aligned} a_0(J_\phi) &= 2m\,\mathrm{Vol}(M,g),\\ a_2(J_\phi) &= \frac{m}{3}\int_M s_g\,dv_g + 2c(m-1)E(\phi),\\ a_4(J_\phi) &= \frac{m}{180}\int_M \left\{5s_g^2 - 2\|\mathrm{Ric}^M\|^2 + 2\|R^M\|^2\right\}dv_g\\ &\quad + \frac{2}{3}c^2(3m-7)\int_M \left\{\left(e(\phi)^T\right)^2 + \left(e(\phi)^\perp\right)^2\right\}dv_g\\ &\quad + \frac{2}{3}c^2\int_M \left(\|\phi^* h^T\|^2 + \|\phi^* h^\perp\|^2\right)dv_g\\ &\quad + \frac{1}{3}c(m-1)\int_M s_g e(\phi)\,dv_g. \end{aligned} \tag{3.28}$$

4. Applications. In this section, we will investigate properties for the Jacobi operator when the spectrum of the harmonic maps coincide in various cases of N.

First we recover a result of [9].

Corollary 4.1 (see [9]). *Let $\phi, \psi : (M,g) \to N^m(c)\times N^m(c)$ be two harmonic maps of an n-dimensional compact Riemannian manifold (M,g) into a $2m$-dimensional Riemannian product manifold N with $m \ge 2$ and $c \ne 0$. If ϕ and ψ are isospectral, then*

$$E(\phi) = E(\psi). \tag{4.1}$$

Proof. The proof follows from Corollary 3.6. □

In Corollary 4.1, if furthermore M has a constant scalar curvature, then one has

$$\begin{aligned}(3m-7)\int_M \left\{(e(\phi)^T)^2+(e(\phi)^\perp)^2\right\}dv_g+2\int_M\left(\|\phi^*h^T\|^2+\|\phi^*h^\perp\|^2\right)dv_g\\ =(3m-7)\int_M\left\{(e(\psi)^T)^2+(e(\psi)^\perp)^2\right\}dv_g+2\int_M\left(\|\psi^*h^T\|^2+\|\psi^*h^\perp\|^2\right)dv_g.\end{aligned} \tag{4.2}$$

The following theorem is an improved version of [9, Corollary 3.3].

Corollary 4.2. *Let $\phi,\psi:(M,g)\to N^{m_1}(c_1)\times N^{m_2}(c_2)$ be two isometric minimal immersions of an n-dimensional compact Riemannian manifold (M,g) into an $m(=m_1+m_2)$-dimensional Riemannian product manifold N. Suppose that $c_1\neq 0$ or $c_2\neq 0$, and either m_1 or m_2 is greater than one. If ϕ and ψ are isospectral, then*

$$E(\phi)=E(\psi). \tag{4.3}$$

Proof. In case $m_1=m_2$ and $c_1=c_2$, this reduces to Corollary 4.1. Thus we may assume $m_1\neq m_2$ or $c_1\neq c_2$. Note that $e(\phi)=n/2$, $n=\dim(M)$ and

$$e(\phi)^\perp=\frac{n}{2}-e(\phi)^T. \tag{4.4}$$

So it follows from Corollary 3.3 that

$$\begin{aligned}a_2(J_\phi)=\frac{m}{6}\int_M s_g\,dv_g+2\{c_1(m_1-1)-c_2(m_2-1)\}\int_M e(\phi)^T dv_g\\ +c_2(m_2-1)n\operatorname{Vol}(M,g).\end{aligned} \tag{4.5}$$

Hence comparing this with $a_2(J_\psi)$, one gets

$$\int_M e(\phi)^T dv_g=\int_M e(\psi)^T dv_g. \tag{4.6}$$

Now from (4.4),

$$\int_M e(\phi)^\perp dv_g=\int_M e(\psi)^\perp dv_g \tag{4.7}$$

and hence $E(\phi)=e(\psi)$. □

In Corollary 4.2, if furthermore M has a constant scalar curvature, then one has

$$\begin{aligned}&\frac{2}{3}\{c_1^2(3m_1-7)+c_2^2(3m_2-7)\}\int_M (e(\phi)^T)^2 dv_g\\ &\quad+\frac{2}{3}c_1^2\int_M\|\phi^*h^T\|^2dv_g+\frac{2}{3}c_2^2\int_M\|\phi^*h^\perp\|^2dv_g\\ &=\frac{2}{3}\{c_1^2(3m_1-7)+c_2^2(3m_2-7)\}\int_M(e(\psi)^T)^2dv_g\\ &\quad+\frac{2}{3}c_1^2\|\psi^*h^T\|^2dv_g+\frac{2}{3}c_2^2\int_M\|\psi^*h^\perp\|^2dv_g.\end{aligned} \tag{4.8}$$

Finally, we will discuss projectively harmonic maps. In general, the composition of two harmonic maps is not necessarily harmonic.

Definition 4.3. We say a harmonic map $\phi: M \to N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$ is *projectively harmonic* if the compositions $\phi_1 = \pi_1 \circ \phi$ and $\phi_2 = \pi_2 \circ \phi$ are harmonic maps, where $\pi_i : N \to N^{m_i}(c_i)$ is the projection map.

Not every harmonic map is in general projectively harmonic (cf. [3]). If $\pi_i : N \to N^{m_i}(c_i)$ is totally geodesic, then $\phi_i = \pi_i \circ \phi$ $(i = 1, 2)$ is harmonic and so in this case every harmonic map $\phi : M \to N$ is projectively harmonic. In particular, in the case of product (i.e., $f \equiv 1$), every harmonic map is automatically projectively harmonic.

Theorem 4.4. *Let $\phi, \psi : M^n \to N = \mathbb{R}^{m_1} \times_f N^{m_2}(c_2)$ be two projectively harmonic maps. If ϕ and ψ are isospectral, then*

$$A(\phi)E(\phi) = A(\psi)E(\psi), \tag{4.9}$$

where

$$A(\phi) = \left\{ \frac{(m_2 - 1)(c_2 - |\nabla f|^2)}{f^2} - \frac{\Delta f}{f} \right\} \circ \phi_1 \tag{4.10}$$

and $A(\psi)$ is similar.

Proof. In case $N = \mathbb{R}^{m_1} \times_f N^{m_2}(c_2)$, $\phi_1 = \pi_1 \circ \phi$ and $\psi_1 = \pi_1 \circ \psi$ are constants since M is compact. Thus, so $e(\phi) = e(\phi)^\perp$ and $e(\psi) = e(\psi)^\perp$. Furthermore, it follows from (3.12) that $\mathrm{Tr}_g(\phi^* Ddf) = 0$. Hence from Theorem 3.2, one gets the results. Note that both $A(\phi)$ and $A(\psi)$ are constants. □

Theorem 4.5. *Let $\phi, \psi : M^n \to N = N^{m_1}(c_1) \times_f N^{m_2}(c_2)$ be two projectively harmonic maps with $m_2 > 1$. If ∇f is parallel, and ϕ and ψ are isospectral, then*

$$E(\phi) = E(\psi). \tag{4.11}$$

Proof. As in the proof of Theorem 4.4, $\phi_1 = \pi_1 \circ \phi$ and $\psi_1 = \pi_1 \circ \psi$ are constants and so $e(\phi) = e(\phi)^\perp$ and $e(\psi) = e(\psi)^\perp$. Moreover by hypothesis, $Ddf = 0$ and so $|\nabla f|$ is constant. The harmonic map is invariant under scaling of the metric, one may assume that $|\nabla f|^2 \neq c_2$. Then by Theorem 3.2, one has

$$a_2(J_\phi) = \frac{m}{6} \int_M s_g \, dv_g + 2(m_2 - 1)(c_2 - |\nabla f|^2) \left(\frac{1}{f^2} \circ \phi_1 \right) E(\phi). \tag{4.12}$$

Note that $((1/f^2) \circ \phi_1)$ is constant. Hence if ϕ and ψ are isospectral, then comparing $a_2(J_\phi)$ with $a_2(J_\psi)$, one gets

$$E(\phi) = E(\psi). \tag{4.13}$$

□

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